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DIFFUSION IN A TEMPORALLY SHRINKABLE MEDIUM

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Abstract

Some media often show shrinkage when the mass inside diffuses, such as wood, soil, clay and concrete. Shrinkage is responsible for developing cracks on clay, wood and concrete during the drying process. This paper is intended to understand the shrinkage better, so

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the defects of materials during and after the drying process can be avoided. A mathematical model of temporally shrinkable medium based on macro scale modeling is discussed. The model is in the form of an integral equation. This model does not allow the shrinkage process to reverse, as the case for concretes and clays after baking. The model is solved numerically by applying a finite difference method. The limiting case of non-shrinkable medium is presented. In this limiting case, the model is just a diffusion equation, and the numerical method is a standard finite difference method.

1. Introduction

Shrinkage is observed in our daily life. Some fruits show shrinkage after they lose the water content. Wood, clays and concrete shrink after drying. Shrinkage is considered to be the most responsible for cracking of such materials. In wood, shrinkage may develop several types of defects after undesired drying process. Such defects are often avoided in some industries. In general, shrinkage is often related to drying of materials. There are a lot of research reports on the simulation of drying, experiments or industrial process that either considering or neglecting shrinkage. Among others are [17] for agricultural products, [3, 6, 18] for wood, and [11] for ceramics. Shrinkage is a main cause of cracks in concrete. Cracks may reduce the strength and the life time of concrete, as reported in the numerical and experimental studies [1, 2]. Although cracks in concrete can be handled by injecting chemical material, but it is still very much to be avoided. The strength and the life time are still not optimal, and the process of repairing is costly and time consuming.

Drying and cracking of clays and soils have been reported in a lot of papers. A review on what affects cracks can be seen in [14], where the cracks were viewed from elasticity theory, the transition between tensile and shear failure, and linear elastic fracture mechanics. As in concrete, cracking in clay also affects the strength which is important for clay facilities as dams' embankment and landfill liners [21]. Moreover, Thusyanthan et al. [21] conducted experimental study where cracks were seen either as pure tension cracks, or as mixed-mode shear-tension cracks. Vogel et al. [22] presented a model that was based on a lattice of Hookean springs of finite strength. The model was to describe the natural crack patterns and the dynamics by evaluating the parameter space. Chemkhi and Zagrouba [8] conducted experimental investigation in the drying of clay material. The finding was that the diffusion coefficient was a function of moisture content. This was obtained by the use of drying curves. Peron et al. [16] conducted experimental and simulation study of the formation of drying crack patterns in soils.

All materials that develop shrinkage presented above are mostly porous media. Theoretically, diffusion in a porous medium is modeled by the porous medium equation, a non-linear type of the heat equation. This equation yields an analytic solution the so-called Barrenblatt's solution [10]. Barrenblatt's solution is a model of a droplet diffusing in such medium. Its limiting case is fundamental solution of the heat equation. The diffusion of droplets will develop interactions as discussed in [4] for one-dimensional case and [5] for two-dimensional medium.

2. Mathematical Modeling

Let us consider material in n-dimensional space, $(q, t) \in R^n \times R^+ \bigcup \{0\}$ for $n = 2, 3$, $q = q(x, t)$ and $q(x, 0) = x$. At $t = 0$, point q inside the material coincides with point x. In other words, $q(t)$ is a trajectory of a point starting at x. Let $\rho(q, t)$ represent the density of liquid at point q and time t of the material. In this assumption, we follow the motion of points inside material. This is a different approach compared to the one discussed in [9], where the fluid and the solid materials were treated separately which resulted in the presence of two state variables for density, namely state variable of solid density and state variable of fluid density.

In this paper, we merely focus on the state variable of fluid density at the moving points of solid material. The flux of the water to the surrounding area is assumed to be proportional to the gradient of ρ with respect to q (Fick's law) plus ρ brought by the rate of motion of q (all in negative direction)

$$v = -K\nabla\rho - \rho\frac{dq}{dt}. \tag{2.1}$$

This is illustrated in Figure 1. K is positive quantity related to diffusion rate. In many text books, K is often considered to be constant. However, it may be a function of q and ρ as observed in industrial data for wood that contained nod or annual ring reported in [3]. For more detailed experiment on clay materials, it has been shown that K was a function of ρ [8]. It was noted in [8] that generally the experimental determination of the diffusion coefficient as a function of moisture content was difficult. However, finding the function of diffusion coefficient is an inverse problem. For the case of linear media, it has been discussed in [19, 20], for more recently for piecewise constant conductivity was discussed in [12].

