Far East J. Theo. Stat. **15**(1) (2005), 115-120

A METHOD FOR CONSTRUCTING TRIVARIATE DISTRIBUTIONS WITH GIVEN BIVARIATE MARGINS

MANUEL ÚBEDA-FLORES

Departamento de Estadística y Matemática Aplicada
Universidad de Almería, Carretera de Sacramento s/n
04120 La Cañada de San Urbano, Almería, Spain
e-mail: mubeda@ual.es

Abstract

In this note, we provide a method for constructing trivariate distributions with given bivariate margins and dependence structure in an easy manner. We also illustrate examples showing that this method can improve other known strategies in certain cases.

1. Introduction

The compatibility of distribution functions has been a problem of interest to statisticians for many years. Dall'Aglio [3] provided important results, and Quesada-Molina and Rodríguez-Lallena [5] applied those general results to copulas. We now review the concept of a copula (for a complete study, see [4]). Let n be a natural number such that $n \geq 2$. An n-dimensional *copula* (briefly n-copula) is a function $C : [0, 1]^n \to [0, 1]$ which satisfies:

(C1) For every $\mathbf{u} = (u_1, u_2, ..., u_n)$ in $[0, 1]^n$, $C(\mathbf{u}) = 0$ if at least one coordinate of $\mathbf{u}$ is 0, and $C(\mathbf{u}) = u_k$ whenever all coordinates of $\mathbf{u}$ are 1 except u_k; and

2000 Mathematics Subject Classification: Primary 62H05; Secondary 62H20.

Key words and phrases: copulas, multivariate positive dependence, ordering.

Received October 22, 2004

(C2) for every $\mathbf{a} = (a_1, a_2, ..., a_n)$ and $\mathbf{b} = (b_1, b_2, ..., b_n)$ in $[0, 1]^n$ such that $a_k \leq b_k$ for all $k = 1, 2, ..., n$, $V_C([\mathbf{a}, \mathbf{b}]) = \sum \operatorname{sgn}(\mathbf{c})C(\mathbf{c}) \geq 0$, where $[\mathbf{a}, \mathbf{b}]$ denotes the *n-box* $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$, the sum is taken over all the *vertices* $\mathbf{c} = (c_1, c_2, ..., c_n)$ of $[\mathbf{a}, \mathbf{b}]$ such that each c_k is equal to either a_k or b_k, and $\operatorname{sgn}(\mathbf{c})$ is 1 if $c_k = a_k$ for an even number of k's, and -1 if $c_k = a_k$ for an odd number of k's.

The importance of copulas as a tool for statistical analysis and modelling stems largely from the observation that the joint distribution H of a set of $n \geq 2$ random variables X_i with marginals F_i can be expressed by $H(\mathbf{x}) = C(F_1(x_1), F_2(x_2), ..., F_n(x_n))$, $\mathbf{x} = (x_1, x_2, ..., x_n) \in [-\infty, \infty]^n$, in terms of a copula C that is uniquely determined on $\operatorname{Ran}F_1 \times \operatorname{Ran}F_2 \times \cdots \times \operatorname{Ran}F_n$.

We now recall the concept of compatibility of three bivariate distributions in terms of copulas (see [5]): Three bivariate copulas $C_{12}(u, v)$, $C_{13}(u, w)$ and $C_{23}(v, w)$ are *compatible* if there exists a 3-copula $C(u, v, w)$ having those 2-copulas as bivariate margins. Finally, let $\prod^n$ denote the copula of independent random variables, i.e., $\prod^n(\mathbf{u}) = \prod_{i=1}^n u_i$; let M^2 denote the 2-copula $M^2(u, v) = \min(u, v)$, $(u, v) \in [0, 1]^2$; and let $\mathbf{X} > \mathbf{x}$ denote the point-wise inequality $(X_1 > x_1, X_2 > x_2, ..., X_n > x_n)$, where $\mathbf{X}$ is a random vector.

2. Construction

In this section, we study a new function in order to provide a procedure for constructing families of 3-copulas with given bivariate margins and dependence structure in an easy manner. The following theorem shows the main result of this section.

Theorem 2.1. *Let C_{12}, C_{13} and C_{23} be three 2-copulas and let C be the function defined by*

$$C(u, v, w) = wC_{12}(u, v) + vC_{13}(u, w) + uC_{23}(v, w) - 2uvw, \quad (u, v, w) \in [0, 1]^3. \tag{1}$$

Then C is a 3-copula if and only if

$$\frac{V_{C_{12}}([u_1, u_2] \times [v_1, v_2])}{(u_2 - u_1)(v_2 - v_1)} + \frac{V_{C_{13}}([u_1, u_2] \times [w_1, w_2])}{(u_2 - u_1)(w_2 - w_1)}$$

$$+ \frac{V_{C_{23}}([v_1, v_2] \times [w_1, w_2])}{(v_2 - v_1)(w_2 - w_1)} \geq 2 \tag{2}$$

for every $u_1, u_2, v_1, v_2, w_1, w_2$ *in* $[0, 1]$ *such that* $u_1 < u_2$, $v_1 < v_2$ *and* $w_1 < w_2$.

