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A SQUEEZED COHERENT STATE

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Abstract

In this paper, we compute the uncertainties in the oscillator's position, momentum, and the mean-square deviations from the average for various linear combinations of number states, squeezed states.

1. Introduction

The term coherent state, also called *Glauber state*, has been introduced by Glauber [2] in 1963. It is not strongly related to the classical term coherence, and refers to a special sort of pure quantum mechanical state of the light field corresponding to a single resonator mode. A squeezed coherent state is any state of the quantum mechanical Hilbert space such that the uncertainty principle is saturated. Squeezed states give measurement results better than those expected from the Heisenberg uncertainty principle, connection with optical interferometers used to measure the relative positions of gravity-wave detectors and in optical communications [1, 4, 5].

We describe a dynamical system in terms of a pair of complex operators a and $a^\dagger$, which we call them as the *annihilation* and *creation operators*.

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These operators, which obey the following commutation relation:

$$[a, a^{\dagger}] = 1, \tag{1}$$

play a fundamental role in descriptions of systems of harmonic oscillators and quantized fields. It is obvious from the algebraic properties of the operators a and $a^{\dagger}$ that we may construct a sequence of states for the harmonic oscillator system. These number states labeled by $|n\rangle$ satisfy the identity

$$\begin{aligned} a|n\rangle &= \sqrt{n}|n-1\rangle, \\ a^{\dagger}|n\rangle &= \sqrt{n+1}|n+1\rangle, \\ a^{\dagger}a|n\rangle &= n|n\rangle \end{aligned} \tag{2}$$

for a nonnegative integer n. They are generated from the state $|0\rangle$ by the rule

$$|n\rangle = \frac{(a^{\dagger})^n}{\sqrt{n!}}|0\rangle. \tag{3}$$

The annihilation and creation operators are defined in terms of the position and momentum operators by

$$a := \frac{x+ip}{\sqrt{2}} \tag{4}$$

and the Hermitian conjugate

$$a^{\dagger} := \frac{x-ip}{\sqrt{2}}. \tag{5}$$

We are interested in calculating the uncertainties in the oscillator's position and momentum for the number states and for various linear combinations of number states. These uncertainties are characterized by the mean-square deviations from the average, or variances:

$$\mathrm{var}(x) := \langle x^2\rangle - \langle x\rangle^2 \tag{6}$$

and

$$\mathrm{var}(p) := \langle p^2 \rangle - \langle p \rangle^2, \tag{7}$$

the symbol $\langle \cdot \rangle$ denotes the quantum-mechanical ensemble average, or expectation value (see [3]). To compute the averages, we should solve (4) and (5) for the position and momentum operators:

$$x = \frac{a + a^\dagger}{\sqrt{2}} \tag{8}$$

and

$$p = \frac{a - a^\dagger}{\sqrt{2}i}. \tag{9}$$

Then for the number states, we obtain

$$\langle x \rangle = \langle p \rangle = 0$$

and

$$\mathrm{var}(x) = \mathrm{var}(p) = n + \frac{1}{2}. \tag{10}$$

In this paper, we mainly consider:

Theorem 1.1. *For a squeezed state* $|\beta\rangle$, *we have*

(a) $\langle x \rangle = 0$,

(b) $\langle p \rangle = 0$,

(c) $\langle x^2 \rangle = \dfrac{|c_0|^2}{2\mu^*\nu^*} \displaystyle\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} \left| -\frac{\nu}{2\mu} \right|^{2n} \{2(2\mu^*\nu^* - \mu^{*2} - \nu^{*2})n - \nu^{*2}\} + \frac{1}{2}$,

(d) $\langle p^2 \rangle = \dfrac{|c_0|^2}{2\mu^*\nu^*} \displaystyle\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} \left| -\frac{\nu}{2\mu} \right|^{2n} \{2(2\mu^*\nu^* + \mu^{*2} + \nu^{*2})n + \nu^{*2}\} + \frac{1}{2}$,

where we express the squeezed state $|\beta\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$ *satisfying*:

Theorem 1.2. *Let* $n \in \mathbb{N}$. *Then equation* (14) *has the solution*:

$$c_{2n} = \frac{\sqrt{(2n)!}}{n!}\left(-\frac{\nu}{2\mu}\right)^n c_0 \quad \textit{and} \quad c_{2n-1} = 0,$$

where μ, ν *and* c_0 *are complex numbers*.

