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ANOTHER FUNCTIONAL EQUATION WITH INVOLUTION RELATED TO THE COSINE FUNCTION

(*This paper is dedicated to the Prasanna K. Sahoo deceased in* 55'*th ISFE*)

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Abstract

Let G be an abelian group, $\mathbb{C}$ be the field of complex numbers, $\alpha \in G$ be any fixed element and $\sigma : G \to G$ be an involution. In this paper, we determine the solutions $f, g : G \to \mathbb{C}$ of the functional equation $f(x + y + \alpha) + g(x + \sigma y + \alpha) = 2f(x)f(y)$ for all $x, y \in G$.

1. Introduction

Let G be an abelian group, $\alpha \in G$ be a fixed element and $\sigma : G \to G$ be an involution. Recall that an involution σ is an endomorphism of the group G satisfying $\sigma(\sigma(x)) = x$ for all $x \in G$. For convenience, we will write $\sigma(x)$ as σx. Let $0 \in G$ be the identity element of G. A multiplicative function on a group G is a function $h : G \to \mathbb{C}$ such that $h(xy) = h(x)h(y)$ for all $x, y \in G$. It is well known that a multiplicative function that vanishes

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at one point vanishes everywhere. Let $\mathbb{C}^{\star}$ be the multiplicative group of nonzero complex numbers. A character $\chi : G \to \mathbb{C}^{\star}$ is a map such that $\chi(xy) = \chi(x)\chi(y)$. Similarly, a function A from the group G into $\mathbb{C}$ is said to be an *additive* if and only if $A(xy) = A(x) + A(y)$ for all $x, y \in G$.

In 1910, Van Vleck [14] (see also [15] and [11]) considered the functional equation

$$f(x - y + \alpha) - f(x + y + \alpha) = 2f(x)f(y), \quad \forall x, y \in \mathbb{R}, \tag{1.1}$$

where $\alpha > 0$ is fixed. He proved that f is a periodic function of period 4α and equation (1.1) implies the cosine functional equation. He selected the continuous periodic solution of the cosine functional of equation of period 4α to obtain one of the continuous nonzero solutions of (1.1), viz.,

$$f(x) = \cos\left(\frac{\pi}{2\alpha}(x - \alpha)\right) = \sin\left(\frac{\pi}{2\alpha}x\right), \quad \forall x \in \mathbb{R}.$$

The other continuous nonzero solutions are all sine functions

$$f_n(x) = (-1)^n \sin\left(\frac{(2n+1)\pi}{2\alpha}x\right), \quad \forall x \in \mathbb{R}, n \in \mathbb{N}$$

having the period an integral fraction of 4α, viz., $\frac{4\alpha}{(2n+1)}$. These continuous nonzero solutions of (1.1) form a countable set (see [12]).

In [5], Kannappan considered the functional equation

$$f(x - y + \alpha) + f(x + y + \alpha) = 2f(x)f(y), \quad \forall x, y \in \mathbb{R} \tag{1.2}$$

and proved the following result: The general solution $f : \mathbb{R} \to \mathbb{C}$ of the functional equation (1.2) is either $f \equiv 0$ or $f(x) = g(x - \alpha)$, where g is an arbitrary solution of the cosine functional equation $g(x + y) + g(x - y) = 2g(x)g(y)$ for all $x, y \in \mathbb{R}$ with period 2α. In [10], Sahoo studied the following generalization:

$$f(x-y+\alpha)+g(x+y+\alpha)=2f(x)f(y) \quad \forall x,\ y \in G \tag{1.3}$$

of the functional equations (1.1) and (1.2) when G is an abelian group and raised the following problem: Given an involution $\sigma : G \to G,$ find all functions $f,\ g : G \to \mathbb{C}$ satisfying

$$f(x+\sigma y+\alpha)+g(x+y+\alpha)=2f(x)f(y) \tag{1.4}$$

for all $x,\ y \in G.$ In [8], the authors provided an answer to the above question posed in [10] by determining the general solutions $f,\ g : G \to \mathbb{C}$ of the functional equation (1.4) for all $x,\ y \in G,$ where G is an abelian group. In 1990, Gajda [4] studied the functional equation

$$f(x+y+\alpha)+f(y-x-\alpha)=2f(x)f(y), \quad \forall x,\ y \in G$$

on a locally compact abelian group G. Gajda's work has been extended to Wilson's functional equation by Fechner [3] in 2009.