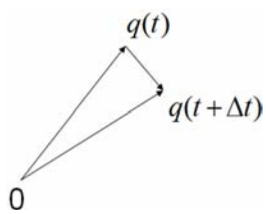

Figure 1. Illustrative plot of the Fick's law of water flux.

On the other hand, let $\Omega(t)$ be any volume in R^n. The total fluid leaving the volume $\Omega(t)$:

$$-\int_{\partial\Omega(t)}\left(K\nabla\rho + \rho\frac{dq}{dt}\right)\cdot \boldsymbol{n}dV, \tag{2.2}$$

where $\boldsymbol{n}$ is outward normal vector of the boundary $\partial\Omega(t)$. Total fluid content inside $\Omega(t)$ is

$$m = \int_{\Omega(t)}\rho dV. \tag{2.3}$$

The change of water content inside $\Omega(t)$:

$$\frac{d\rho}{dt} = \int_{\Omega(t)} \frac{d\rho}{dt} dV + \int_{\Omega'(t)} \rho dV, \tag{2.4}$$

where $\Omega'(t)$ is the derivative of $\Omega(t)$ with respect to *t*.

Assuming the mass conserved, the increase of mass in $\Omega(t)$ of equation (2.4) is equal to the mass entering the boundary which is negative of (2.2). Hence, the mass conservation for the liquid yields an integral equation in the form

$$\int_{\partial\Omega(t)} \left(K\nabla\rho + \rho \frac{dq}{dt} \right) \cdot \boldsymbol{n} dV = \int_{\Omega(t)} \frac{d\rho}{dt} dV + \int_{\Omega'(t)} \rho dV. \tag{2.5}$$

By applying Gauss Divergence Theorem, we obtain

$$\int_{\Omega(t)} \left(\frac{d\rho}{dt} - \nabla \cdot (K\nabla\rho) \right) dV = \int_{\partial\Omega(t)} \rho \frac{dq}{dt} \cdot \boldsymbol{n} dV - \int_{\Omega'(t)} \rho dV. \tag{2.6}$$

For the case of nondeformable material or no shrinkage, (2.6) becomes

$$\int_{\Omega(t)} \left(\frac{d\rho}{dt} - \nabla \cdot (K\nabla\rho) \right) dV = 0. \tag{2.7}$$

Moreover, by assuming the integrand is sufficiently smooth and equation (2.7) is satisfied for any deformable region Ω, we have

$$\frac{d\rho}{dt} = \nabla \cdot (K\nabla\rho). \tag{2.8}$$

Hence, the model introduced in (2.6) is a generalization of the well-known diffusion equation of the heat equation for constant *K*.

For one-dimensional case, the region $\Omega(t)$ is in the form of interval $I(t) = [a_I(t), b_I(t)]$. Equation (2.6) becomes a simpler form

$$\int_{I(t)} \left(\frac{d\rho}{dt} - \partial_q (K\partial_q \rho) \right) dq = (\rho(b_I, t) b_I' - \rho(a_I, t) a_I') - \int_{I'(t)} \rho dq. \tag{2.9}$$

3. Numerical Formulation

The discussion for the numerical simulation focuses on the one-dimensional case. Some lumber industries have more issues on drying woods where the length and width are much larger than the thickness. Such geometrical shape of woods often develops defects such as bow, cup, twist, crook and crack. For this geometrical wood, the diffusion of water to the direction of width is more dominant than to the other directions (see [3] and the references therein). Concrete drying is also often dealt with such geometrical shape. In this case, one-dimensional model performs as good as the two- or three-dimensional models.

We consider initial interval $[0, 1]$. Otherwise, any close interval can be normalized in this form. Keeping the left boundary fixed, the right boundary may depend on t. Hence, the interval at t is $[0, b(t)]$. The initial condition is given by

$$\rho(x, 0) = g(x) \text{ for } 0 < x < 1. \tag{3.1}$$

The boundary conditions are

$$\rho(0, t) = f_1(t) \text{ and } \rho(b(t), t) = f_2(t). \tag{3.2}$$

Such boundary conditions apply for the wood drying in kiln chamber of industries by setting the equilibrium moisture content.

Divide the interval $[0, b(t)]$ into M evenly spatial partitions, as illustrated in Figure 2:

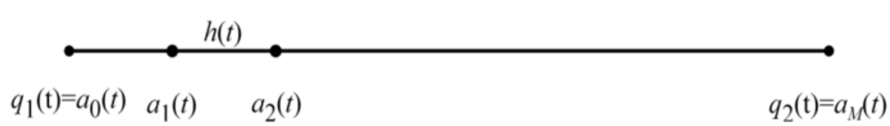

Figure 2. Illustrative plot of spatial partitions.

