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WEAK CONTRACTION CONDITION FOR COMPATIBLE MAPPINGS INVOLVING CUBIC TERMS OF THE METRIC FUNCTION

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Abstract

In this paper, we prove a common fixed point theorem for compatible mappings satisfying the generalized ϕ-weak contraction condition involving cubic terms.

1. Introduction

For the last four decades there has been a considerable interest to study common fixed point for a pair (or family) of mappings satisfying contractive conditions in metric spaces. Several interesting and elegant results were

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obtained in this direction by various authors. It was the turning point in the fixed point theory literature when the notion of commutativity mappings was used by Jungck [4] to obtain a generalization of Banach's fixed point theorem for a pair of mappings. This result was further generalized, extended and unified using various types of contractions and minimal commutative mappings.

Fixed point theorems statements basically involve sufficient conditions for the existence of fixed points. Therefore, one of the central concerns in fixed point theory is to find a minimal set of sufficient conditions which ensures the guarantee of fixed points or common fixed points. Common fixed point theorems for contraction type mappings necessarily require a commutativity condition, a condition on the containment of ranges of the mappings, continuity of one or more mappings besides a contraction condition. Mostly fixed point or common fixed point theorems attempt to weaken the above described condition. The study of common fixed points of pair of self mappings satisfying contractive type conditions becomes more interesting when we extend such studies to the class of noncommuting contractive type mapping pair.

In 1969, Boyd and Wong [2] replaced the constant k in Banach contractive condition by an upper semi-continuous function as follows:

Let (X, d) be a complete metric space and $\psi : [0, \infty) \to [0, \infty)$ be upper semi-continuous from the right such that $0 \leq \psi(t) < t$ for all $t > 0.$ If $T : X \to X$ satisfies $d(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$, then it has a unique fixed point $x \in X$ and $\{T_n x\}$ converges to x for all $x \in X$.

The first ever attempt to relax the commutativity of mappings to a smaller subset of the domain of mappings was initiated by Sessa [9] in 1982 who gave the notion of weak commutativity. One can notice that the notion of weak commutativity is a point property, while the notion of compatibility is an iterate of sequence. Two self mappings f and g of a metric space (X, d) are said to be *weakly commuting* if $d(fgx, gfx) \leq d(gx, fx)$ for all $x \in X$.

Further, in 1986, Jungck [5] introduced more generalized commutativity, so-called compatibility. Clearly commuting and weakly commuting mappings are compatible but converse need not be true (see [6]).

In 1997, Alber and Guerre-Delabriere [1] introduced the concept of weak contraction and in 2001, Rhoades [8] had shown that the results of Alber and Guerre-Delabriere [1] are also valid in complete metric spaces.

A mapping $T : X \to X$ is said to be ϕ-*weak contraction* if for each $x, y \in X$, there exists a function $\phi : [0, \infty) \to [0, \infty)$, $\phi(t) > 0$ for all $t > 0$ and $\phi(0) = 0$ such that

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)).$$

In 2013, Murthy and Vara Prasad [7] introduced a new type of inequality having cubic terms that extended and generalized the results of Alber and Guerre-Delabriere [1] and others cited in the literature of fixed point theory. Further Jain et al. [3] extended and generalized the result of Murthy and Vara Prasad [7] for pairs of mappings.

In this paper, we extend and generalize the result of Jain et al. [3] for two pairs of mappings satisfying the generalized ϕ-weak contractive condition involving various combination of the metric function.

2. Preliminaries

In this section, we give some basic definitions and results that are useful for proving our main results.

In 1986, Jungck [5] introduced the notion of compatible mappings as follows:

Definition 2.1. Two self mappings f and g on a metric space (X, d) are called *compatible* if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

Proposition 2.2 [5]**.** *Let S and T be compatible mappings of a metric space* (X, d) *into itself. If* $St = Tt$ *for some* $t \in X$, *then* $STt = SSt = TTt = TSt$.

Proposition 2.3 [5]**.** *Let S and T be compatible mappings of a metric space* (X, d) *into itself. Suppose that* $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ *for some* $t \in X$. *Then*

(i) $\lim_{n\to\infty} TSx_n = St$ *if S is continuous at t*;

(ii) $\lim_{n\to\infty} STx_n = Tt$ *if T is continuous at t*;

(iii) $STt = TSt$ *and* $St = Tt$ *if S and T are continuous at t.*

3. Fixed Points for Compatible Mappings

In 2013, Murthy and Vara Prasad [7] proved the following result:

Theorem 3.1. *Let T be a mapping of a complete metric space* (X, d) *into itself satisfying the following*:

$$\begin{aligned}
&[1 + pd(x, y)]d^2(Tx, Ty) \\
&\leq p\max\{1/2\,[d^2(x, Tx)d(y, Ty) + d(x, Tx)d^2(y, Ty)], \\
&\qquad d(x, Tx)d(x, Ty)d(y, Tx),\ d(x, Ty)d(y, Tx)d(y, Ty)\} \\
&\quad + m(x, y) - \phi(m(x, y)),
\end{aligned}$$

where

$$\begin{aligned}
m(x, y) = \max\{&d^2(x, y),\ d(x, Tx)d(y, Ty),\ d(x, Ty)d(y, Tx), \\
&1/2\,[d(x, Tx)d(x, Ty) + d(y, Tx)d(y, Ty)]\},
\end{aligned}$$

$p \geq 0$ *is a real number and* $\phi : [0, \infty) \to [0, \infty)$ *is a continuous function with* $\phi(t) = 0$ *iff* $t = 0$ *and* $\phi(t) > 0$ *for each* $t > 0$.