Other similar functional equations solved in literature are

$$f(x+y+\alpha)f(x-y+\alpha)=f(x)^2-f(y)^2 \tag{1.5}$$

and

$$f(x+y+\alpha)f(x-y+\alpha)=f(x)^2+f(y)^2-1, \tag{1.6}$$

for all $x,\ y \in \mathbb{R}.$ The functional equation (1.5) was considered by Kannappan in [6] (see also [7]) while (1.6) was considered by Etigson in [2]. The functional equations (1.1)-(1.6) are examples of functional equations with restricted arguments where at least one of the variables is restricted to a certain discrete subset of the domain of the other variable(s). In particular, the subset may consist of a single element.

The main goal of this paper is to determine the general solution of the functional equation

$$f(x+y+\alpha)+g(x+\sigma y+\alpha)=2f(x)f(y) \tag{1.7}$$

for all $x, y \in G$. The functional equation (1.1) is a special case of the functional equation (1.7), where $g = \phi$, $f = -\phi$ and $G = \mathbb{R}$ with $\sigma(y) = -y$. If $G = \mathbb{R}$, and $g = \phi$ with $\sigma(y) = -y$, then the functional equation (1.7) reduces to the functional equation (1.2) studied by Kannappan in [5]. Hence the solution of (1.1) and (1.2) can be obtained from the results obtained in this paper. Although, the functional equation (1.7) looks similar to (1.4), their solutions are different. Using some ideas from [8], we determined the general solutions of (1.7) without using any regularity conditions on the unknown functions.

2. Preliminary Results

It is easy to see that if ϕ is the zero function and ψ is an arbitrary function, then they are the solutions of the functional equation

$$\phi(x + y) + \phi(x + \sigma y) = 2\phi(x)\psi(y) \tag{2.1}$$

for all $x, y \in G$. The following result from [1] gives the solution of (2.1) when ϕ is not identically zero.

Lemma 1. *Let G be an abelian group. Let $\phi, \psi : G \to \mathbb{C}$ satisfy the functional equation* (2.1) *for all $x, y \in G$, where $\sigma : G \to G$ is an involution and let $\phi \neq 0$. Then there exists a character $h : G \to \mathbb{C}^{\star}$ such that*

$$\psi = \frac{h + h \circ \sigma}{2}.$$

If $h \neq h \circ \sigma$, then ϕ has the form

$$\phi = ah + bh \circ \sigma$$

for some $a, b \in \mathbb{C}\backslash\{0\}$. If $h = h \circ \sigma$, then ϕ has the form

$$\phi = h[A - A \circ \sigma + \gamma],$$

where $A : G \to \mathbb{C}$ is an additive function and $\gamma \in \mathbb{C}$.

The following result was proved by the authors in [9] and will be used while proving the main result in the next section.

Lemma 2. *Let G be an abelian group, $\alpha \in G$ be a fixed element and $\sigma : G \to G$ be an involution. Then any nonzero solution $f : G \to \mathbb{C}$ of*

$$f(x + \sigma y + \alpha) - f(x + y + \alpha) = 2f(x)f(y) \tag{2.2}$$

for all $x, y \in G$ is given by

$$f = -\frac{1}{2}\chi(\alpha)[\chi - \chi \circ \sigma], \tag{2.3}$$

where χ is a character on G satisfying $\chi(\sigma\alpha) = -\chi(\alpha)$. Conversely, f in (2.3) *defines a nonzero solution of* (2.2).

The following lemma is known as Artin's lemma and can be found in Corollary 3.20 in [13].

Lemma 3. *The set of characters on a group G is a linearly independent subset of the vector space of all complex-valued functions on G.*

3. Main Result

Now we are ready to present the main result of this paper.

