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QUANTUM COVARIANCES AND SCHRÖDINGER TYPE OF UNCERTAINTY RELATION

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Abstract

In this paper, we study relations between quantum covariances, anti-symmetric covariance and symmetric covariance. From the relation, we derive a type of Schrödinger uncertainty relation and this can be seen as a refinement of the relation, which is given in [5].

1. Introduction

Heisenberg uncertainty relation and Schrödinger uncertainty relation have been reinterpreted using quantum covariances as studied in [3-5] and many other papers. In this paper, we first review the various definitions of covariances given in [6]. Using the properties of operator monotone functions, we study relations between those covariances such as covariance,

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quantum covariance, antisymmetric covariance and symmetric covariance. Such relations will give us a useful inequality which leads to a refinement of Schrödinger uncertainty relation which is given in [5] and [2].

2. Operator Monotone Functions

Let $\mathrm{M}_n = \mathrm{M}_n(\mathbb{C})$ be the set of all $n \times n$ complex matrices and let $\mathrm{M}_n^{\mathrm{sa}} = \{A \in \mathrm{M}_n \,|\, A^* = A\}$ be the set of all self-adjoint matrices, endowed with the Hilbert-Schmidt scalar product $\langle A, B\rangle_{\mathrm{HS}} = \mathrm{Tr}(A^*B),\ A, B \in M_n.$ Let $\mathcal{D}_n$ be the set of strictly positive matrices in M_n and let $\mathcal{D}_n^1 \subset \mathcal{D}_n$ be the set of positive density matrices; $D_n^1 = \{\rho \in \mathcal{D}_n \,|\, \mathrm{Tr}\,\rho = 1\}.$ Note that $\mathcal{D}_n$ and $\mathcal{D}_n^1$ are differentiable manifolds and the tangent space of $\mathcal{D}_n$ at ρ can be identified with $\mathrm{M}_n^{\mathrm{sa}}$ and the tangent space of $\mathcal{D}_n^1$ at ρ is $\mathrm{M}_n^{\mathrm{sa},0} = \{A \in \mathrm{M}_n^{\mathrm{sa}} \,|\, \mathrm{Tr}\,A = 0\}.$

A function $f : (0, +\infty) \to \mathbb{R}^+$ is said to be *operator monotone* if, for any natural number *n* and $A, B \in \mathrm{M}_n^{\mathrm{sa}}$ such that $0 \le A \le B,$ the inequalities $0 \le f(A) \le f(B)$ hold. An operator monotone function f is called *normalized* if $f(1) = 1$ and *symmetric* if $f(x) = xf\left(\frac{1}{x}\right).$ Let $\mathcal{F}_{\mathrm{op}}$ be the set of symmetric, normalized operator monotone functions $f : (0, +\infty) \to \mathbb{R}^+.$ Examples of elements in $\mathcal{F}_{\mathrm{op}}$ are the following:

$$f_{\mathrm{rd}}(x) := \frac{2x}{1+x} \quad \text{and} \quad f_{\mathrm{sd}}(x) := \frac{1+x}{2}.$$

For any $f \in \mathcal{F}_{\mathrm{op}},$ for all $x > 0,$

$$f_{\mathrm{rd}}(x) \le f(x) \le f_{\mathrm{sd}}(x). \tag{1}$$

For $f \in \mathcal{F}_{\text{op}}$, define $f(0) = \lim_{x \to 0} f(x)$. Then for any $f \in \mathcal{F}_{\text{op}}$ and $x \geq 0$, we have

$$f(0)(1 + x) \leq f(x) \leq \frac{1}{2}(1 + x). \tag{2}$$

Define

$$\mathcal{F}_{\text{op}}^{r} := \{f \in \mathcal{F}_{\text{op}} \mid f(0) \neq 0\}, \quad \mathcal{F}_{\text{op}}^{n} := \{f \in \mathcal{F}_{\text{op}} \mid f(0) = 0\}.$$

Obviously one has $\mathcal{F}_{\text{op}} = \mathcal{F}_{\text{op}}^{r} \cup \mathcal{F}_{\text{op}}^{n}$ and $\mathcal{F}_{\text{op}}^{r} \cap \mathcal{F}_{\text{op}}^{n} = \varnothing$.