2. Proofs of Theorem 1.1 and Theorem 1.2

We show that a quite general property of a nonlinear device is to create negative and positive frequency output phasors B and B^* that are each combinations of input phasors A and A^*. That is,

$$B = \mu A + \nu A^* + \lambda AA^*$$

and

$$B^* = \mu^* A^* + \nu^* A + \lambda^* A^* A,$$

where μ, ν, and λ are complex numbers. We can write the following quantum-mechanical operators to represent these output phasors:

$$b = \mu a + \nu a^\dagger + \lambda a a^\dagger \tag{11}$$

and

$$b^\dagger = \mu^* a^\dagger + \nu^* a + \lambda^* a^\dagger a.$$

If a squeezed state $|\beta\rangle$ is to be an eigen-function of b with eigenvalue β, then by (2) and (11), we have

$$\begin{aligned} &\beta_0 c_0 |0\rangle + \beta_1 c_1 |1\rangle + \beta_2 c_2 |2\rangle + \cdots \\ &= \sum_{n=0}^{\infty} \beta_n c_n |n\rangle \\ &= \beta|\beta\rangle \\ &= b|\beta\rangle \end{aligned}$$

$$= (\mu a + \nu a^{\dagger} + \lambda a a^{\dagger}) \sum_{n=0}^{\infty} c_n | n\rangle$$

$$= \mu \sum_{n=1}^{\infty} c_n \sqrt{n} | n-1\rangle + \nu \sum_{n=0}^{\infty} c_n \sqrt{n+1} | n+1\rangle + \lambda \sum_{n=0}^{\infty} c_n (n+1) | n\rangle$$

$$= \mu(c_1 | 0\rangle + c_2 \sqrt{2} |1\rangle + c_3 \sqrt{3} | 2\rangle + \cdots) + \nu(c_0 |1\rangle + c_1 \sqrt{2} | 2\rangle + \cdots)$$

$$+ \lambda(c_0 | 0\rangle + 2c_1 |1\rangle + 3c_2 | 2\rangle + \cdots)$$

$$= (\mu c_1 + \lambda c_0) | 0\rangle + (\mu\sqrt{2} c_2 + \nu c_0 + 2\lambda c_1) |1\rangle + (\mu\sqrt{3} c_3 + \nu\sqrt{2} c_1 + 3\lambda c_2) | 2\rangle$$

$$+ \cdots$$

and so

$$\beta_0 c_0 = \mu c_1 + \lambda c_0,$$

$$\beta_1 c_1 = \mu\sqrt{2} c_2 + \nu c_0 + 2\lambda c_1, \tag{12}$$

in general

$$\beta_{n-1} c_{n-1} = \mu\sqrt{n} c_n + \nu\sqrt{n-1}\, c_{n-2} + n\lambda c_{n-1}.$$

This shows that

$$c_n = \frac{(\beta_{n-1} - n\lambda) c_{n-1} - \nu\sqrt{n-1} c_{n-2}}{\mu\sqrt{n}}.$$