Theorem 4. *Let G be an abelian group, $\alpha \in G$ be a fixed element and $\sigma : G \to G$ be an involution. Any solution $f, g : G \to \mathbb{C}$ of the functional equation*

$$f(x + y + \alpha) + g(x + \sigma y + \alpha) = 2f(x)f(y) \quad \forall x, y \in G \tag{3.1}$$

has one of the following forms, in which h denotes a character of G.

$$f = \gamma, \quad g = \gamma(2\gamma - 1) \tag{3.2}$$

where $\gamma \in \mathbb{C}$, or

$$f = \frac{1}{2}h(\alpha)[h - h \circ \sigma], \quad g = -f, \tag{3.3}$$

or

$$f = dh, \quad g = dh[2dh(\alpha)^{-1} - 1], \tag{3.4}$$

where $d \in \mathbb{C}$, $2dh(\alpha)^{-1} \neq 1$, *and* $h = h \circ \sigma$, *or*

$$f = \frac{1}{2} h(\alpha) h, \quad g = 0, \tag{3.5}$$

where $d \in \mathbb{C}$, $2dh(\alpha)^{-1} = 1$, *and* $h = h \circ \sigma$, *or finally*

$$f = \frac{1}{2}[h(\alpha)h + h(\sigma\alpha)h \circ \sigma], \quad g = \frac{1}{2}[h(\sigma\alpha)h + h(\alpha)h \circ \sigma], \tag{3.6}$$

where $h \neq h \circ \sigma$.

Conversely, each of the five forms defines a solution (f, g) *of the functional equation* (3.1).

Proof. It is easy but tedious to check that the solutions listed in (3.2), (3.3), (3.4), (3.5), and (3.6) satisfy the functional equation (3.1). Next, we show that these are the only solutions of (3.1).

If $f(x) = \gamma$ for all $x \in G$, then we get (3.2) from (3.1).

From now on, we assume that f is nonconstant. Let $y = 0$ in (3.1). Then we have

$$f(x + \alpha) + g(x + \alpha) = 2f(x)f(0).$$

This implies

$$g(x) = 2df(x - \alpha) - f(x), \quad \forall x \in G, \tag{3.7}$$

where $d := f(0)$ is a constant in $\mathbb{C}$. Using (3.7) in (3.1), we see that

$$f(x + \sigma y + \alpha) - f(x + y + \alpha) = 2f(x)f(y) - 2df(x + \sigma y) \tag{3.8}$$

for all $x, y \in G$. Letting $F(x) := -f(x)$ for all $x \in G$ in (3.8), we obtain

$$F(x + \sigma\alpha + \alpha) - F(x + y + \alpha) = 2F(x)F(y) + 2dF(x + \sigma y) \tag{3.9}$$

for all $x, y \in G$.

Case 1. Suppose $d = 0$. Then (3.9) becomes

$$F(x + \sigma\alpha + \alpha) - F(x + y + \alpha) = 2F(x)F(y) \tag{3.10}$$

for all $x, y \in G$. Thus, from Lemma 2, (3.7) and definition of $F(x)$, we obtain

$$f(x) = \frac{1}{2}h(\alpha)[h(x) - h(\sigma x)] \quad \text{and} \quad g(x) = -f(x),$$

where $h : G \to \mathbb{C}^*$ is a character with $h(\alpha) = -h(\sigma\alpha)$. This is the asserted solution (3.3) when $h(\alpha) = -h(\sigma\alpha)$.