Then T has a unique fixed point in X.

Now we extend and generalize Theorem 3.1 for pairs of compatible mappings as follows:

Theorem 3.2. *Let S, T, A and B be four mappings of a complete metric space* (X, d) *into itself satisfying the following conditions*:

(*C*1) $S(X) \subset B(X)$ *and* $T(X) \subset A(X)$;

(*C*2)

$$[1 + pd(Ax, By)]d^2(Sx, Ty)$$

$$\leq p\max\{1/2[d^2(Ax, Sx)d(By, Ty) + d(Ax, Sx)d^2(By, Ty)],$$

$$d(Ax, Sx)d(Ax, Ty)d(By, Sx),\ d(Ax, Ty)d(By, Sx)d(By, Ty)\}$$

$$+ m(Ax, By) - \phi(m(Ax, By))$$

for all $x, y \in X$, *where*

$$m(Ax, By) = \max\{d^2(Ax, By),\ d(Ax, Sx)d(By, Ty),\ d(Ax, Ty)d(By, Sx),$$

$$1/2[d(Ax, Sx)d(Ax, Ty) + d(By, Sx)d(By, Ty)]\},$$

$p \geq 0$ *is a real number and* $\phi : [0, \infty) \to [0, \infty)$ *is a continuous function with* $\phi(t) = 0$ *iff* $t = 0$ *and* $\phi(t) > 0$ *for each* $t > 0$;

(*C*3) *one of S, T, A and B is continuous*.

Assume that the pairs (A, S) *and* (B, T) *are compatible*. *Then S, T, A and B have a unique common fixed point in X*.

Proof. Let $x_0 \in X$ be an arbitrary point. From (*C*1), we can find x_1 such that $Sx_0 = Bx_1 = y_0$ for this x_1 one can find $x_2 \in X$ such that $Tx_1 = Ax_2 = y_1$. Continuing in this way one can construct a sequence $\{y_n\}$ such that

$$y_{2n} = Sx_{2n} = Bx_{2n+1},\ y_{2n+1} = Tx_{2n+1} = Ax_{2n+2}, \quad n \geq 0. \tag{3.1}$$

For brevity, we write $\alpha_{2n} = d(y_{2n}, y_{2n+1})$.

First we prove that $\{\alpha_{2n}\}$ is non increasing sequence and converges to zero.

Case I. If n is even, taking $x = x_{2n}$ and $y = x_{2n+1}$ in ($C2$), we get

$$\begin{aligned}
&[1 + pd(Ax_{2n}, Bx_{2n+1})]d^2(Sx_{2n}, Tx_{2n+1}) \\
&\leq p \max\{1/2\,[d^2(Ax_{2n}, Sx_{2n})d(Bx_{2n+1}, Tx_{2n+1}) \\
&\qquad + d(Ax_{2n}, Sx_{2n})d^2(Bx_{2n+1}, Tx_{2n+1})], \\
&\qquad d(Ax_{2n}, Sx_{2n})d(Ax_{2n}, Tx_{2n+1})d(Bx_{2n+1}, Sx_{2n}), \\
&\qquad d(Ax_{2n}, Tx_{2n+1})d(Bx_{2n+1}, Sx_{2n})d(Bx_{2n+1}, Tx_{2n+1})\} \\
&\quad + m(Ax_{2n}, Bx_{2n+1}) - \phi(m(Ax_{2n}, Bx_{2n+1})),
\end{aligned}$$

where

$$\begin{aligned}
&m(Ax_{2n}, Bx_{2n+1}) \\
&= \max\{(d^2(Ax_{2n}, Bx_{2n+1}), d(Ax_{2n}, Sx_{2n})d(Bx_{2n+1}, Tx_{2n+1}), \\
&\qquad d(Ax_{2n}, Tx_{2n+1})d(Bx_{2n+1}, Sx_{2n}), 1/2[d(Ax_{2n}, Sx_{2n})d(Ax_{2n}, Tx_{2n+1}) \\
&\qquad + d(Bx_{2n+1}, Sx_{2n})d(Bx_{2n+1}, Tx_{2n+1})]\}.
\end{aligned}$$