Proof. For every u, v, w, t in $[0, 1]$, it is immediate that $C(0, v, w) = C(u, 0, w) = C(u, v, 0) = 0$ and $C(1, 1, t) = C(1, t, 1) = C(t, 1, 1) = t$, whence condition (C1) is satisfied. To verify condition (C2), let $u_1, u_2, v_1, v_2, w_1, w_2$ be in $[0, 1]$ such that $u_1 \leq u_2$, $v_1 \leq v_2$ and $w_1 \leq w_2$. If $u_1 = u_2$ or $v_1 = v_2$ or $w_1 = w_2$, then we have $V_C([u_1, u_2] \times [v_1, v_2] \times [w_1, w_2]) = (w_2 - w_1)V_{C_{12}}([u_1, u_2] \times [v_1, v_2]) + (v_2 - v_1)V_{C_{13}}([u_1, u_2] \times [w_1, w_2]) + (u_2 - u_1)V_{C_{23}}([v_1, v_2] \times [w_1, w_2]) - 2(u_2 - u_1)(v_2 - v_1)(w_2 - w_1) = 0$; in another case, inequality (2) holds. Conversely, we only need to follow the same steps backwards. Finally, we have $C(1, v, w) = wC_{12}(1, v) + vC_{13}(1, w) + C_{23}(v, w) - 2vw = C_{23}(v, w)$, and, in a same manner, $C(u, 1, w) = C_{13}(u, w)$ and $C(u, v, 1) = C_{12}(u, v)$, which completes the proof.

Remark 2.1. Observe that if, at least, two of the three bivariate margins in (1) are $\prod^2$, then the function given by (1) is always a 3-copula. Furthermore, if $C_{12} = C_{13} = C_{23} = \prod^2$ in Theorem 2.1, then the 3-copula given by (1) is $\prod^3$.

We now investigate a partial ordering and a certain positive dependence property on the 3-copulas defined by (1). Given two n-copulas C_1 and C_2, C_1 is said *more concordant* than C_2 if $C_1(\mathbf{u}) \geq C_2(\mathbf{u})$ and $\overline{C}_1(\mathbf{u}) \geq \overline{C}_2(\mathbf{u})$ for all $\mathbf{u}$ in $[0, 1]^n$, where $\overline{C}(\mathbf{u}) = \mathrm{P}[\mathbf{U} > \mathbf{u}]$ and $\mathbf{U}$ is a random vector with n-copula C. When $C_2 = \prod^n$, C_1 is said *positively orthant dependent* (POD). In the bivariate case, C is said *positively* *quadrant dependent* (PQD) if $C(u, v) \geq uv$ for every (u, v) in $[0, 1]^2$. For more details, see [4]. Thus, we have:

Theorem 2.2. *Let* C_{12}, C_{13} *and* C_{23}, *and* D_{12}, D_{13} *and* D_{23} *be the three bivariate margins of two* 3*-copulas* C *and* D, *respectively, defined by* (1) *and such that* $C_{ij} \geq D_{ij}$, $1 \leq i < j \leq 3$. *Then* C *is more concordant than* D.

Proof. Let C and D be two 3-copulas as in the hypothesis. Then, for every (u, v, w) in $[0, 1]^3$, $C(u, v, w) \geq D(u, v, w)$ if and only if $w(C_{12}(u, v) - D_{12}(u, v)) + v(C_{13}(u, w) - D_{13}(u, w)) + u(C_{23}(v, w) - D_{23}(v, w)) \geq 0$. On the other hand, $\overline{C}(u, v, w) \geq \overline{D}(u, v, w)$ is equivalent to $(1 - w)(C_{12}(u, v) - D_{12}(u, v)) + (1 - v)(C_{13}(u, w) - D_{13}(u, w)) + (1 - u)(C_{23}(v, w) - D_{23}(v, w)) \geq 0$, whence the result follows.

Corollary 2.3. *If* C_{12}, C_{13} *and* C_{23} *are three* 2*-copulas such that are PQD, then the* 3*-copula defined by* (1) *is POD.*

As an application of our results, we provide the following example.

Example 2.1. Consider the semiparametric family of 2-copulas given by $C_\theta(u, v) = uv + \theta\phi(u)\phi(v)$, for every (u, v) in $[0, 1]^2$, with $\theta \in [0, 1]$, where ϕ is a function defined on $[0, 1]$ such that $\phi(0) = \phi(1) = 0$ and satisfying the Lipschitz condition $|\phi(u) - \phi(v)| \leq |u - v|$ for all (u, v) in $[0, 1]^2$. This family of copulas can model higher dependence than copulas with polynomial sections, preserving dependence properties (for more details, see [1]). If $C_{12}(u, v) = uv + \theta_1\phi_1(u)\phi_1(v)$, $C_{13}(u, w) = uw + \theta_2\phi_2(u)\phi_2(w)$ and $C_{23}(v, w) = vw + \theta_3\phi_3(v)\phi_3(w)$, $(u, v, w) \in [0, 1]^3$, where $\theta_1, \theta_2, \theta_3 \in [0, 1]$ and the functions ϕ_1, ϕ_2 and ϕ_3 satisfy the above conditions, then the function C defined by (1) is a 3-copula if and only if $\theta_1\phi_1'(p)\phi_1'(q) + \theta_2\phi_2'(p)\phi_2'(r) + \theta_3\phi_3'(q)\phi_3'(r) \geq -1$, $(p, q, r) \in (0, 1)^3$. Thus, if $\sum_{i=1}^{3}\theta_i \leq 1$, then C is a 3-copula. Furthermore, since (for $\theta \geq 0$) C_θ is