Case 2. Next, suppose $d \neq 0$. Replace y with σy in (3.9) to obtain

$$F(x + y + \alpha) - F(x + \sigma y + \alpha) = 2F(x)F(\sigma y) - 2dF(x + y) \tag{3.11}$$

for all $x, y \in G$. By adding (3.11) to (3.9), we see that

$$d[F(x + y) + F(x + \sigma y)] = -F(x)[F(y) + F(\sigma y)]. \tag{3.12}$$

Define $\phi : G \to \mathbb{C}$ by $\phi(x) = \frac{F(x)}{d}$, then (3.12) can be rewritten as

$$\phi(x + y) + \phi(x + \sigma y) = 2\phi(x)H(y), \tag{3.13}$$

where

$$H(y) = -\frac{\phi(y) + \phi(\sigma y)}{2}. \tag{3.14}$$

From Lemma 1, the solution of (3.13) is given by

$$\phi(x) = \begin{cases} h(x)[A(x - \sigma x) + \gamma] & \text{if } h = h \circ \sigma \\ ah(x) + bh(\sigma x) & \text{if } h \neq h \circ \sigma \end{cases} \tag{3.15}$$

and

$$H(x) = \frac{h(x) + h(\sigma x)}{2}, \tag{3.16}$$

where $h : G \to \mathbb{C}^*$ is a character, $A : G \to \mathbb{C}$ is an additive function and $a, b, \gamma \in \mathbb{C}$ are constants. Hence, from the definitions of ϕ and F, we get

$$f(x) = \begin{cases} -dh(x)[A(x - \sigma x) + \gamma] & \text{if } h = h \circ \sigma \\ -d[ch(x) + dh(\sigma x)] & \text{if } h \neq h \circ \sigma, \end{cases} \tag{3.17}$$

where $c, d, \gamma \in \mathbb{C}$ are constants. Letting $a = -c$ and $b = -d$, we have

$$f(x) = \begin{cases} -dh(x)[A(x - \sigma x) + \gamma] & \text{if } h = h \circ \sigma \\ -d[ah(x) + bh(\sigma x)] & \text{if } h \neq h \circ \sigma, \end{cases} \tag{3.18}$$

where $a, b, \gamma \in \mathbb{C}$ are constants. Interchanging x and y in (3.8), we have

$$f(y + x + \alpha) - f(y + \sigma x + \alpha) = 2f(y)f(x) - 2df(y + \sigma x). \tag{3.19}$$

By comparing (3.8) and (3.19), we obtain

$$f(x + \sigma y + \alpha) - 2df(x + \sigma y) = f(y + \sigma x + \alpha) - 2df(y + \sigma x).$$

The substitution of $y = 0$ in the previous equation yields

$$f(x + \alpha) = f(\sigma x + \alpha) + 2d[f(x) - f(\sigma x)] \tag{3.20}$$

for all $x \in G$.

Subcase 2.1. Suppose $h = h \circ \sigma$. Using (3.14) and (3.16), we see that

$$h(x) + h(\sigma x) = -\phi(x) - \phi(\sigma x).$$

Since $h = h \circ \sigma$, the use of (3.15) in the last equation yields

$$2h(x) = -h(x)[A(x - \sigma x) + \gamma + A(\sigma x - x) + \gamma]$$

which simplifies to

$$2h(x)(1 + \gamma) = 0.$$

Hence $\gamma = -1$. This means our solution for f is of the form

$$f(x) = -dh(x)[A(x - \sigma x) - 1], \tag{3.21}$$

where $h : G \to \mathbb{C}^*$ is a character and $A : G \to \mathbb{C}$ is an additive function. Using (3.21), we obtain

$$f(x + \alpha) = -dh(x)h(\alpha)[A(x - \sigma x) + A(\alpha - \sigma\alpha) - 1] \tag{3.22}$$

and

$$f(\sigma x + \alpha) = -dh(x)h(\alpha)[A(\alpha - \sigma\alpha) - A(x - \sigma x) - 1]. \tag{3.23}$$

Thus, (3.20), (3.21), (3.22) and (3.23) yield

$$[h(\alpha) - 2d]A(x - \sigma x) = 0, \quad \forall x \in G. \tag{3.24}$$

Suppose $2dh(\alpha)^{-1} \neq 1.$ Then (3.24) yields $A(x - \sigma x) = 0$ for all $x \in G$ and *f* in (3.21) reduces to

$$f(x) = dh(x) \tag{3.25}$$

when $h = h \circ \sigma.$ Using (3.25) in (3.7), we see

$$g(x) = 2df(x - \alpha) - f(x) = dh(x)[2dh(\alpha)^{-1} - 1]. \tag{3.26}$$

Thus we get the solution (3.4).