Using (3.1), we have

$$\begin{aligned}
&[1 + pd(y_{2n-1}, y_{2n})]d^2(y_{2n}, y_{2n+1}) \\
&\leq p \max\{1/2\,[d^2(y_{2n-1}, y_{2n})d(y_{2n}, y_{2n+1}) \\
&\qquad + d(y_{2n-1}, y_{2n})d^2(y_{2n}, y_{2n+1})], \\
&\qquad d(y_{2n-1}, y_{2n})d(y_{2n-1}, y_{2n+1})d(y_{2n}, y_{2n}), \\
&\qquad d(y_{2n-1}, y_{2n+1})d(y_{2n}, y_{2n})d(y_{2n}, y_{2n+1})\} \\
&\quad + m(y_{2n-1}, y_{2n}) - \phi(m(y_{2n-1}, y_{2n})),
\end{aligned} \tag{3.2}$$

where

$$m(y_{2n-1}, y_{2n})$$

$$= \max\{d^2(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n})d(y_{2n}, y_{2n+1}),$$

$$d(y_{2n-1}, y_{2n+1})d(y_{2n}, y_{2n}),$$

$$1/2[d(y_{2n-1}, y_{2n})d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n})d(y_{2n}, y_{2n+1})]\}.$$

On using $\alpha_{2n} = d(y_{2n}, y_{2n+1})$ in (3.2), we have

$$[1 + p\alpha_{2n-1}]\alpha_{2n}^2 \leq p \max\{1/2[\alpha_{2n-1}^2\alpha_{2n} + \alpha_{2n-1}\alpha_{2n}^2], 0, 0\}$$

$$+ m(y_{2n-1}, y_{2n}) - \phi(m(y_{2n-1}, y_{2n})), \tag{3.3}$$

where

$$m(y_{2n-1}, y_{2n})$$

$$= \max\{\alpha_{2n-1}^2, \alpha_{2n-1}\alpha_{2n}, 0, 1/2[\alpha_{2n-1}d(y_{2n-1}, y_{2n+1}) + 0]\}.$$

By triangular inequality and using property of ϕ, we get

$$d(y_{2n-1}, y_{2n+1}) \leq d(y_{2n-1}, y_{2n})d(y_{2n}, y_{2n+1})$$

$$= \alpha_{2n-1} + \alpha_{2n}$$

and

$$m(y_{2n-1}, y_{2n}) \leq \max\{\alpha_{2n-1}^2, \alpha_{2n-1}\alpha_{2n}, 0, 1/2[\alpha_{2n-1}(\alpha_{2n-1} + \alpha_{2n}), 0]\}.$$

If $\alpha_{2n-1} < \alpha_{2n}$, then (3.3) reduces to

$$p\alpha_{2n}^2 \leq p\alpha_{2n}^2 - \phi\alpha_{2n}^2,$$

which is a contradiction. Hence, $\alpha_{2n}^2 \leq \alpha_{2n-1}^2$ implies that $\alpha_{2n} \leq \alpha_{2n-1}$.

In a similar way, if n is odd, then we can obtain $\alpha_{2n+1} < \alpha_{2n}$. It follows that the sequence $\{\alpha_{2n}\}$ is decreasing.

Let $\lim_{n\to\infty} \alpha_{2n} = r$ for some $r \geq 0$. Suppose that $r > 0$. Then from inequality $(C2)$, we have

$$[1 + pd(Ax_{2n}, Bx_{2n+1})]d^2(Sx_{2n}, Tx_{2n+1})$$

$$\leq p \max\{1/2 [d^2(Ax_{2n}, Sx_{2n})d(Bx_{2n+1}, Tx_{2n+1})$$

$$+ d(Ax_{2n}, Sx_{2n})d^2(Bx_{2n+1}, Tx_{2n+1})],$$

$$d(Ax_{2n}, Sx_{2n})d(Ax_{2n}, Tx_{2n+1})d(Bx_{2n+1}, Sx_{2n}),$$

$$d(Ax_{2n}, Tx_{2n+1})d(Bx_{2n+1}, Sx_{2n})d(Bx_{2n+1}, Tx_{2n+1})\}$$

$$+ m(Ax_{2n}, Bx_{2n+1}) - \phi(m(Ax_{2n}, Bx_{2n+1})),$$

where

$$m(Ax_{2n}, Bx_{2n+1})$$

$$= \max\{(d^2(Ax_{2n}, Bx_{2n+1}), d(Ax_{2n}, Sx_{2n})d(Bx_{2n+1}, Tx_{2n+1}),$$

$$d(Ax_{2n}, Tx_{2n+1})d(Bx_{2n+1}, Sx_{2n}),$$

$$1/2 [d(Ax_{2n}, Sx_{2n})d(Ax_{2n}, Tx_{2n+1})$$

$$+ d(Bx_{2n+1}, Sx_{2n})d(Bx_{2n+1}, Tx_{2n+1})]\}.$$

Now by using (3.3), triangular inequality and property of ϕ and proceed limits $n \to \infty$, we get

$$[1 + pr]r^2 \leq pr^3 + r^2 - \phi(r^2).$$

Then $\phi(r^2) \leq 0$, since r is positive, by property of ϕ, we get $r = 0$, we conclude that

$$\lim_{n\to\infty} \alpha_{2n} = \lim_{n\to\infty} d(y_{2n}, y_{2n-1}) = r = 0. \tag{3.4}$$

