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ON SMALLEST ORDER OF TRIANGLE-FREE GRAPHS WITH PRESCRIBED $(3, k)$-DEFECTIVE CHROMATIC NUMBER

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Abstract

Let m be a positive integer and let k be a non-negative integer. A k-defective chromatic number $\chi_k(G)$ is the least positive integer m such that G is (m, k)-defective-colorable. Let $f(m, k)$ be the smallest order of a triangle-free G such that $\chi_k(G) = m$. It is known that $f(4, 0) = 11$, $f(5, 0) = 22$, $32 \leq f(6, 0) \leq 40$, $f(3, 1) = 9$, $f(3, 2) = 13$ and $3k + 7 \leq f(3, k) \leq 4k + 5$. This paper proves that $f(3, k) = 4k + 5$ for any integer $k \geq 2$.

1. Introduction

Let m be a positive integer and let k be a non-negative integer. An (m, k)-defective-coloring of graph G is an assignment of colors $1, 2, ..., m$ to the vertices of G such that the maximum degree of the induced subgraph

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on vertices receiving the same color is at most k. A k-defective chromatic number $\chi_k(G)$ is the least positive integer m such that G is (m, k)-defective-colorable.

If $S \subset V(G)$ and $\Delta(G[S]) \leq k,$ then S is called *k-independent*. Therefore, if $\chi_k(G) = m,$ then the vertices of G is divided into m disjoint subset V_i such that $G[V_i]$ is k-independent, $1 \leq i \leq m.$

Let $f(m, k)$ be the smallest order of a triangle-free G such that $\chi_k(G) = m.$ It is a difficult problem to determine $f(m, k)$ (see Toft [12]). However, a partial solution had been established. Chvátal [4] proved that $f(4, 0) = 11.$ Finding $f(5, 0)$ is not easy. Four papers devoted to find $f(5, 0).$ First, Avis [2] proved that $f(5, 0) \geq 19,$ and then, Hanson and MacGillivray [7] proved that $f(5, 0) \geq 20,$ Grinstead et al. [5] proved that $21 \leq f(5, 0) \leq 22,$ and finally Jensen and Royle [9] proved that $f(5, 0) = 22.$ Recently, Goedgebeur [6] proved that $32 \leq f(6, 0) \leq 40.$ It is not known $f(m, 0)$ for $m \geq 7.$

The partial solutions to the case $f(3, k)$ had also been established by some authors. Simanihuruk et al. [11] and Achuthan et al. [1] proved that $f(3, k) = 4k + 5$ for $k = 1$ and $k = 2,$ respectively, and characterized all the triangle-free of order $f(3, k).$ Simanihuruk [10] characterized all triangle-free graph G of order $4k + 5$ for any integer $k \geq 0$ and $\Delta(G) \geq 3k + 1.$ In addition, Achuthan et al. [1] proved that $3k + 7 \leq f(3, k) \leq 4k + 5.$ This paper proves that $f(3, k) = 4k + 5$ for any integer $k \geq 2.$

In Section 2, we provide some preliminary results to prove the main result. In Section 3, we provide the main results of this paper.

The reader is referred to Chartrand and Lesniak [3] for the basic definition and notation do not cover in this paper.

2. Preliminary Results

In this section, we provide some previous results and develop some lemmas to prove the main result of this paper.

Using the results of Simanihuruk et al. [11], Simanihuruk [10] proved the following theorem.

Theorem 2.1 [10]**.** *Let G be a triangle-free graph of order* $4k + 5$ *with* $\Delta(G) \geq 3k + 1$. *If* $k \geq 0$ *is an integer, then* $\chi_k(G) = 3$ *if and only if* $k = 0$ *and* $G \cong C_5$ *or* (ii) $k = 1$ *and* $G \cong G_i$, *where* G_i *is a triangle-free graph in Figure* 1, $i = 1, 2, 3, 4$.