It is clear that the above recursion relation provides

$$\beta_{n-1} = n\lambda \quad \text{and} \quad |\mu| > |\nu|, \tag{13}$$

so that the sequence of c_n converges, we can write

$$c_n = -\frac{\nu\sqrt{n-1} c_{n-2}}{\mu\sqrt{n}}. \tag{14}$$

Proof of Theorem 1.2. Now equation (14) deduces that

$$c_2 = -\frac{\nu\sqrt{1}}{\mu\sqrt{2}}c_0,$$

$$c_4 = -\frac{\nu\sqrt{3}}{\mu\sqrt{4}}c_2,$$

$$\vdots$$

$$c_{2n} = -\frac{\nu\sqrt{2n-1}}{\mu\sqrt{2n}}c_{2n-2}.$$

Multiplying both sides, we have

$$\begin{aligned} c_2c_4\cdots c_{2n} &= \left(-\frac{\nu}{\mu}\right)^n \frac{\sqrt{1}}{\sqrt{2}}\frac{\sqrt{3}}{\sqrt{4}}\cdots\frac{\sqrt{2n-1}}{\sqrt{2n}}c_0c_2\cdots c_{2n-2} \\ &= \left(-\frac{\nu}{\mu}\right)^n \sqrt{\frac{1\cdot 2\cdot 3\cdots(2n-1)\cdot 2n}{2^2\cdot 4^2\cdots(2n)^2}}c_0c_2\cdots c_{2n-2} \\ &= \left(-\frac{\nu}{\mu}\right)^n \frac{\sqrt{(2n)!}}{2\cdot 4\cdots(2n)}c_0c_2\cdots c_{2n-2} \\ &= \left(-\frac{\nu}{\mu}\right)^n \frac{\sqrt{(2n)!}}{2^n\cdot n!}c_0c_2\cdots c_{2n-2} \end{aligned}$$

and so

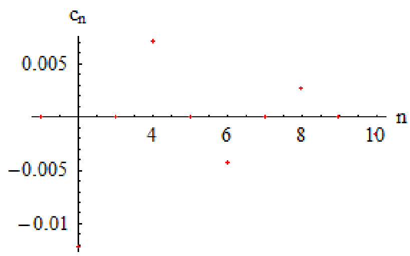
$$c_{2n} = \frac{\sqrt{(2n)!}}{n!}\left(-\frac{\nu}{2\mu}\right)^n c_0.$$

Finally, by (12)-(14), we conclude that

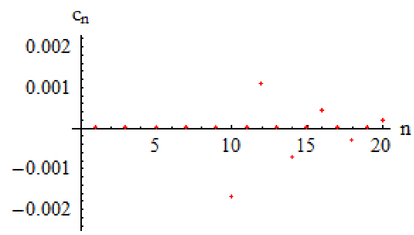
$$c_{2n-1} = 0. \qquad \square$$

Now, for instance we put $\mu = 1.5$, $\nu = 1$, $c_0 = 0.026$ with increasing n to visualize some values of c_n in Theorem 1.2 at the pictures (A), (B) and

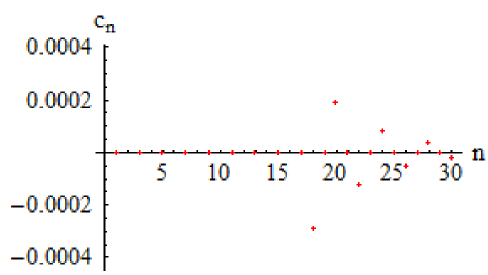
(C) in Figure 1. Throughout Figure 1, we conclude that c_{2n} vibrates versus n and as n is bigger then c_{2n} approaches to zero. And we let $\mu = 1.5$, $\nu = -1$, $c_0 = 0.026$ with various n in the pictures (A), (B) and (C) in Figure 2. Then Figure 2 implies that for bigger n, the values of c_{2n} decrease slowly and approach to zero.



(A) c_n versus $n(1 \le n \le 10)$.

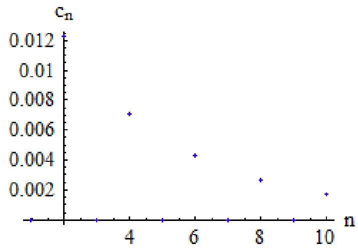


(B) c_n versus $n(1 \le n \le 20)$.

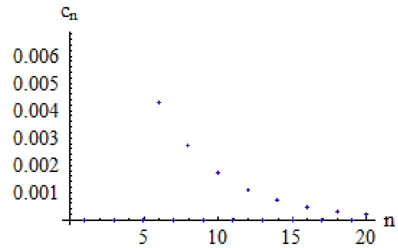


(C) c_n versus $n(1 \le n \le 30)$.