Now we show that $\{y_n\}$ is a Cauchy sequence. Suppose that we assume that $\{y_n\}$ is not a Cauchy sequence. For given $\varepsilon > 0$, we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k, $n(k) > m(k) > k$,

$$d(y_{m(k)}, y_{n(k)}) \geq \varepsilon, \ d(y_{m(k)}, y_{n(k)-1}) < \varepsilon.$$

Now

$$\varepsilon \leq d(y_{m(k)}, y_{n(k)}) \leq d(y_{m(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{n(k)}).$$

Letting $k \to \infty$, we get

$$\lim_{k\to\infty} d(y_{m(k)}, y_{n(k)}) = \varepsilon. \tag{3.5}$$

Now from the triangular inequality we have,

$$| \, d(y_{n(k)}, y_{m(k)+1}) - d(y_{m(k)}, y_{n(k)}) \, | \leq d(y_{m(k)}, y_{m(k)+1}).$$

Taking limits as $k \to \infty$ and using (3.4) and (3.5), we have

$$\lim_{k\to\infty} d(y_{n(k)}, y_{m(k)+1}) = \varepsilon.$$

Again from the triangular inequality, we have

$$| \, d(y_{m(k)}, y_{n(k)+1}) - d(y_{m(k)}, y_{n(k)}) \, | \leq d(y_{n(k)}, y_{n(k)+1}).$$

Taking limits as $k \to \infty$ and using (3.4) and (3.5), we have

$$\lim_{k\to\infty} d(y_{m(k)}, y_{n(k)+1}) = \varepsilon.$$

Similarly on using triangular inequality, we have

$$| \, d(y_{m(k)+1}, y_{n(k)+1}) - d(y_{m(k)}, y_{n(k)}) \, |$$

$$\leq d(y_{m(k)}, y_{m(k)+1}) + d(y_{n(k)}, y_{n(k)+1}).$$

Taking limit as $k \to \infty$ in the above inequality and using (3.4) and (3.5), we have

$$\lim_{k\to\infty} d(y_{n(k)+1}, y_{m(k)+1}) = \varepsilon.$$

On putting $x = x_{m(k)}$ and $y = x_{n(k)}$ in ($C2$), we get

$$
\begin{aligned}
&[1 + pd(Ax_{m(k)}, Bx_{n(k)})]d^2(Sx_{m(k)}, Tx_{n(k)}) \\
&\leq p \max\{1/2\,[d^2(Ax_{m(k)}, Sx_{m(k)})d(Bx_{n(k)}, Tx_{n(k)}) \\
&\qquad + d(Ax_{m(k)}, Sx_{m(k)})d^2(Bx_{n(k)}, Tx_{n(k)})], \\
&\qquad d(Ax_{m(k)}, Sx_{m(k)})d(Ax_{m(k)}, Tx_{n(k)})d(Bx_{n(k)}, Sx_{m(k)}), \\
&\qquad d(Ax_{m(k)}, Tx_{n(k)})d(Bx_{n(k)}, Sx_{m(k)})d(Bx_{n(k)}, Tx_{n(k)})\} \\
&\quad + m(Ax_{m(k)}, Bx_{n(k)}) - \phi(m(Ax_{m(k)}, Bx_{n(k)})),
\end{aligned}
$$

where

$$
\begin{aligned}
&m(Ax_{m(k)}, Bx_{n(k)}) \\
&= \max\{d^2(Ax_{m(k)}, Bx_{n(k)}), d(Ax_{m(k)}, Sx_{m(k)})d(Bx_{n(k)}, Tx_{n(k)}), \\
&\qquad d(Ax_{m(k)}, Tx_{n(k)})d(Bx_{n(k)}, Sx_{m(k)}), \\
&\qquad 1/2\,[d(Ax_{m(k)}, Sx_{m(k)})d(Ax_{m(k)}, Tx_{n(k)}) \\
&\qquad + d(Bx_{n(k)}, Sx_{m(k)})d(Bx_{n(k)}, Tx_{n(k)})]\}.
\end{aligned}
$$