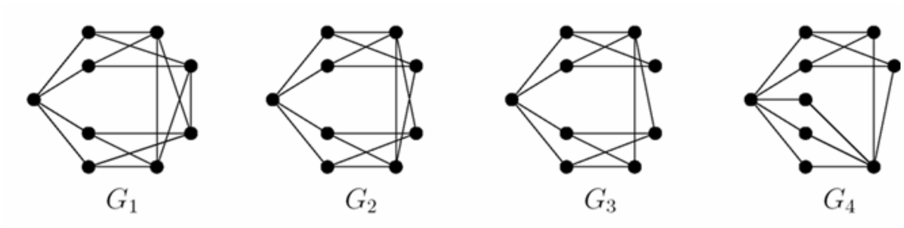

Figure 1. Graphs G_1, G_2, G_3 and G_4.

By Theorem 2.1, we have immediately the following.

Lemma 2.1. *Let G be a connected triangle-free graph of order* $4k + 5$ *and* $k \geq 2$ *be an integer*. *If* $\Delta(G) \geq 3k + 1$, *then* $\chi_k(G) \leq 2$.

A central theorem in the studying of the *k*-defective chromatic number $\chi_k(G)$ is a determination of the upper bound of $\chi_k(G)$. Hopkins and Staton [8] proved the upper bound of $\chi_k(G)$ as a function of $\Delta(G)$ and *k* as in the following theorem:

Theorem 2.2 [8]**.** *For a graph G with maximum degree* Δ, *we have* $\chi_k(G) \leq \left\lceil \frac{\Delta + 1}{k + 1} \right\rceil$.

By Theorem 2.2, we have immediately the following:

Lemma 2.2. *Let G be a graph*. *If* $\chi_k(G) = 3$, *then* $\Delta(G) \geq 2k + 2$.

By Lemmas 2.1 and 2.2, we only need to show that $\chi_k(G) \leq 2$ for $2k+2 \leq \Delta(G) \leq 3k$ and any triangle-free of order $4k+4$. If $x \in V(G)$, then $\Delta(G - N(x)) \geq k+1$, otherwise $\chi_k(G) \leq 2$. This fact and $2k+2 \leq \Delta(G) \leq 3k$ give the initial structure of triangle-free G of order $4k+4$. We describe this initial structure in Structure 1 by introducing some additional notations. Structure 1 is developed from Achuthan et al. [1] and Simanihuruk [10].

Structure 1. Let G be a triangle-free graph of order $4k+5$. Let $u \in V(G)$ such that $\deg(u) = \Delta(G), U = N_G(u), H = G - U - u$. Moreover, we define $O = \{o_1, o_2, ..., o_t\}$ to be a set of vertices in H such that $\deg_H \geq k+1$, A_i and F_i be the corresponding neighbor of o_i in U and H, respectively. In addition, we define $A = A_1 \cup A_2 \cup \cdots \cup A_t, F = F_1 \cup F_2 \cup \cdots \cup F_t$, $B = U - A$, $I = A - A_i, J = F - F_1$ and $Z = V(H) - O - F$. Without loss of generality we assume that $\deg_H(o_1) = \Delta(H)$. The structure of G is now shown in Figure 2.

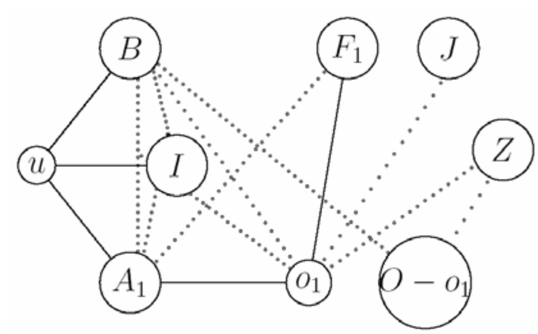

Figure 2. The structure of triangle-free graph G.

From now on we assume that the triangle-free G has the properties and notation in Structure 1.

The following lemma is representation of Lemma 2.3 in Achuthan et al. [1] for any triangle-free graph of order $4k+5$. For the purpose and the completeness of this paper, we rewrite it for any triangle-free graph of order $4k+4$.