For $f \in \mathcal{F}_{\text{op}}^{r}$ and $x > 0$, let

$$\tilde{f}(x) = \frac{1}{2}\left[(x + 1) - (x - 1)^2 \frac{f(0)}{f(x)}\right].$$

Then $\tilde{f} \in \mathcal{F}_{\text{op}}^{n}$ and this gives the bijection between $\mathcal{F}_{\text{op}}^{r}$ and $\mathcal{F}_{\text{op}}^{n}$. For example, $\tilde{f}_{\text{sd}}(x) = f_{\text{rd}}(x)$ and hence

$$\tilde{f}_{\text{sd}}(x) \leq f(x) \leq f_{\text{sd}}(x).$$

For $f, g \in \mathcal{F}_{\text{op}}^{r}$,

$$\begin{aligned}
\tilde{f} \leq \tilde{g} &\Rightarrow \frac{1}{2}\left[(x + 1) - (x - 1)^2 \frac{f(0)}{f(x)}\right] \leq \frac{1}{2}\left[(x + 1) - (x - 1)^2 \frac{g(0)}{g(x)}\right] \\
&\Rightarrow \frac{f(0)}{f(x)} \geq \frac{g(0)}{g(x)} \\
&\Rightarrow \frac{f(x)}{f(0)} \leq \frac{g(x)}{g(0)}.
\end{aligned} \tag{3}$$

For each $f \in \mathcal{F}_{\text{op}}$, we define the *mean function* $m_f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ by

$$m_f(x, y) = yf(xy^{-1}) \text{ for } x, y \in \mathbb{R}^+.$$

For example, if $f(x) = f_{\mathrm{rd}}(x) = \frac{2x}{1+x}$, then $m_f(x, y) = \frac{2xy}{x+y}$ and if $f(x) = \frac{1+x}{2}$, then $m_f(x, y) = \frac{x+y}{2}$. By definition, for all $f, g \in \mathcal{F}_{\mathrm{op}}$ and $x, y \in \mathbb{R}^+$,

$$f \le g \Rightarrow m_f(x, y) \le m_g(x, y).$$

Thus, by (1), (2), we have for each $f \in \mathcal{F}_{\mathrm{op}}$,

$$\frac{2xy}{x+y} \le m_f(x, y) \le \frac{x+y}{2}, \tag{4}$$

$$f(0)(x+y) \le m_f(x, y) \le \frac{x+y}{2}. \tag{5}$$

We may extend the mean function to $\mathbb{R}^+ \cup \{0\}$ and we have for $f \in \mathcal{F}_{\mathrm{op}}^r$,

$$\begin{aligned} m_{\tilde{f}}(x, y) &= y\tilde{f}(xy^{-1}) \\ &= \frac{y}{2}\left[\frac{x}{y} + 1 - \left(\frac{x}{y} - 1\right)^2 \frac{f(0)}{f(x)}\right] \\ &= \frac{x+y}{2} - \frac{f(0)(x-y)^2}{2m_f(x, y)}. \end{aligned} \tag{6}$$

Proposition 1. *Let* $f \in \mathcal{F}_{\mathrm{op}}$. *Then for each* $x, y \in \mathbb{R}^+$ *such that* $x - y \ge 0$, *we have*

$$\frac{x+y}{2} - m_{\tilde{f}}(x, y) \le \frac{1}{2}(x-y). \tag{7}$$

Proof. By (6), $\frac{x+y}{2} - m_{\tilde{f}}(x, y) = \frac{f(0)(x-y)^2}{2m_f(x, y)}$ and by (2), we have $f(0)(x+y) \le m_f(x, y)$. Thus,

$$\frac{1}{m_f(x, y)} \leq \frac{1}{f(0)(x+y)} \Rightarrow \frac{f(0)}{2m_f(x, y)} \leq \frac{1}{2(x+y)}$$

$$\Rightarrow \frac{f(0)(x-y)}{2m_f(x, y)} \leq \frac{x-y}{2(x+y)} \leq \frac{1}{2}$$

and hence

$$\frac{f(0)(x-y)^2}{2m_f(x, y)} \leq \frac{1}{2}(x-y). \qquad \square$$

Proposition 2. *For* $f \in \mathcal{F}_{op}$, *we have the following inequality*:

$$f(0)^2(x-y)^2 \leq \frac{f(0)(x-y)^2}{2m_f(x, y)} \frac{f(0)(x+y)^2}{2m_f(x, y)}. \tag{8}$$