Using (3.1), we obtain

$$
\begin{aligned}
&[1 + pd(y_{m(k)-1}, y_{n(k)-1})]d^2(y_{m(k)}, y_{n(k)}) \\
&\leq p \max\{1/2\,[d^2(y_{m(k)-1}, y_{m(k)})d(y_{n(k)-1}, y_{n(k)}) \\
&\qquad + d(y_{m(k)-1}, y_{m(k)})d^2(y_{n(k)-1}, y_{n(k)})], \\
&\qquad d(y_{m(k)-1}, y_{m(k)})d(y_{m(k)-1}, y_{n(k)})d(y_{n(k)-1}, y_{m(k)}), \\
&\qquad d(y_{m(k)-1}, y_{n(k)})d(y_{n(k)-1}, y_{m(k)})d(y_{n(k)-1}, y_{n(k)})\} \\
&\quad + m(Ax_{m(k)}, Bx_{n(k)}) - \phi(m(Ax_{m(k)}, Bx_{n(k)})),
\end{aligned}
$$

where

$$
\begin{aligned}
& m(Ax_{m(k)}, Bx_{n(k)}) \\
&= \max\{d^2(y_{m(k)-1}, y_{n(k)-1}), d(y_{m(k)-1}, y_{m(k)})d(y_{n(k)-1}, y_{n(k)}), \\
&\qquad d(y_{m(k)-1}, y_{n(k)})d(y_{n(k)-1}, y_{m(k)}), \\
&\qquad 1/2\,[d(y_{m(k)-1}, y_{m(k)})d(y_{m(k)-1}, y_{n(k)}) \\
&\qquad + d(y_{n(k)-1}, y_{m(k)})d(y_{n(k)-1}, y_{n(k)})]\}.
\end{aligned}
$$

Letting $k \to \infty$, we get

$$
\begin{aligned}
[1 + \rho\varepsilon]\varepsilon^2 &\le p\max\{1/2\,[0 + 0], 0, 0\} + \varepsilon^2 - \phi(\varepsilon^2) \\
&= \varepsilon^2 - \phi(\varepsilon^2),
\end{aligned}
$$

which is a contradiction. Thus, $\{y_n\}$ is a Cauchy sequence in X. Since (X, d) is a complete metric space. Hence, $\{y_n\}$ converges to a point z as $n \to \infty$. Consequently, the subsequences $\{Sx_{2n}\}$, $\{Ax_{2n}\}$, $\{Tx_{2n+1}\}$ and $\{Bx_{2n+1}\}$ also converge to the same point z.

Now suppose that A is continuous. Then $\{AAx_{2n}\}$ and $\{ASx_{2n}\}$ converges to Az as $n \to \infty$. Since the mappings A and S are compatible on X, it follows from Proposition 2.3 that $\{SAx_{2n}\}$ converges to Az as $n \to \infty$.

Now we claim that $z = Az$. For this put $x = Ax_{2n}$ and $y = x_{2n+1}$ in equation ($C2$), we get

$$
\begin{aligned}
& [1 + pd(AAx_{2n}, Bx_{2n+1})]d^2(SAx_{2n}, Tx_{2n+1}) \\
&\le p\max\{1/2\,[d^2(AAx_{2n}, SAx_{2n})d(Bx_{2n+1}, Tx_{2n+1}) \\
&\qquad + d(AAx_{2n}, SAx_{2n})d^2(Bx_{2n+1}, Tx_{2n+1})], \\
&\qquad d(AAx_{2n}, SAx_{2n})d(AAx_{2n}, Tx_{2n+1})d(Bx_{2n+1}, SAx_{2n}),
\end{aligned}
$$

$$d(AAx_{2n}, Tx_{2n+1})d(Bx_{2n+1}, SAx_{2n})d(Bx_{2n+1}, Tx_{2n+1})\}$$

$$+ m(AAx_{2n}, Bx_{2n+1}) - \phi(m(AAx_{2n}, Bx_{2n+1})),$$

where

$$m(AAx_{2n}, Bx_{2n+1})$$

$$= \max\{d^2(AAx_{2n}, Bx_{2n+1}), d(AAx_{2n}, SAx_{2n})d(Bx_{2n+1}, Tx_{2n+1}),$$

$$d(AAx_{2n}, Tx_{2n+1})d(Bx_{2n+1}, SAx_{2n}),$$

$$1/2[d(AAx_{2n}, SAx_{2n})d(AAx_{2n}, Tx_{2n+1})$$

$$+ d(Bx_{2n+1}, SAx_{2n})d(Bx_{2n+1}, Tx_{2n+1})]\}.$$

Letting $n \to \infty$, we have

$$m(Az, z) = \max\{d^2(Az, z), d(Az, z)d(z, z), d(Az, z)d(z, Az),$$

$$1/2[d(Az, Az)d(Az, z) + d(z, Az)d(z, z)]\}$$

$$= d^2(Az, z).$$

Hence, we have

$$[1 + pd(Az, z)]d^2(Az, z)$$

$$\leq p \max\{1/2[0 + 0], 0, 0\} + d^2(Az, z) - \phi(d^2(Az, z)).$$

Thus, we get $d^2(Az, z) = 0$ and hence $Az = z$.