Consider interval partition I_m at time t_n. Approximating $\int_{I_m} \frac{d\rho}{dt} dq$ using forward method, we have

$$\int_{I_m} \frac{d\rho}{dt} dq = \frac{\rho_m^{n+1} h^n - \rho_m^n h^n}{\Delta t}. \tag{3.3}$$

Applying central difference for $\int_{I_m} \partial_q (K \partial_q \rho) dq$, we have

$$\int_{I_m} \partial_q (K \partial_q \rho) dq = K_{m+1}^n \left(\frac{\rho_{m+1}^n - \rho_m^n}{h^n} \right) - K_m^n \left(\frac{\rho_m^n - \rho_{m-1}^n}{h^n} \right). \tag{3.4}$$

Now, from (3.3) and (3.4), explicit numerical scheme of equation (2.9) is in the form

$$\rho_m^{n+1} = \rho_m^n \frac{h^n}{h^{n+1}} + \frac{\Delta t}{h^{n+1}} \begin{pmatrix} K_{m+1}^n \left(\frac{\rho_{m+1}^n - \rho_m^n}{h^n} \right) - K_m^n \left(\frac{\rho_m^n - \rho_{m-1}^n}{h^n} \right) \\ + \rho_{m+1}^n \cdot q_{m+1}^n - \rho_m^n \cdot q_m^n \end{pmatrix}. \tag{3.5}$$

For nondeformable material and constant diffusion rate, (3.5) is just a standard finite difference formula for the heat equation

$$\rho_m^{n+1} = \rho_m^n + \frac{\Delta t}{h^2} K (\rho_{m+1}^n - 2\rho_m^n + \rho_{m-1}^n). \tag{3.6}$$

This can be easily found in standard text books such as [15].

4. Results and Discussion

The numerical simulation is implemented for $K = 1$, $h^1 = 0.05$ and $\Delta t = 0.0006$. The initial condition is assumed to be uniform and the value is 1. It is considered that the material is wet with the uniform density of fluid. In kiln chamber for drying wood, this is achieved by spraying the wood at the beginning of drying process. Any uniform conditions can be normalized to this, the boundary conditions are constant, and less than the initial condition. In this case, the boundary conditions are taken to be 0. Again, any constant boundary conditions can be normalized to 0. The computation is run in Scilab. Figure 3 shows the results of computation representing the interval length and the state variable in this interval for various times. The length of interval started from one unit to 0.8 unit after 0.2 unit of time. At the same time, the value of the state variable decreases, and approaches the boundary condition. Such process represents the drying and shrinkage of wood without defects, which is in the need of the industries.

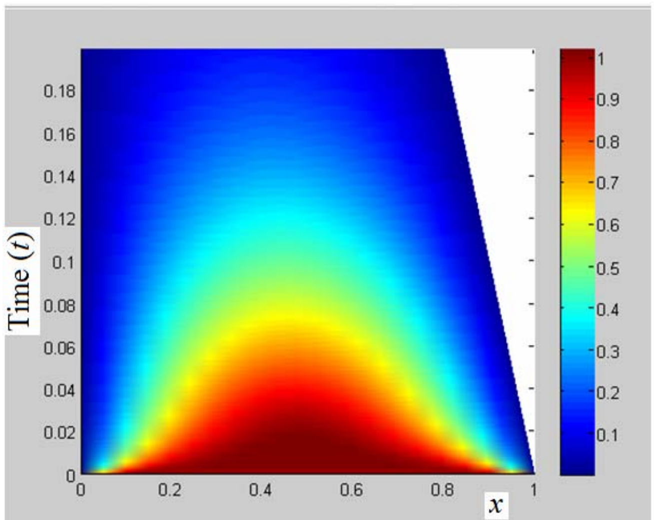

Figure 3. The interval length and the value of state variable.

This approach does not allow for the material to increase in size after shrinkage. Some materials as concrete and clay after baking experience such phenomena. Concrete and clay after baking do not increase in size after shrinkage, the shrinkage process is not reversible. On the other hand, in materials such as wood, the shrinkage process may be reversible. Wood may increase in size again after shrinkage when the liquid density inside the wood increases. For this material, model discussed in [9] is more desirable.

5. Concluding Remark

Modeling of diffusion process with irreversible shrinkable material has been discussed. Such process occurs in the drying of concrete and clay after baking. The state variable which represents the fluid density inside material is a function of time and space. The motion of spatial point inside material allows the material to shrink. The model is in the form of an integral equation. For the case of nondeformable material and constant diffusion rate, the model is nothing else but the heat equation.