Next, suppose $2dh(\alpha)^{-1} = 1.$ Then (3.21) yields

$$f(x) = dh(\alpha)h(x)[A(x - \sigma x) - 1], \tag{3.27}$$

when $h = h \circ \sigma.$ Using (3.27) in (3.7), we see

$$g(x) = 2df(x - \alpha) - f(x) = dh(\alpha)A(\alpha - \sigma\alpha)h(x). \tag{3.28}$$

Letting the forms of *f* and *g* from (3.27) and (3.28) in (3.1) and simplifying, we obtain

$$h(x)h(y)h(\alpha)^2 A(x - \sigma x)A(y - \sigma y) = 0$$

for all $x, y \in G.$ Since *h* is character, thus we have

$$A(x - \sigma x)A(y - \sigma y) = 0$$

for all $x, y \in G$. Hence $A(x - \sigma x) = 0$ for all $x \in G$ and for this case

$$f(x) = \frac{1}{2} h(\alpha) h(x) \quad \text{and} \quad g(x) = 0, \quad \forall x \in G.$$

Thus we get the solution (3.5).

Subcase 2.2. Suppose $h \neq h \circ \sigma$. We have by (3.18) that f is a linear combination of h and $h \circ \sigma$, namely

$$f(x) = adh(x) + bdh(\sigma x) \tag{3.29}$$

for all $x \in G$. Letting $x = 0$ in (3.29) and noting that $f(0) = d \neq 0$, we see that

$$b = 1 - a. \tag{3.30}$$

It follows from (3.7) that g is also a linear combination of h and $h \circ \sigma$ and it is given by

$$g(x) = Ah(x) + Bh(\sigma x), \tag{3.31}$$

where

$$A = 2ad^2 h(\alpha)^{-1} - ad \quad \text{and} \quad B = 2bd^2 h(\sigma\alpha)^{-1} - bd. \tag{3.32}$$

Using (3.29) and (3.31), we get

$$\begin{aligned} 0 &= f(x + y + \alpha) + g(x + \sigma y + \alpha) - 2f(x)f(y) \\ &= [(adh(\alpha) - 2a^2d^2)h(y) + (Ah(\alpha) - 2abd^2)h(\sigma y)]h(x) \\ &\quad + [(Bh(\sigma\alpha) - 2abd^2)h(y) + (bdh(\sigma\alpha) - 2b^2d^2)h(\sigma y)]h(\sigma x) \end{aligned}$$

for all $x, y \in G$. By Artin's lemma, h and $h \circ \sigma$ are linearly independent and thus the above computation yields

$$(adh(\alpha) - 2a^2d^2)h(y) + (Ah(\alpha) - 2abd^2)h(\sigma y) = 0, \tag{3.33}$$

$$(Bh(\sigma\alpha) - 2abd^2)h(y) + (bdh(\sigma\alpha) - 2b^2d^2)h(\sigma y) = 0 \tag{3.34}$$

for all $y \in G$. Again by Artin's lemma $h(y)$ and $h(\sigma y)$ are linearly independent and hence the above equations (3.33) and (3.34) yield

$$adh(\alpha) - 2a^2d^2 = 0, \tag{3.35}$$

$$Ah(\alpha) - 2abd^2 = 0, \tag{3.36}$$

$$Bh(\sigma\alpha) - 2abd^2 = 0, \tag{3.37}$$

$$bdh(\sigma\alpha) - 2b^2d^2 = 0. \tag{3.38}$$

From (3.36) and (3.37), we see that

$$Ah(\alpha) = Bh(\sigma\alpha) = 2abd^2. \tag{3.39}$$

Next, we expressed a, b and d in terms of $h(\alpha)$ and $h(\sigma\alpha)$. Using (3.32) and (3.30) in (3.39) and simplifying, we have