Lemma 2.3. *Let G be a connected triangle-free graph of order* $4k + 4$ *and* $k \geq 2$ *be an integer. If* $\chi_k(G) = 3,$ *then*

$$\deg_H(o_1) = |F_1| = \Delta(H) \geq k + 1, \tag{1}$$

$$k + 2 \leq |V(H)| \leq 2k + 1. \tag{2}$$

Proof. Let $\chi_k(G) = 3$ and suppose $\deg_H(o_1) = |F_1| = \Delta(H) \leq k$ and consider the partitions of $V(G)$ into two disjoint partitions W_1 and $W_2,$ where $W_1 = \{u\} \cup V(H)$ and $W_2 = N_G(u)$. It is clear that W_1 and W_2 are k-independent sets. Therefore $\chi_k(G) \leq 2,$ a contradiction. Hence (1).

The lower bound of (2) follows from (1) and hence we need to show it is a upper bound. By Lemma 2.2, we have $\Delta(G) = |U| \geq 2k + 2.$ Therefore $|V(H)| \leq 2k + 1.$ Hence the upper bound of (2) and the lemma. □

Lemma 2.4. *Let G be a connected triangle-free graph of order* $4k + 4$ *and let* $k \geq 2$ *be an integer. If* $\chi_k(G) = 3,$ *then the set O is* 0-*independent*.

Proof. *Let* $\chi_k(G) = 3.$ Suppose the set O is not 0-independent. This implies $|V(H)| \geq 2k + 2,$ a contradiction to the upper bound of (2). Hence the set O is 0-independent and the lemma. □

Lemma 2.5. *Let G be a triangle-free graph of order* $4k + 4$ *and let* $k \geq 2$ *be an integer. If* $\Delta(G) = 2k + 2 + q,$ $0 \leq q \leq k - 2,$ *and* $\chi_k(G) = 3,$ *then*

$$\text{there is an } A_i \text{ such that } k + 1 \leq |A_i| \leq k + 1 + q, \tag{3}$$

$$k + 1 \leq |I \cup B| \leq k + 1 + q, \tag{4}$$

$$k + 1 \leq |F_i| \leq k + 1 + q, \tag{5}$$

$$k + 1 \leq |F| \leq 2k - 1. \tag{6}$$

Proof. Let $\chi_k(G) = 3.$ Suppose $|A_i| \leq k$ for all $1 \leq i \leq t$. By (1), we have $\Delta(H) = |F_1| \geq k + 1.$ Therefore $|O| \leq k,$ otherwise $V(H) \geq 2k + 2,$

a contradiction to the upper bound of (2). Let us consider the partition of $V(G)$ into disjoint subset $W_1 = \{u\} \cup F \cup Z$ and $W_2 = O \cup A \cup B$. Since O is 0-independent (Lemma 2.4), it is clear that W_1 and W_2 are k-independent. Therefore $\chi_k(G) \leq 2$, a contradiction. Hence the lower bound of (3).

The upper bound of (3) follows from the fact that $\Delta(G) = 2k + 2 + q \geq \deg_H(o_i) + \deg_U(o_i) \geq k + 1 + |A_i|$. Hence $|A_i| \leq k + 1 + q$ and the upper bound of (3).

The lower bound of (4) follows from the upper bound of (3) and the fact that $\Delta(G) = |A_i| + |I \cup B| = 2k + 2 + q$ with $0 \leq q \leq k - 2$. Similarly, the upper bound of (4) follows from the lower bound of (3) and the fact that $\Delta(G) = |A_i| + |I \cup B| = 2k + 2 + q$ with $0 \leq q \leq k - 2$.

The lower bound of (5) follows from the definition of the set O and F_i in Structure 1. Next, we will show the upper bound of (5). Suppose $|F_i| \geq k + 2 + q$. Then $\deg_H(o_i) \geq 2k + 3 + q$, a contradiction to $\Delta(G) = 2k + 2 + q$. Hence $|F_i| \leq k + 1 + q$.