Proof. By (4), $\frac{1}{x+y} \leq \frac{1}{m_f(x, y)}$. Thus,

$$f(0)(x+y) \leq \frac{f(0)(x+y)^2}{2m_f(x, y)}$$

and

$$\frac{f(0)(x-y)^2}{2m_f(x, y)} \frac{f(0)(x+y)^2}{2m_f(x, y)} \geq \frac{f(0)(x-y)^2}{2m_f(x, y)} \cdot f(0)(x+y)$$

$$\geq f(0)^2(x-y)^2. \qquad \square$$

3. Monotone Metrics and Covariances

For each $f \in \mathcal{F}_{\mathrm{op}}$, one may associate the *monotone family of metrics* on the manifold $\mathcal{D}_n$ [7, 8]. For $A, B \in M_n^{sa}$ and $\rho \in \mathcal{D}_n^1$, the metric is defined by

$$\langle A, B \rangle_{\rho, f} = \mathrm{Tr}(Am_f(L_\rho, R_\rho)(B)),$$

where $L_\rho(A) = \rho A$ and $R_\rho(A) = A\rho$.

By extending operator monotone functions to complex analytic functions on a neighborhood of $\mathbb{R}^+$, one may have the Riesz-Dunford operator calculus (cf. [1]):

$$f(\rho) = \frac{1}{2\pi i}\int_C f(\xi)(\xi - \rho)^{-1} d\xi,$$

where the spectrum of ρ is contained in C. The operators L_ρ and R_ρ are positive definite with respect to the Hilbert-Schmidt inner product on M_n. Then

$$f(L_\rho) = \int_C f(\xi)(\xi \,\mathrm{Id} - L_\rho)^{-1} d\xi$$

and

$$m_f(L_\rho, R_\rho)(B) = \frac{1}{(2\pi i)^2}\iint_C m_f(\xi, \eta)(\xi - \rho)^{-1} B(\eta - \rho)^{-1} d\xi d\eta.$$

Along the lines of [6], we have

Proposition 3. *Let* $\rho = \mathrm{Diag}(\lambda_1, \lambda_2, ..., \lambda_n) \in \mathcal{D}_n^1$ *and* $A, B \in \mathrm{M}_n^{\mathrm{sa}}$. *Then for all* $f \in \mathcal{F}_{op}$, *we have*

$$\langle A, B\rangle_{\rho, f} = \sum_{k, l=1} m_f(\lambda_k, \lambda_l)^{-1} A_{lk} B_{kl}. \tag{9}$$

Proof. For $A \in \mathrm{M}_n$, we can write

$$A = \sum_{i, j=1}^{n} A_{ij} E_{ij},$$

where E_{ij} is the $n \times n$ matrix which is all entries except (ij)-entry are zero. First we have

$$\begin{aligned}\langle E_{ij}, E_{kl}\rangle_{\rho, f} &= \mathrm{Tr}(E_{ij} m_f(L_\rho, R_\rho)^{-1}(E_{kl})) \\ &= \frac{1}{(2\pi i)^2}\mathrm{Tr}\iint_C \frac{1}{m_f(\xi, \eta)} E_{ij}(\xi-\rho)^{-1}E_{kl}(\eta-\rho)^{-1}d\xi d\eta \\ &= \delta_{kj}\delta_{li}\frac{1}{m_f(\xi, \eta)}. \end{aligned} \tag{10}$$

Since $A = \sum_{i, j=1}^{n} A_{ij}E_{ij}$ and $B = \sum_{k, l=1}^{n} B_{kl}E_{kl}$, we have

$$\begin{aligned}\langle A, B\rangle_{\rho, f} &= \sum_{i, j, k, l=1}^{n}\delta_{kj}\delta_{li}\frac{1}{m_f(\xi, \eta)}A_{ij}B_{kl} \\ &= \sum_{k, l=1} A_{lk}B_{kl}m_f(\lambda_k, \lambda_l)^{-1}. \qquad \square\end{aligned}$$