PQD if and only if either $\phi(u) \geq 0$ for all u in $[0, 1]$ or $\phi(u) \leq 0$ for all u in $[0, 1]$, as a consequence of Corollary 2.3, we have that C is POD if $\sum_{i=1}^{3} \theta_i \leq 1$, and either $\phi_i(u) \geq 0$ for all u in $[0, 1]$ or $\phi_i(u) \leq 0$ for all u in $[0, 1]$, $i = 1, 2, 3$.

3. Comparison with other Methods

In [5], a method for constructing trivariate distributions is provided seeking conditions for two 2-copulas C_1 and C_2 under which $C_2(C_1(u, v), w)$ is a 3-copula (observe that, in this case, the three bivariate margins are C_1, C_2 and C_3). In such a case, it is said that C_1 is *directly compatible* with C_2. In [2], another method is based on the fact that the function defined by

$$F(u, v, w) = wC_{12}\left(\frac{C_{13}(u, w)}{w}, \frac{C_{23}(v, w)}{w}\right), \quad (u, v, w) \in [0, 1]^3 \tag{3}$$

is a 3-copula having C_{12}, C_{13} and C_{23} as bivariate margins.

In the following examples we show that, in certain cases, our method provides a wider range of families of 3-copulas with given bivariate margins than the ones obtained by the two above strategies.

Example 3.1. Consider the known Farlie-Gumbel-Morgenstern family of 2-copulas given by $E_\lambda(u, v) = uv[1 + \lambda(1 - u)(1 - v)]$, $(u, v) \in [0, 1]^2$, with $\lambda \in [-1, 1]$. If $C_{12} = C_{13} = E_\lambda$ and $C_{23} = \Pi^2$, then the function C defined by (1) is a 3-copula if and only if $\lambda(g(u_1, u_2)[g(v_1, v_2) + g(w_1, w_2)]) \geq -1$, where $g(u, v) = [v(1 - v) - u(1 - u)]/(v - u)$. Hence, since $|g(u, v)| \leq 1$ for all (u, v) in $[0, 1]^2$, we can conclude that C is a 3-copula if and only if $|\lambda| \leq 1/2$ (observe that if $\lambda \in [0, 1/2]$, then C is POD). However, Π^2 is directly compatible with E_λ if and only if $|\lambda| \leq 1/3$ (see [5, Example 3.10]).

Example 3.2. Let $\alpha \in [0, 1]$ and let C_α be the 2-copula given by $C_\alpha(u, v) = \alpha uv + (1 - \alpha)M^2(u, v)$, $(u, v) \in [0, 1]^2$. If $C_{12} = \Pi^2$ and $C_{13} = C_{23} = C_\alpha$, then the inequality (2) is equivalent to $\alpha + (1 - \alpha)V_{M^2}(J)/V_{\Pi^2}(J) \geq 1/2$ for any 2-box J. Thus, a sufficient condition for the function given by (1) to be a 3-copula is that $\alpha \geq 1/2$. However, this result is not true, in general, using the function given by (3): For instance, if $\alpha = 1/2$, then $V_F([0.01, 0.4] \times [0.01, 0.3] \times [0.4, 0.6]) \cong -0.018$. Furthermore, Π^2 is directly compatible with C_α if and only if $\alpha = 1$ (see [5, Example 3.8]).

Acknowledgements

This work was partially supported by the Ministerio de Ciencia y Tecnologia (Spain) and FEDER, under research project BFM 2003-06522. The author wishes to thank Roger B. Nelsen for his helpful comments.

References

[1] C. Amblard and S. Girard, Symmetry and dependence properties within a semiparametric family of bivariate copulas, J. Nonparametr. Stat. 14 (2002), 715-727.

[2] A. Chakak and K. J. Koehler, A strategy for constructing multivariate distributions, Commun. Statist.-Simulation 24 (1995), 537-550.

[3] G. Dall'Aglio, Fréchet classes and compatibility of distribution functions, Sympos. Math. 9 (1972), 131-150.

[4] R. B. Nelsen, An Introduction to Copulas, Springer, New York, 1999.

[5] J. J. Quesada-Molina and J. A. Rodríguez-Lallena, Some advances in the study of the compatibility of three bivariate copulas, J. Ital. Statist. Soc. 3 (1994), 397-417.

■