$$4ad - 2d = a[h(\alpha) + h(\sigma\alpha)] - h(\sigma\alpha). \tag{3.40}$$

From (3.35) and (3.38), see that

$$h(\alpha) = 2ad \quad \text{and} \quad h(\sigma\alpha) = 2bd. \tag{3.41}$$

Adding $h(\alpha)$ to $h(\sigma\alpha)$ in (3.41) and using $b = 1 - a$, we obtain

$$d = \frac{1}{2}[h(\alpha) + h(\sigma\alpha)]. \tag{3.42}$$

Since $h(\sigma\alpha) \neq 0$, this implies that $h(\alpha) + h(\sigma\alpha) \neq 0$. Using d from (3.42) in (3.40), we have

$$a = \frac{h(\alpha)}{h(\alpha) + h(\sigma\alpha)} \quad \text{and} \quad b = 1 - a = \frac{h(\sigma\alpha)}{h(\alpha) + h(\sigma\alpha)}. \tag{3.43}$$

Using (3.42) and (3.43), the form of f in (3.29) can be rewritten as

$$f(x) = \frac{1}{2}[h(\alpha)h(x) + h(\sigma\alpha)h(\sigma x)]$$

which is included in (3.6). Using (3.42), (3.43) and (3.32), the constants A and B given in (3.32) can be rewritten as

$$A = 2ad^2h(\alpha)^{-1} - ad = \frac{1}{2}h(\sigma\alpha)$$

and

$$B = 2bd^2h(\sigma\alpha)^{-1} - bd = \frac{1}{2}h(\alpha).$$

Thus g given in (3.31) takes the form

$$g(x) = \frac{1}{2}[h(\sigma\alpha)h(x) + h(\alpha)h(\sigma x)]$$

which is included in the asserted solution (3.6). This completes the Case 2.

Since no more cases are left, the proof of the theorem is now complete. $\square$

The following corollary presents the general solution of (1.2) with an involution on an abelian group G generalizing the result of Kannappan in [5].

Corollary 5. *Let G be an abelian group, $\sigma : G \to G$ be an involution, and $\alpha \in G$ be a fixed element of G. The function $\ell : G \to \mathbb{C}$ satisfies the functional equation*

$$\ell(x + y + \alpha) + \ell(x + \sigma y + \alpha) = 2\ell(x)\ell(y), \quad \forall x, y \in G \tag{3.44}$$

if and only if

$$\ell = \frac{1}{2}h(\alpha)[h + h \circ \sigma], \tag{3.45}$$

where $h : G \to \mathbb{C}$ is a multiplicative function satisfying $h(\alpha) = h(\sigma\alpha)$.

Proof. In solution (3.2) of Theorem 4, letting $g(x) = f(x) = \ell(x)$, we obtain $2\gamma(\gamma - 1) = 0$ and thus $\ell(x) = 0$ or $\ell(x) = 1$ for all $x \in G$ which is included in (3.45).

Letting $g(x) = f(x) = \ell(x)$ in the solution (3.3), we get $\ell = f = 0$. This solution is included in (3.45).

Letting $g(x) = f(x) = \ell(x)$ in the solution (3.4) when $h = h \circ \sigma$, we get $h(\alpha) = d$. Hence $\ell(x) = h(\alpha)h(x)$ when $h = h \circ \sigma$ and this is included in the solution (3.45).

Letting $g(x) = f(x) = \ell(x)$ in the solution (3.5) when $h = h \circ \sigma$ and $2dh(\alpha)^{-1} = 1$, we get $\ell = f = 0$. This solution is included in (3.45).

Letting $g(x) = f(x) = \ell(x)$ in the solution (3.6) when $h \neq h \circ \sigma$, we get

$$[h(\alpha) - h(\sigma\alpha)]h - [h(\alpha) - h(\sigma\alpha)]h \circ \sigma = 0.$$

Hence, by Artin's lemma, the characters h and $h \circ \sigma$ are linearly independent and thus we have $h(\alpha) = h(\sigma\alpha)$. For the case when $h \neq h \circ \sigma$, we have from (3.6),

$$\ell(x) = \frac{1}{2} h(\alpha)[h(x) + h(\sigma x)]$$

which is the asserted solution (3.45). □

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