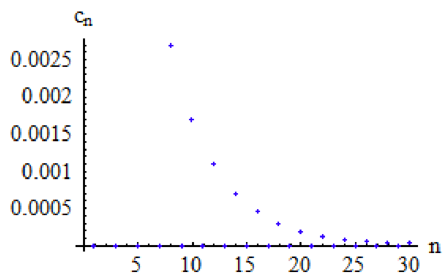
Figure 1. c_n versus n with $\mu = 1.5$, $\nu = 1$, $c_0 = 0.026$.



(A) c_n versus $n(1 \le n \le 10)$.



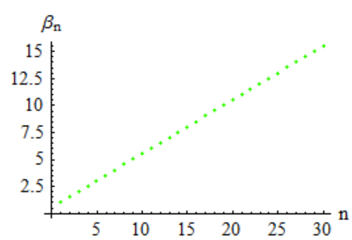
(B) c_n versus $n(1 \le n \le 20)$.



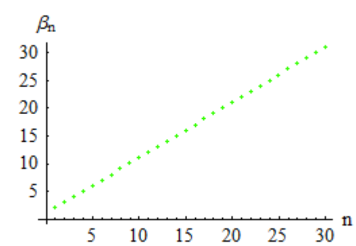
(C) c_n versus $n(1 \le n \le 30)$.

Figure 2. c_n versus n with $\mu = 1.5$, $\nu = -1$, $c_0 = 0.026$.

We can also see that by (13) the eigenvalues of β_n are plotted as Figure 3 which shows that eigenvalues are proportional to n.



(A) β_n versus n with $\lambda = 0.5$.



(B) β_n versus n with $\lambda = 1$.

Figure 3. β_n versus $n(1 \le n \le 30)$ with $\lambda = 0.5$ and $\lambda = 1$.

In the following lemma, we calculate the expectation value of the number operator in a squeezed state:

Lemma 2.1. *For the number operator* $a^{\dagger}a$ *and a squeezed state* $|\beta\rangle$, *we have*

$$\langle\beta|\, a^{\dagger}a\, |\beta\rangle = 2|\, c_0\,|^2 \sum_{n=1}^{\infty} \frac{(2n)!}{n((n-1)!)^2} \left| -\frac{\nu}{2\mu} \right|^{2n}.$$

Proof. From (2) and Theorem 1.2, we deduce that

$$\begin{aligned}
\langle\beta|\, a^{\dagger}a\, |\beta\rangle &= \langle m| \sum_{m=0}^{\infty} c_m \cdot a^{\dagger}a \cdot \sum_{n=0}^{\infty} c_n |\, n\rangle \\
&= \sum_{m,\, n=0}^{\infty} c_m^* c_n n \langle m\,|\, n\rangle \\
&= \sum_{m,\, n=0}^{\infty} c_m^* c_n n \delta_{m,\, n} \\
&= \sum_{n=0}^{\infty} n|\, c_n\,|^2 \\
&= \sum_{n=0}^{\infty} 2n|\, c_{2n}\,|^2 + \sum_{n=0}^{\infty} (2n+1)|\, c_{2n+1}\,|^2 \\
&= 2\sum_{n=1}^{\infty} n|\, c_{2n}\,|^2 \\
&= 2\sum_{n=1}^{\infty} \frac{(2n)!}{n((n-1)!)^2} \left| -\frac{\nu}{2\mu} \right|^{2n} |\, c_0\,|^2. \qquad \square
\end{aligned}$$

Corollary 2.2. *For a squeezed state* $|\beta\rangle$, *we have*

(a) $\langle\beta|\, a\, |\beta\rangle = 0$,

(b) $\langle\beta|\, a^{\dagger}\, |\beta\rangle = 0$.