Numerical scheme based on finite difference method has been derived for the model. Again, for the case of nondeformable material and constant diffusion rate, the numerical scheme is the well-known standard finite difference method for the heat equation. A simulation of an initial and boundary value problem where the material shrinks about 20 percent is also presented. Such simulation is the need for drying process of wood in kiln chamber of industries for the case of no defects, where the wood does not change in shape but merely the size.

Future work will be on improving the model by considering several aspects. At first, it will take into account the driving force for the shrinkage. Does it depend on the density of the solid itself, or also the density of fluid altogether? Second, it will consider state variables of the fluid density and solid density in the model. Are the state variables separated in the model, or do they interact? Third, it will deal with the comparison with experiments or industrial process.

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References

[1] F. Barpi and S. Valente, Lifetime evaluation of concrete structures under sustained post-peak loading, Eng. Frac. Mech. 72 (2005), 2427-2443.

[2] Z. P. Bazant and Y. Xiang, Crack growth and lifetime of concrete under long time loading, J. Eng. Mech. 123(4) (1997), 350-358.

[3] E. Cahyono, Y. Soeharyadi and Mukhsar, A smooth diffusion rate model of wood drying: a simulation toward more efficient process in industry, J. Teknik Industri 10(1) (2008), 1-10.

[4] E. Cahyono and Y. Soeharyadi, On the interaction of Barenblatt's solutions to the porous medium equation, Int. J. Math. Sci. Eng. Appl. 4(2) (2010), 187-198.

[5] E. Cahyono, La Hamimu, L. O. Ngkoimani and J. Safani, Interaction of droplet diffusions governed by 2-D porous medium equation, Appl. Math. Sci. 127(8) (2014), 6303-6311.

[6] E. Cahyono, Mathematical problem appearing in industrial lumber drying: a review, Sci. J. Appl. Math. Stat. 2(1) (2014), 26-30.

[7] N. J. Carino and J. R. Clifton, Prediction of cracking in reinforced concrete structures, building and fire research laboratory, National Institute of Standards and Technology, Gaithersburg, 1995 (unpublished).

[8] S. Chemkhi and F. Zagrouba, Water diffusion coefficient in clay material from drying data, Desalination 185 (2005), 491-498.

[9] S. Chemkhi and F. Zagrouba, Development of a Darcy-flow model applied to simulate the drying of shrinking media, Braz. J. Chem. Eng. 25(03) (2008), 503-514.

[10] L. C. Evans, Partial differential equations, Graduate Studies in Mathematics, Vol. 19, American Mathematical Society, Providence, Rhode Island, 1998.

[11] Z. X. Gong and A. S. Mujumdar, Drying of clay and nonclay media: heat and mass transfer and quality aspects, Dry. Technol. 16(6) (1998), 1119-1152.

[12] S. Gutman and A. G. Ramm, Inverse problem for a heat equation with piecewise-constant conductivity, J. Appl. Math. Informatics 28(3-4) (2010), 651-661.

[13] K. Khalfaoui, S. Chemkhi and F. Zagrouba, Modeling and stress analysis during drying of a deformable and saturated porous medium, Dry. Technol. 31(10) (2013), 1124-1137.

[14] P. H. Morris, J. Graham and D. J. Williams, Cracking in drying soils, Can. Geotech. J. 29(2) (1992), 263-277.

[15] K. W. Morton and D. F. Mayers, Numerical Solution of Partial Differential Equations, Cambridge University Press, Cambridge, 1996.

[16] H. Peron, L. Laloui, L.-B. Hu and T. Huecke, Formation of drying crack patterns in soils: a deterministic approach, Acta Geotechnica (2012), 7 pages. doi:10.1007/s11440-012-0184-5.

[17] P. Perre and B. K. May, A numerical drying model that accounts for the coupling between transfers and solid mechanics: case of highly deformable products, Dry. Technol. 19(8) (2001), 1629-1643.

[18] P. Perre and J. Passard, A physical and mechanical model able to predict the stress field in wood over a wide range of drying conditions, Dry. Technol. 22(1-2) (2004), 27-44.

[19] A. G. Ramm, An inverse problem for the heat equation, J. Math. Anal. Appl. 264 (2001), 691-697.

[20] A. G. Ramm, An inverse problem for the heat equation II, Appl. Anal. 81(4) (2002), 929-937.

[21] N. I. Thusyanthan, W. A. Take, S. P. G. Madabhushi and M. D. Bolton, Crack initiation in clay observed in beam bending, Géotechnique 57(7) (2007), 581-594.

[22] H.-J. Vogel, H. Hoffmann, A. Leopold and K. Roth, Studies of crack dynamics in clay soil II, A physically based model for crack formation, Geoderma 125 (2005), 213-223.