Next, we will show (6). The lower bound of (6) follows from (5), since $F_i \subset F$. Next, we will show the upper bound of (6). Suppose $|F| \geq 2k$. Notice that $|V(H)| \leq 2k + 1$. Therefore $|F| \leq 2k$. Hence $|F| = 2k$ and $|O| = 1$. Since $|F| = 2k$ and $|O| = 1$, we have $\Delta(G) = |F| + |A_1| \leq 3k$. This and $|F| = 2k$ imply $|A_1| \leq k$, a contradiction to the lower bound of (3). Hence $|F| \leq 2k - 1$ and the lemma. □

We notice that

$$\Delta(G) = |A_i| + |I \cup B| = 2k + 2 + q, \quad 0 \leq q \leq k - 2. \tag{7}$$

Using (7), the inequalities (3), (4) and (5) can be written in the following form:

$$|A_i| = k + 1 + q_1, \quad 0 \le q_1 \le q \le k - 2, \tag{8}$$

$$|I \cup B| = k + 1 + q - q_1, \quad 0 \le q - q_1 \le k - 2. \tag{9}$$

Lemma 2.6. *Let G be a triangle-free graph of order $4k + 4$ and let $k \ge 2$ be an integer. Let $\Delta(G) = 2k + 2 + q$, $0 \le q \le k - 2$ and $\chi_k(G) = 3$. If $|F| = k + 1$, then $|Z| \ge 1$.*

Proof. Let $\Delta(G) = 2k + 2 + q, 0 \le q \le k - 2, \chi_k(G) = 3$, and $|F| = k + 1$. Suppose $|Z| = 0$. We will first show $|B| \ge k + 1$ in Figure 2. Suppose $|B| \le k$. Let $W_1 = A_i \cup F$ and $W_2 = \{u\} \cup B \cup O$. It is easy to verify that W_1 and W_2 are k-independent. Therefore $\chi_k(G) \le 2$, a contradiction. Hence $|B| \ge k + 1$. This implies $|A_i \cup I| \le 2k - 1$. Now let A_1 and A_2 be the partitions of $|A_i \cup I|$ such that $A_1 \cap A_2 = \emptyset$, $|A_1| = k$ and $|A_2| \le k - 1$. Let $W_1 = A_2 \cup B \cup O$ and $W_2 = \{u\} \cup A_1 \cup F$. It is easy to verify that W_1 and W_2 are k-independent. Therefore $\chi_k(G) \le 2$, a contradiction. In all the cases, we arrive in a contradiction. Hence the supposition is false. Therefore $|Z| \ge 1$ and the lemma. □

Lemma 2.7. *Let G be a triangle-free graph of order $4k + 4$ and let $k \ge 2$ be an integer. If $\Delta(G) = 2k + 2 + q, 1 \le q \le k - 2$, then $\chi_k(G) \le 2$.*

Proof. Let $\Delta(G) = 2k + 2 + q, 0 \le q \le k - 2$. Suppose $\chi_k(G) = 3$. Let A_i and F_i be the corresponding neighbors of o_i in U and H, respectively, such that $|A_i| \ge k + 1$ and $I = A - A_i$.

By (9), we have $|I \cup B| = k + 1 + q - q_1$, $0 \le q - q_1 \le k - 2$. Let B_{11} and B_{12} be disjoint partitions of $I \cup B$ such that $|B_{11}| = k$ and $|B_{12}| = 1 + q - q_1$, $0 \le q - q_1 \le k - 2$. Notice that $1 \le |B_{12}| \le k - 1$. Moreover, let $F_{i1} \subset F_i$ such that $|F_{i1}| = k$ and $F_{i2} = F - F_{i1}$. Using this notation the new structure of G in Figure 2 is presented in Figure 3.