Next we will show that $Sz = z$. For this put $x = z$ and $y = x_{2n+1}$ in $(C2)$,

$$[1 + pd(Az, Bx_{2n+1})]d^2(Sz, Tx_{2n+1})$$

$$\leq p \max\{1/2[d^2(Az, Sz)d(z, z) + d(Az, Sz)d^2(z, z)],$$

$$d(Az, Sz)d(Az, z)d(z, Sz), d(Az, z)d(z, Sz)d(z, z)\}$$

$$+ m(Az, z) - \phi(m(Az, z)),$$

where

$$m(Az, z) = \max\{d^2(Az, z), d(Az, Sz)d(z, z), d(Az, z)d(z, Sz),$$
$$1/2[d(Az, Sz)d(Az, z) + d(z, Sz)d(z, z)]\} = 0.$$

Hence, we get

$$[1 + pd(z, z)]d^2(Sz, z) \leq p\max\{1/2[0 + 0], 0, 0\} + 0 - \phi(0).$$

Thus, we get $d^2(Sz, z) = 0.$ This implies that $Sz = z.$ Since $S(X) \subset B(X)$ and hence there exists a point $u \in X$ such that $z = Sz = Bu.$

We claim that $z = Tu.$ For this, we put $x = z$ and $y = u$ in $(C2)$, we get

$$[1 + pd(Az, Bu)]d^2(Sz, Tu)$$
$$\leq p\max\{1/2[d^2(Az, Sz)d(Bu, Tu) + d(Az, Sz)d^2(Bu, Tu)],$$
$$d(Az, Sz)d(Az, Tu)d(Bu, Sz), d(Az, Tu)d(Bu, Sz)d(Bu, Tu)\}$$
$$+ m(Az, Bu) - \phi(m(Az, Bu)),$$

where

$$m(Az, Bu) = \max\{d^2(Az, Bu), d(Az, Sz)d(Bu, Tu), d(Az, Tu)d(Bu, Sz),$$
$$1/2[d(Az, Sz)d(Az, Tu) + d(Bu, Sz)d(Bu, Tu)]\}$$
$$= \max\{d^2(z, z), d(z, z)d(z, Tu), d(z, Tu)d(z, z),$$
$$1/2[d(z, z)d(Az, Tu) + d(z, z)d(z, Tu)]\}$$
$$= 0.$$

Hence, we have

$$[1 + pd(z, z)]d^2(z, Tu)$$
$$\leq p\max\{1/2[d^2(z, z)d(z, Tu) + d(z, z)d^2(z, Tu)],$$
$$d(z, z)d(z, Tu)d(z, z), d(z, Tu)d(z, z)d(z, Tu)\} + 0 - \phi(0),$$

which implies that $z = Tu$. Since (B, T) is compatible in X and $Bu = Tu = z$, by Proposition 2.2, we have $BTu = TBu$ and hence $Bz = BTu = TBu = Tz$. Also, we have

$$\begin{aligned}
&[1 + pd(Az, Bz)]d^2(Sz, Tz) \\
&\leq p \max\{1/2[d^2(Az, Sz)d(Bz, Tz) + d(Az, Sz)d^2(Bz, Tz)], \\
&\qquad d(Az, Sz)d(Az, Tz)d(Bz, Sz), d(Az, Tz)d(Bz, Sz)d(Bz, Tz)\} \\
&\quad + m(Az, Bz) - \phi(m(Az, Bz)),
\end{aligned}$$

where

$$\begin{aligned}
m(Az, Bz) &= \max\{d^2(Az, Bz), d(Az, Sz)d(Bz, Tz), d(Az, Tz)d(Bz, Sz), \\
&\qquad 1/2[d(Az, Sz)d(Az, Tz) + d(Bz, Sz)d(Bz, Tz)]\} \\
&= d^2(z, Bz).
\end{aligned}$$

Hence, we become

$$\begin{aligned}
&[1 + pd(z, Bz)]d^2(z, Bz) \\
&\leq p \max\{1/2[0 + 0], 0, 0\} + d^2(z, Bz) - \phi(d^2(z, Bz)),
\end{aligned}$$

which implies that $z = Bz$. Hence, $z = Bz = Tz = Az = Sz$. Therefore, z is a common fixed point of S, T, A and B.

Similarly we can also complete the proof when B is continuous.

Next suppose that S is continuous. Then $\{SSx_{2n}\}$ and $\{SAx_{2n}\}$ converge to Sz as $n \to \infty$. Since the mappings A and S are compatible on X, it follows from Proposition 2.3 that $\{ASx_{2n}\}$ converges to Sz as $n \to \infty$.