For $A, B \in \mathrm{M}_n^{\mathrm{sa}}$ and $\rho = \mathrm{Diag}(\lambda_1, ..., \lambda_n)$, the *covariance* of A and B is defined by

$$\begin{aligned}\mathrm{Cov}_\rho(A, B) &= \frac{1}{2}\mathrm{Tr}(\rho AB + \rho BA) - \mathrm{Tr}(\rho A)\mathrm{Tr}(\rho B) \\ &= \sum_{k, l=1}^{n}\frac{\lambda_k + \lambda_l}{2}A_{lk}B_{kl} - \mathrm{Tr}(\rho A)\mathrm{Tr}(\rho B) \\ &= \sum_{k, l=1}^{n}\frac{\lambda_k + \lambda_l}{2}[A_0]_{lk}[B_0]_{kl}, \end{aligned} \tag{11}$$

where $A_0 = A - \mathrm{Tr}(\rho A)I$. In what follows, we only consider the matrices satisfying $\mathrm{Tr}(\rho A) = 0$. Let $\mathrm{Cov}_\rho(A, A) = V_\rho(A)$. Then by the Cauchy-Schwarz inequality, we get

$$V_\rho(A)V_\rho(B) \geq |\,\mathrm{Cov}_\rho(A, B)\,|^2.$$

The Heisenberg uncertainty relation is given by

$$V_\rho(A)V_\rho(B) \geq \frac{1}{4}|\operatorname{Tr}(\rho[A, B])|^2$$

and the Schrödinger uncertainty relation is then

$$V_\rho(A)V_\rho(B) - |\operatorname{Cov}_\rho(A, B)|^2 \geq \frac{1}{4}|\operatorname{Tr}(\rho[A, B])|^2,$$

where $[A, B] = AB - BA.$

For $f \in \mathcal{F}_{\mathrm{op}}$, the *quantum f-covariance* of *A* and *B*, introduced in [5], is

$$\begin{aligned}\operatorname{Cov}_\rho^f(A, B) &= \operatorname{Tr}(Am_f(L_\rho, R_\rho)(B)) \\ &= \sum_{k,l=1}^{n} m_f(\lambda_k, \lambda_l)A_{lk}B_{kl},\end{aligned} \tag{12}$$

where the local form of the quantum covariance can be obtained as in the proof of Proposition 2.

The *antisymmetric f-covariance*, which was given in [6], is defined as

$$I_\rho^f(A, B) = \frac{f(0)}{2}\langle i[\rho, A], i[\rho, B]\rangle_{\rho, f}. \tag{13}$$

Proposition 4. *For* $A, B \in \mathrm{M}_n^{\mathrm{sa}}$ *and* $\rho = \operatorname{Diag}(\lambda_1, \lambda_2, ..., \lambda_n) \in \mathcal{D}_n^1$,

$$I_\rho^f(A, B) = \operatorname{Cov}_\rho(A, B) - \operatorname{Cov}_\rho^{\tilde{f}}(A, B). \tag{14}$$

Proof. Since $\rho = \operatorname{Diag}(\lambda_1, \lambda_2, ..., \lambda_n)$, we have

$$\begin{aligned}I_\rho^f(A, B) &= \frac{f(0)}{2}\langle i[\rho, A], i[\rho, B]\rangle_{\rho, f} \\ &= \frac{f(0)}{2}\operatorname{Tr}([\rho, A]m_f(L_\rho, R_\rho)([\rho, B]) \\ &= \frac{f(0)}{2}\sum_{k,l=1}^{n}[\rho, A]_{kl}[\rho, B]_{lk}m_f(\lambda_k, \lambda_l)\end{aligned}$$

$$= \frac{f(0)}{2} \sum_{k,l=1}^{n} \frac{(\lambda_k - \lambda_l)^2}{m_f(\lambda_k, \lambda_l)} A_{lk} B_{kl}$$

$$= \sum_{k,l=1}^{n} \frac{f(0)(\lambda_k - \lambda_l)^2}{2m_f(\lambda_k, \lambda_l)} A_{lk} B_{kl}$$

$$= \sum_{k,l=1}^{n} \left[\frac{\lambda_k + \lambda_l}{2} - m_{\tilde{f}}(\lambda_k, \lambda_l)\right] A_{lk} B_{kl}$$

$$= \mathrm{Cov}_\rho(A, B) - \mathrm{Cov}_\rho^{\tilde{f}}(A, B). \qquad \square$$

By Proposition 1, we get

$$| I_\rho^f(A, B) | = \left| \sum_{k,l=1}^{n} \frac{\lambda_k + \lambda_l}{2} - m_{\tilde{f}}(\lambda_k, \lambda_l) \right| | A_{lk} || B_{kl} |$$

$$\leq \frac{1}{2} \left| \sum_{k,l=1}^{n} | \lambda_k - \lambda_l || A_{lk} || B_{kl} | \right| = \frac{1}{2} | \mathrm{Tr}(\rho[A, B]) |. \qquad (15)$$

Finally, we improve the inequality given in [5].