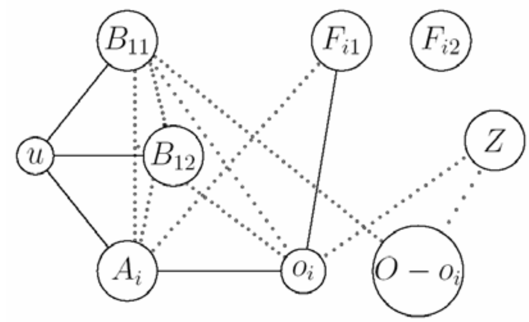

Figure 3. The new structure of triangle-free graph G in Figure 2.

Let $P = O \cup F_{i2} \cup Z$.

Claim 1. If $x \in F_{i2}$, then x is not adjacent to at least one vertex of P.

First, let $|F_{i2}| = 1$. If this is a case, then $F = F_i$ and $|F| = |F_i| = k + 1$. By Lemma 2.6, we have $|Z| \geq 1$. Since $|Z| \geq 1$ and $|F| = |F_i| = k + 1$, we have a vertex $r \in F_i$ such that the vertex r is not adjacent to vertex $z \in Z$, otherwise any vertex of Z is adjacent to $k + 1$ vertices of F_i, a contradiction to the definition of Z. Thus in the case $|F_{i2}| = 1$, we can always chose one vertex r of F_i to be the element of F_{i2} such that r is not adjacent to vertex $z \in Z$. Hence Claim 1 in this case. Second let $|F_{i2}| \geq 2$. Let $f_m \in F_{i2}$ such that $x \neq f_m$. We notice that there is a vertex $o_m \in O$ such that $(o_m, f_m) \in E(H)$. It is clear that the vertex x is not adjacent to either the vertex o_m or the vertex f_m, otherwise G has a triangle. Hence Claim 1.

Case 1. $|O - o_i| \geq 1$.

Now let $W_1 = A_i \cup B_{11} \cup F_{i1}$ and $W_2 = \{u\} \cup B_{12} \cup O \cup F_{i2} \cup Z$. We will show that W_1 and W_2 are k-independent.

Let us consider the set $W_1 = A_i \cup B_{11} \cup F_{i1}$ and let $x \in W_1$.

First, let $x \in A_i$. Clearly, the vertex x is not adjacent to any vertex of $B_{11} \cup F_{i1}$. Therefore $\deg(x) = 0 \leq k$. Second let $x \in B_{11}$. The vertex x may be adjacent to at most k vertices of F_{i1}, since $|F_{i1}| = k$. Therefore $\deg(x) \leq k$. Similarly, if $x \in F_{i1}$, then the vertex x is not adjacent to any vertex of A_i, but the vertex x may be adjacent to at most k vertices of B_{11}, since $|B_{11}| = k$. Therefore $\deg(x) \leq k$. Hence the set W_1 is k-independent.

Let us consider the set $W_2 = \{u\} \cup B_{12} \cup O \cup F_{i2} \cup Z$. We will show that W_2 is k-independent. Let $P = O \cup F_{i2} \cup Z$.

Claim 2. If $x \in B_{12}$, then x is not adjacent to at least two vertices of P.

It is clear that the vertex x is not adjacent to the vertex o_i. Let $o_j \in O - \{o_i\}$ and $f_j \in F_{i2}$ such that $(o_j, f_j) \in E(H)$. Now the vertex x is not adjacent to either the vertex o_j or the vertex f_j. Hence the vertex $x \in B_{12}$ is not adjacent to at least two vertices of P. Hence Claim 2.

Now using Claim 1 and Claim 2, we will show that $W_2 = \{u\} \cup B_{12} \cup O \cup F_{i2} \cup Z$ is k-independent. We notice that $|P| = |V(H)| - |F_{i1}|$. Using the fact $|V(H)| = 2k + 1 - q$ and $|F_{i1}| = k$, we have $|P| = k + 1 - q$. Hence $|B_{12} \cup P| = k + 2 - q_1$, since $|B_{12}| = 1 + q - q_1$. Using this fact we will show W_2 is k-independent. First let $x \in P$. Using Clam 1, we have $\deg(x) = |B_{12} \cup P| - 2 = k - q_1 \leq k$, since $|B_{12} \cup P| = k + 2 - q_1$. Second let $x \in B_{12}$. Using Claim 2 and $|P| = k + 1 - q$, we have $\deg(x) = |\{u\}| + |P| - 2 = 1 + |P| - 2 = k - q \leq k$. Hence the set W_2 is k-independent.