Proof. (a) By (2) and Theorem 1.2, we obtain

$$\langle\beta|\, a\, |\beta\rangle = \langle m|\sum_{m=0}^{\infty} c_m \cdot a \cdot \sum_{n=0}^{\infty} c_n|\, n\rangle = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} c_m^* c_n \sqrt{n} \cdot \langle m|\, n-1\rangle$$

$$= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} c_m^* c_n \sqrt{n}\,\delta_{m,\, n-1}$$

$$= \sum_{n=1}^{\infty} c_{n-1}^* c_n \sqrt{n}$$

$$= 0.$$

(b) In a similar manner to part (a), we note that

$$\langle\beta|\, a^{\dagger}\, |\beta\rangle = \langle m|\sum_{m=0}^{\infty} c_m \cdot a^{\dagger} \cdot \sum_{n=0}^{\infty} c_n|\, n\rangle = \sum_{m,\, n=0}^{\infty} c_m^* c_n \sqrt{n+1} \cdot \langle m|\, n+1\rangle$$

$$= \sum_{m,\, n=0}^{\infty} c_m^* c_n \sqrt{n+1}\,\delta_{m,\, n+1}$$

$$= \sum_{n=0}^{\infty} c_{n+1}^* c_n \sqrt{n+1}$$

$$= 0. \qquad \square$$

Proof of Theorem 1.1. (a) From (8) and Corollary 2.2, we observe that

$$\langle x\rangle = \langle\beta|\, x\, |\beta\rangle = \langle\beta|\, \frac{a + a^{\dagger}}{\sqrt{2}}\, |\beta\rangle = \frac{1}{\sqrt{2}}(\langle\beta|\, a\, |\beta\rangle + \langle\beta|\, a^{\dagger}\, |\beta\rangle) = 0.$$

(b) By (9) and Corollary 2.2, it is clear.

(c) Employing (1), (2) and (8), we note that

$$\langle x^2\rangle = \langle\beta|\ x^2\ |\beta\rangle = \langle\beta|\left(\frac{a+a^\dagger}{\sqrt{2}}\right)^2 |\beta\rangle$$

$$= \frac{1}{2}\langle\beta|aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger|\beta\rangle$$

$$= \frac{1}{2}\langle\beta|aa + 2a^\dagger a + 1 + a^\dagger a^\dagger|\beta\rangle$$

$$= \frac{1}{2}\Bigg(\langle m|\sum_{m=0}^{\infty} c_m \cdot aa \cdot \sum_{n=0}^{\infty} c_n|n\rangle + 2\langle m|\sum_{m=0}^{\infty} c_m \cdot a^\dagger a \cdot \sum_{n=0}^{\infty} c_n|n\rangle$$

$$+ \langle m|\sum_{m=0}^{\infty} c_m \sum_{n=0}^{\infty} c_n|n\rangle + \langle m|\sum_{m=0}^{\infty} c_m \cdot a^\dagger a^\dagger \sum_{n=0}^{\infty} c_n|n\rangle\Bigg)$$

$$= \frac{1}{2}\Bigg(\sum_{m=0}^{\infty}\sum_{n=2}^{\infty} c_m^* c_n \sqrt{n}\sqrt{n-1}\,\delta_{m,\,n-2} + 2\sum_{m,\,n=0}^{\infty} c_m^* c_n n \delta_{m,\,n}$$

$$+ \sum_{m,\,n=0}^{\infty} c_m^* c_n \delta_{m,\,n} + \sum_{m,\,n=0}^{\infty} c_m^* c_n \sqrt{n+1}\sqrt{n+2}\,\delta_{m,\,n+2}\Bigg)$$

$$= \frac{1}{2}\Bigg(\sum_{n=2}^{\infty}\sqrt{n(n-1)}\,c_{n-2}^* c_n + 2\sum_{n=0}^{\infty} n c_n^* c_n + \sum_{n=0}^{\infty} c_n^* c_n$$

$$+ \sum_{n=0}^{\infty}\sqrt{(n+1)(n+2)}\,c_{n+2}^* c_n\Bigg)$$

$$= \frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty} c_n\left(\sqrt{n(n-1)}\,c_{n-2}^* + 2n c_n^* + \sqrt{(n+1)(n+2)}\,c_{n+2}^*\right)$$

then by Theorem 1.2, since c_{2n-1} does not exist,

$$\langle x^2\rangle = \frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty} c_{2n}\left(\sqrt{2n(2n-1)}\,c_{2n-2}^{*} + 4nc_{2n}^{*} + \sqrt{(2n+1)(2n+2)}\,c_{2n+2}^{*}\right). \tag{15}$$