Now we claim that $z = Sz$. For this put $x = Sx_{2n}$ and $y = x_{2n+1}$ in $(C2)$, we get

$$[1 + pd(ASx_{2n},\ Bx_{2n+1})]d^2(SSx_{2n},\ Tx_{2n+1})$$

$$\leq p\max\{1/2[d^2(ASx_{2n},\ SSx_{2n})d(Bx_{2n+1},\ Tx_{2n+1})$$

$$+ d(ASx_{2n},\ SSx_{2n})d^2(Bx_{2n+1},\ Tx_{2n+1})],$$

$$d(ASx_{2n},\ SSx_{2n})d(ASx_{2n},\ Tx_{2n+1})d(Bx_{2n+1},\ SSx_{2n}),$$

$$d(ASx_{2n},\ Tx_{2n+1})d(Bx_{2n+1},\ SSx_{2n})d(Bx_{2n+1},\ Tx_{2n+1})\}$$

$$+ m(ASx_{2n},\ Bx_{2n+1}) - \phi(m(ASx_{2n},\ Bx_{2n+1})),$$

where

$$m(ASx_{2n},\ Bx_{2n+1})$$

$$= \max\{d^2(ASx_{2n},\ Bx_{2n+1}),\ d(ASx_{2n},\ SSx_{2n})d(Bx_{2n+1},\ Tx_{2n+1}),$$

$$d(ASx_{2n},\ Tx_{2n+1})d(Bx_{2n+1},\ SSx_{2n}),$$

$$1/2[d(ASx_{2n},\ SSx_{2n})d(ASx_{2n},\ Tx_{2n+1})$$

$$+ d(Bx_{2n+1},\ SSx_{2n})d(Bx_{2n+1},\ Tx_{2n+1})]\}.$$

Letting $n \to \infty$, we get

$$m(Sz,\ z) = \max\{d^2(Sz,\ z),\ d(Sz,\ z)d(z,\ z),\ d(Sz,\ z)d(z,\ Sz),$$

$$1/2[d(Sz,\ Sz)d(Sz,\ z) + d(z,\ Sz)d(z,\ z)]\}$$

$$= d^2(Sz,\ z).$$

Hence, we become

$$[1 + pd(Sz,\ z)]d^2(Sz,\ z)$$

$$\leq p\max\{1/2[0 + 0],\ 0,\ 0\} + d^2(Sz,\ z) - \phi(d^2(Sz,\ z)).$$

Thus, we get $d^2(Sz,\ z) = 0$ implies that $Sz = z$. Since $S(X) \subset B(X)$ and hence there exists a point $v \in X$ such that $z = Sz = Bv$.

We claim that $z = Tv$. For this, we put $x = Sx_{2n}$ and $y = v$ in $(C2)$, we get

$$[1 + pd(ASx_{2n}, Bv)]d^2(SSx_{2n}, Tv)$$

$$\leq p \max\{1/2[d^2(ASx_{2n}, SSx_{2n})d(Bv, Tv) + d(ASx_{2n}, SSx_{2n})d^2(Bv, Tv)],$$

$$d(ASx_{2n}, SSx_{2n})d(ASx_{2n}, Tv)d(Bv, SSx_{2n}),$$

$$d(ASx_{2n}, Tv)d(Bv, SSx_{2n})d(Bv, Tv)\}$$

$$+ m(ASx_{2n}, Bv) - \phi(m(ASx_{2n}, Bv)),$$

where

$$m(ASx_{2n}, Bv)$$

$$= \max\{d^2(ASx_{2n}, Bv), d(ASx_{2n}, SSx_{2n})d(Bv, Tv),$$

$$d(ASx_{2n}, Tv)d(Bv, SSx_{2n}), 1/2[d(ASx_{2n}, SSx_{2n})d(ASx_{2n}, Tv)$$

$$+ d(Bv, SSx_{2n})d(Bv, Tv)]\}.$$

Letting $n \to \infty$, we get

$$m(z, Bz) = \max\{d^2(z, z), d(z, z)d(z, Tv), d(z, Tv)d(z, z),$$

$$1/2[d(z, z)d(z, Tv) + d(z, z)d(z, Tv)]\} = 0.$$

Hence, we become

$$[1 + pd(z, z)]d^2(z, Tv)$$

$$\leq p \max\{1/2[d^2(z, z)d(z, Tv) + d(z, z)d^2(z, Tv)],$$

$$d(z, z)d(z, Tv)d(z, z), d(z, Tv)d(z, z)d(z, Tv)\} + 0 - \phi(0),$$

which implies that $z = Tv$. Since (B, T) is compatible on X and $Bv = Tv = z$, by Proposition 2.2, we have $BTv = TBv$ and hence $Bz = BTv = TBv = Tv$.

Now we put $x = x_{2n}$ and $y = z$ in $(C2)$.

$$[1 + pd(Ax_{2n}, Bz)]d^2(Sx_{2n}, Tz)$$

$$\leq p\max\{1/2\,[d^2(Ax_{2n}, Sx_{2n})d(Bz, Tz) + d(Ax_{2n}, Sx_{2n})d^2(Bz, Tz)],$$

$$d(Ax_{2n}, Sx_{2n})d(Ax_{2n}, Tz)d(Bz, Sx_{2n}),$$

$$d(Ax_{2n}, Tz)d(Bz, Sx_{2n})d(Bz, Tz)\}$$

$$+ m(Ax_{2n}, Bz) - \phi(m(Ax_{2n}, Bz)),$$

where

$$m(Ax_{2n}, Bz)$$

$$= \max\{d^2(Ax_{2n}, Bz), d(Ax_{2n}, Sx_{2n})d(Bz, Tz), d(Ax_{2n}, Tz)d(Bz, Sx_{2n}),$$