We have proved that the sets W_1 and W_2 are k-independent. Therefore $\chi_k(G) \leq 2$, a contradiction to the supposition.

Case 2. $|O - o_i| = 0$.

Notice that $|O| = 1$, $A_i = A_1$ and $|F_i| = |F_1| = |F| \geq k + 1$. Therefore $|Z| \leq k - q - 2$.

Case 2.1. Z has an edge.

Claim 3. If $x \in B_{12}$, then x is not adjacent to at least two vertices of P.

It is clear that the vertex x is not adjacent to the vertex o_i. Let $y, z \in Z$ such that $(y, z) \in E(H)$. Now the vertex x is not adjacent to either the vertex y or the vertex z. Hence the vertex $x \in B_{12}$ is not adjacent to at least two vertices of P. Hence Claim 3.

Let $W_1 = A_i \cup B_{11} \cup F_{i1}$ and $W_2 = \{u\} \cup B_{12} \cup O \cup F_{i2} \cup Z$. Using Claims 1 and 3, we can verify in similar line to that of in Case 1 that W_1 and W_2 are k-independent. Therefore $\chi_k(G) \leq 2$, a contradiction to the supposition.

Case 2.2. Z has no edge.

By (8), we can partition A_i into disjoint subset A_{i1} and A_{i2} such that $|A_{i1}| = k$ and $|A_{i2}| = 1 + q_1$, $0 \leq q_1 \leq k - 2$.

Case 2.2.a. There is no edge between Z and F_i.

Let $W_1 = O \cup A_{i1} \cup B$ and $W_2 = \{u\} \cup A_{i2} \cup F_i \cup Z$. It is easy to verify that W_1 and W_2 are k-independent. Therefore $\chi_k(K) \leq 2$, a contradiction to the supposition.

Case 2.2.b. There is no edge between Z and B.

Let $W_1 = O \cup Z \cup A_{i1} \cup B$ and $W_2 = \{u\} \cup A_{i2} \cup F_i$. It is easy to verify that W_1 and W_2 are k-independent. Therefore $\chi_k(G) \leq 2$, a contradiction to the supposition.

Case 2.2.c. There are some edges between Z and F_i, and between Z and B.

Let $F_{i1} \subset F_i$ such that $|F_{i1}| = k$ and $F_{i2} = F - F_{i1}$. Moreover, we choose F_{i2} such that there are some edges between Z and F_{i2}. Similarly, let $B_{11} \subset B$ such that $|B_{11}| = k$ and $B_{12} = B - B_{11}$. Moreover, we choose

B_{12} such that there are some edges between Z and B_{12}. By (9) and the fact $I = \emptyset$, we have $|B_{12}| = 1 + q - q_1$.

Claim 4. If $x \in B_{12}$, then x is not adjacent to at least two vertices of P.

It is clear that the vertex x is not adjacent to the vertex o_i. Let $z \in Z$ and $y \in F_{i2}$ such that $(z, y) \in E(H)$. Now the vertex x is not adjacent to either the vertex z or the vertex y. Hence the vertex $x \in B_{12}$ is not adjacent to at least two vertices of P. Hence Claim 4.

Claim 5. If $x \in F_{i2}$, then x is not adjacent to at least one vertex of B_{12}. Let $z \in Z$ and $y \in B_{12}$ such that $(z, y) \in E(G)$. Now the vertex x is not adjacent to either the vertex z or the vertex y. Hence the vertex $x \in F_{i2}$ is not adjacent to at least one vertex of P. Hence Claim 5.

Claim 6. If $x \in Z$, then x is not adjacent to o_i. It follows from the definition of O and Z.