Now applying

$$\begin{aligned}\sqrt{2n(2n-1)}\,c_{2n-2}^{*} &= \sqrt{2n(2n-1)}\cdot\frac{\sqrt{(2n-2)!}}{(n-1)!}\left(-\frac{\nu^{*}}{2\mu^{*}}\right)^{n-1} c_0^{*}\\ &= \frac{\sqrt{(2n)!}}{n!}\left(-\frac{\nu^{*}}{2\mu^{*}}\right)^{n} c_0^{*}\cdot\left(-\frac{2\mu^{*}}{\nu^{*}}\,n\right)\\ &= -\frac{2\mu^{*}}{\nu^{*}}\,nc_{2n}^{*}\end{aligned}$$

and

$$\begin{aligned}&\sqrt{(2n+1)(2n+2)}\,c_{2n+2}^{*}\\ &= \sqrt{(2n+1)(2n+2)}\cdot\frac{\sqrt{(2n+2)!}}{(n+1)!}\left(-\frac{\nu^{*}}{2\mu^{*}}\right)^{n+1} c_0^{*}\\ &= \frac{\sqrt{(2n)!}}{n!}\left(-\frac{\nu^{*}}{2\mu^{*}}\right)^{n} c_0^{*}\,\frac{(2n+1)(2n+2)}{n+1}\left(-\frac{\nu^{*}}{2\mu^{*}}\right)\\ &= -\frac{\nu^{*}}{\mu^{*}}(2n+1)c_{2n}^{*},\end{aligned}$$

to equation (15) we conclude that

$$\begin{aligned}\langle x^2\rangle &= \frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty}|\,c_{2n}\,|^2\left(-\frac{2\mu^{*}}{\nu^{*}}\,n + 4n - \frac{\nu^{*}}{\mu^{*}}(2n+1)\right)\\ &= \frac{1}{2} + \frac{1}{2\mu^{*}\nu^{*}}\sum_{n=0}^{\infty}|\,c_{2n}\,|^2\left(2(2\mu^{*}\nu^{*} - \mu^{*2} - \nu^{*2})n - \nu^{*2}\right).\end{aligned}$$

(d) It is obvious by part (c). □

Using (6), (7), and Theorem 1.1, we obtain

$$\mathrm{var}(x) = \langle x^2 \rangle - \langle x \rangle^2$$

$$= \frac{|c_0|^2}{2\mu^*\nu^*} \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} \left| -\frac{\nu}{2\mu} \right|^{2n} \{2(2\mu^*\nu^* - \mu^{*2} - \nu^{*2})n - \nu^{*2}\} + \frac{1}{2}$$

and

$$\mathrm{var}(p) = \langle p^2 \rangle - \langle p \rangle^2$$

$$= \frac{|c_0|^2}{2\mu^*\nu^*} \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} \left| -\frac{\nu}{2\mu} \right|^{2n} \{2(2\mu^*\nu^* + \mu^{*2} + \nu^{*2})n + \nu^{*2}\} + \frac{1}{2}$$

for a squeezed state $|\beta\rangle$ thus we can compare these values with (10) for a number state $|n\rangle$.

3. Conclusion

In this article, we add an input term λAA^* newly to the previous input phasors as follows:

$$B = \mu A + \nu A^* + \lambda AA^*$$

and

$$B^* = \mu^* A^* + \nu^* A + \lambda^* A^* A,$$

where μ, ν and λ are complex numbers. So we can obtain quantum-mechanical operators to represent output phasors:

$$b = \mu a + \nu a^\dagger + \lambda a a^\dagger$$

and

$$b^\dagger = \mu^* a^\dagger + \nu^* a + \lambda^* a^\dagger a.$$

Through these operators, we can compute oscillator's position, momentum, and the mean-square deviations from the average for squeezed states.

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