$$1/2\,[d(Ax_{2n}, Sx_{2n})d(Ax_{2n}, Tz) + d(Bz, Sx_{2n})d(Bz, Tz)]\}.$$

Letting $n \to \infty$, we get

$$m(z, Tz) = d^2(z, Tz).$$

Hence, we reduce to

$$[1 + pd(z, Tz)]d^2(z, Tz)$$

$$\leq p\max\{1/2\,[0 + 0], 0, 0\} + d^2(z, Tz) - \phi(d^2(z, Tz)).$$

This gives $z = Tz$. Since $T(X) \subset A(X)$ and hence there exists a point $w \in X$ such that $z = Tz = Aw$.

We claim that $z = Sw$. For this, we put $x = w$ and $y = z$ in $(C2)$, we get

$$[1 + pd(Aw, Bz)]d^2(Sw, Tz)$$

$$\leq p\max\{1/2\,[d^2(Aw, Sw)d(Bz, Tz) + d(Aw, Sw)d^2(Bz, Tz)],$$

$$d(Aw, Sw)d(Aw, Tz)d(Bz, Sw), d(Aw, Tz)d(Bz, Sw)d(Bz, Tz)\}$$

$$+ m(Aw, Bz) - \phi(m(Aw, Bz)),$$

where

$$m(Aw, Bz) = \max\{d^2(Aw, Bz), d(Aw, Sw)d(Bz, Tz),$$
$$d(Aw, Tz)d(Bz, Sw),$$
$$1/2[d(Aw, Sw)d(Aw, Tz) + d(Bz, Sw)d(Bz, Tz)]\}$$
$$= \max\{(d^2(z, z), d(z, Sw)d(Tz, Tz), d(z, z)d(z, Sw),$$
$$1/2[d(z, Sw)d(z, z) + d(z, Sw)d(Tz, Tz)]\}$$
$$= 0.$$

Hence, we have

$$[1 + pd(z, z)]d^2(Sw, z)$$
$$\leq p\max\{1/2[d^2(z, Sw)d(z, z) + d(z, Sw)d^2(z, z)],$$
$$d(z, Sw)d(z, z)d(z, Sw), d(z, z)d(z, Sw)d(z, z)\} + 0 - \phi(0),$$

which implies that $Sw = z$. Since (S, A) is compatible on X, $Sw = Aw = z$, by Proposition 2.2, we have $ASw = SAw$ and hence $Az = ASw = SAw = Sz$. That is, $z = Az = Sz = Bz = Tz$. Therefore, z is a common fixed point of S, T, A and B.

Similarly we can complete the proof when T is continuous.

Finally, in order to prove the uniqueness, suppose that z and w $(z \neq w)$ are two common fixed points of S, T, A and B.

Put $x = z$ and $y = w$ in $(C2)$.

$$[1 + pd(z, w)]d^2(z, w) = [1 + pd(Az, Bw)]d^2(Sz, Tw)$$
$$\leq p\max\{0, 0, 0\} + m(Az, Bw) - \phi(m(Az, Bw))$$
$$= d^2(z, w) = \phi(d^2(z, w)).$$

Thus, we have $d^2(z, w) = 0$ and hence $z = w$. Therefore, S, T, A and B have a unique common fixed point in X. This completes the proof. □

If we put $p = 0$ in Theorem 3.2, then we have the following result:

Corollary 3.3. *Let S, T, A and B be four mappings of a complete metric space (X, d) into itself satisfying* (C1), (C3) *and the following condition*:

$$d^2(Sx, Ty) \leq m(Ax, By) - \phi(m(Ax, By))$$

for all $x, y \in X$, *where*

$$\begin{aligned} &m(Ax, By) \\ &= \max\{d^2(Ax, By), d(Ax, Sx)d(By, Ty), d(Ax, Ty)d(By, Sx), \\ &\quad 1/2[d(Ax, Sx)d(Ax, Ty) + d(By, Sx)d(By, Ty)]\}, \end{aligned}$$

$\phi : [0, \infty) \to [0, \infty)$ *is a continuous function with* $\phi(t) = 0$ *iff* $t = 0$ *and* $\phi(t) > 0$ *for each* $t > 0$.

Assume that the pairs (A, S) *and* (B, T) *are compatible*. *Then S, T, A and B have a unique common fixed point in X*.

Remark 3.4. If we put $A = B = I$ (the identity mapping) and $S = T$ in Theorem 3.2, we get the required result (Theorem 3.1) of Murthy and Vara Prasad [7].

Remark 3.5. If we put $A = B$ and $S = T$ in Theorem 3.2, we get the result of Jain et al. [3]. Further, if we put $S = I$ (the identity mapping) in Jain et al. [3, Theorem 2.2], we get the required result (Theorem 3.1) of Murthy and Vara Prasad [7].

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