Let $W_1 = A_i \cup B_{11} \cup F_{i1}$ and $W_2 = \{u\} \cup B_{12} \cup O \cup F_{i2} \cup Z$. We will show that W_1 and W_2 are k-independent.

It had been proved in Case 1 that $W_1 = A_i \cup B_{11} \cup F_{i1}$ is k-independent.

Next let us consider the set $W_2 = \{u\} \cup B_{12} \cup O \cup F_{i2} \cup Z$. Recall that $P = O \cup F_{i2} \cup Z$ and $|P| = k + 1 - q$. Hence $|B_{12} \cup P| = k + 2 - q_1$, since $|B_{12}| = 1 + q - q_1$. Using this fact, we will show W_2 is k-independent. First let $x \in \{u\}$. Clearly,

$$\deg(x) = |B_{12}| = \deg(x) = |\{u\}| + |P| - 2 = 1 + |P| - 2 = k - q \le k.$$

Second let $x \in B_{12}$. Using Claim 4 and $|P| = k + 1 - q$, we have

$$\deg(x) = |\{u\}| + |P| - 2 = 1 + |P| - 2 = k - q \le k.$$

Third let $x \in Z$. Using Claim 6 and the fact $|B_{12} \cup P| = k + 2 - q_1$, we have $\deg(x) = k + 2 - q_1 - 2 \le k$. Fourth let $x \in F_{i2}$. Using Claim 5 and the fact $|B_{12} \cup P| = k + 2 - q_1$, we have $\deg(x) = k + 2 - q_1 - 2 \leq k$. Finally, let $x \in O$. It is clear that every vertex of O is not adjacent to any vertex of B_{12}. Using this and the fact $|B_{12} \cup P| \leq k + 2 - q_1$, we have $\deg(x) = k + 2 - q_1 - 2 \leq k$. Hence the set W_2 is k-independent.

We have proved that the sets W_1 and W_2 are k-independent in this case. Therefore $\chi_k(G) \leq 2$, a contradiction to the supposition.

In all the cases, we arrive in a contradiction. Hence the supposition $\chi_k(G) = 3$ is false. Therefore $\chi_k(G) \leq 2$ and the lemma. □

Finally, we present the following result of Achuthan et al. [1] to prove the main result of this paper.

Theorem 2.3. *If* $k \geq 2$ *is an integer, then* $3k + 7 \leq f(3, k) \leq 4k + 5$.

3. Main Results

First we show that every triangle-free graph of order $4k + 4$ is $(2, k)$-defective-colorable and then prove the main result.

Theorem 3.1. *Let G be a triangle-free graph of order* $4k + 4$. *If* $k \geq 2$ *is an integer, then* $\chi_k(G) \leq 2$.

Proof: Let G be a triangle-free graph of order $4k + 4$ and let $k \geq 2$ be an integer. If $\Delta(G) \leq 2k + 1$, then by Theorem 3.1, $\chi_k(G) \leq 2$. If $\Delta(G) \geq 3k + 1$, then by Lemma 2.2, $\chi_k(G) \leq 2$. If $2k + 2 \leq \Delta(G) \leq 3k$, then by Lemma 2.7, $\chi_k(G) \leq 2$. Hence the theorem. □

By Theorem 3.1 and the minimality of $f(3, k)$, we conclude the following result.

Theorem 3.2. *If* $k \geq 2$ *is an integer, then* $f(3, k) \geq 4k + 5$.

We are now ready to prove the main result of this paper.

Theorem 3.3. *If* $k \geq 2$ *is an integer*, *then* $f(3, k) = 4k + 5$.

Proof. Let $k \geq 2$ be an integer. By Theorems 3.2 and 2.3, we conclude that $f(3, k) = 4k + 5$. Hence the theorem. □

By Theorem 3.3, we conclude that the smallest order of triangle-free graph *G* such that $\chi_k(G) = 3$ is $4k + 5$ for any integer $k \geq 2$.

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