Theorem 1. *Let* $A, B \in \mathrm{M}_n^{\mathrm{sa}}$ *and* $\rho = \mathrm{Diag}(\lambda_1, \lambda_2, ..., \lambda_n) \in \mathcal{D}_n^1$. *Suppose that* $f, g \in \mathcal{F}_{op}^r$ *such that* $\tilde{f} \leq \tilde{g}$. *Then*

$$\mathrm{Cov}_\rho(A, B) \leq \frac{1}{2f(0)} \mathrm{Cov}_\rho^f(A, B) \leq \frac{1}{2g(0)} \mathrm{Cov}_\rho^g(A, B).$$

Proof. By (3), we have $\frac{f(x)}{f(0)} \leq \frac{g(x)}{g(0)}$ and by the property of mean function

$$\frac{1}{f(0)} m_f(x, y) \leq \frac{1}{g(0)} m_g(x, y).$$

Thus, the second equality follows. By (2), $f(0)(1+x) \leq f(x)$. Thus,

$$\frac{f(0)(1+x)}{2} \leq \frac{f(x)}{2} \Rightarrow \frac{(1+x)}{2} \leq \frac{f(x)}{2f(0)}$$

and hence the first inequality follows from

$$\frac{x+y}{2} \leq \frac{m_f(x, y)}{2f(0)}. \qquad \square$$

The *symmetric f-covariance*, which was introduced in [6], is given as

$$J_\rho^f(A, B) = \frac{f(0)}{2}\langle\{\rho, A\}, \{\rho, B\}\rangle_{\rho, f}, \tag{16}$$

where $\{A, B\} = AB + BA$. The local form of the symmetric f-covariance can written as

$$J_\rho^f(A, B) = \frac{f(0)}{2}\sum_{k,l=1}^{n}\frac{(\lambda_k + \lambda_l)^2}{m_f(\lambda_k, \lambda_l)} A_{lk}B_{kl}. \tag{17}$$

Now we prove the main theorem which can be seen as a Schrödinger type of uncertainty relation.

Theorem 2. *Let* $A, B \in \mathrm{M}_n^{\mathrm{sa}}$ *and* $\rho = \mathrm{Diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathcal{D}_n^1$. *For all* $f \in \mathcal{F}_{op}^r$, *we have*

$$f(0)^2 | I_\rho^f(A, B)|^2 \leq \frac{1}{4} I_\rho^f(A) J_\rho^f(B).$$

Proof. By (15) and Proposition 2,

$$\begin{aligned}
& f(0)^2 | I_\rho^f(A, B)|^2 \\
& \leq \frac{1}{4}\sum_{k,l=1} f(0)^2 |\lambda_k - \lambda_l|^2 |A_{lk}|^2 |B_{kl}|^2 \\
& \leq \frac{1}{4}\sum_{k,l=1} \frac{f(0)(\lambda_k - \lambda_l)^2}{2m_f(\lambda_k, \lambda_l)} \frac{f(0)(\lambda_k + \lambda_l)^2}{2m_f(\lambda_k, \lambda_l)} |A_{lk}|^2 |B_{kl}|^2
\end{aligned}$$

$$\leq \frac{1}{4} \sum_{k, l=1} \frac{f(0)(\lambda_k - \lambda_l)^2}{2m_f(\lambda_k, \lambda_l)} | A_{lk} |^2 \frac{1}{4} \sum_{k, l=1} \frac{f(0)(\lambda_k + \lambda_l)^2}{2m_f(\lambda_k, \lambda_l)} | B_{kl} |^2$$

$$\leq \frac{1}{4} I_\rho^f(A) J_\rho^f(B). \qquad \square$$

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