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WEAKLY *p*-RADICAL SUBMODULES OVER NON-COMMUTATIVE RINGS

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Abstract

Some important results were found on weakly prime submodules over non-commutative rings. In this paper, we generalize these results on weakly primary submodules over non-commutative rings. We also introduce the concept of weakly p-radical submodule and study some properties of primary radical of a submodule and show that for an R-module M that satisfies the ACC on weakly p-radical submodules, every weakly p-radical submodule is the weakly p-radical of a finitely generated submodule.

1. Introduction

Throughout this paper, all rings are associative rings with identity and are not necessarily commutative and all modules are unitary right R-modules. Some work has previously been conducted related to concepts of prime submodules, including the concept of weakly prime submodules over non- commutative rings introduced and studied primarily by Callialp and Farzalipour in [7].

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In particular, we refer to a number of papers concerning prime submodules which have been studied by various authors, see, for example [9, 11, 12]. Moreover, weakly prime ideals in a commutative ring with nonzero identity have been introduced and studied by Anderson and Smith in [2]. The structure of weakly primary ideals in a commutative ring has been studied by Atani and Farzalipour in [5]. The structure of weakly prime ideals over non-commutative rings has been studied by Hirano et al. [10].

The study of prime submodules is extended in many ways, such as weakly prime submodules, primary submodules, graded prime submodules, and n-absorbing submodules, see [5, 8, 13, 14]. The motivation of this paper is to continue the study of the family of primary submodules, and also to extend the results of Atani and Farzalipour [5] and Smith [14] to the weakly primary submodules over non-commutative rings. In fact, a number of results concerning weakly primary submodules over non-commutative rings are also mentioned.

In Section 2, we introduce the definition of the weakly primary submodule. A proper submodule N of an R-module M is called a *weakly primary submodule* of M if whenever $r \in R$ and $m \in M$ with $rRm \subseteq N$, then either $m \in M$ or $r \in \sqrt{(N : M)}$.

We give some results about the weakly primary submodule and provide a characterization of weakly primary submodule as: a proper submodule N of an R-module M is weakly primary submodule of M if and only if for any ideal I of R and for any submodule K of M with $0 \neq IK \subset N$ either $I \subseteq \sqrt{(N : M)}$ or $K \subseteq N$. Also, it is shown in Theorem 2.9 that M_1 and M_2 are R-modules, with $M = M_1 \oplus M_2$ and $N \subseteq M_1 \oplus M_2$. Then if $N = K \oplus M_2$ (or $N = M_1 \oplus K$) is a weakly primary submodule of M for some submodule K of M_1, then K is a weakly primary submodule of M_1 (resp. K is a weakly primary submodule of M_2).

The concepts of primary radical of a submodule and the p-radical submodule over commutative rings have been introduced and studied by

Abulebda in [3]. In Section 3, we introduce the concept of weakly p-radical submodule over non-commutative rings as follows: let N be a submodule of an R-module M. If there exist weakly primary submodules containing N, then the intersection of all weakly primary submodules containing N is the weakly primary radical submodule of N, denoted by $wprad(N)$. If there is no weakly primary submodule containing N, then $wprad(N) = M$. In particular, $wprad(M) = M$. We say that a submodule N is *weakly p-radical submodule* if $wprad(N) = N$. We study some properties of primary radical of a submodule and show that for an R-module M that satisfies the ACC on weakly p-radical submodules, then every weakly p-radical submodule is the weakly p-radical of a finitely generated submodule.

Some results in this paper which will be identified by placing (*) sign at them are quite similar to some in [1].

2. Weakly Primary Submodule over Non-commutative Ring

The weakly prime submodules over non-commutative ring have been studied by Callialp and Farzalipour in [7].

Definition 2.1. Let M be a left R-module. A proper submodule N of M is called a *weakly prime submodule* of M if whenever $r \in R$ and $m \in M$ with $0 \neq rRm \subseteq N$, then either $m \in M$ or $r \in (N; M)$.

Definition 2.2. Let R be an associative ring with identity and M be a unitary right R-module, and N be a submodule of M. Then $\sqrt{(N : M)} = \{r \in R \mid r^n M \subseteq N$ for some positive integer $n\}$ is called the *radical* of N over R.

Definition 2.3. Let M be a left R-module. A proper submodule N of M is called a *primary submodule* of M if whenever $r \in R$ and $m \in M$ with $rRm \subseteq N$, then either $m \in M$ or $r \in \sqrt{(N : M)}$.

Now we introduce the definition of the weakly primary submodule:

Definition 2.4. Let M be a left R-module. A proper submodule N of M is called a *weakly primary submodule* of M if whenever $r \in R$ and $m \in M$ with $0 \neq rRm \subseteq N$, then either $m \in M$ or $r \in \sqrt{(N : M)}$.

Remark 2.5. (a) Every weakly prime submodule is a weakly primary submodule. The converse is not true as in the following example:

Let $R = Z_8 \oplus Z_8$ and M be an R-module as $M = (Z_8 \oplus Z_8) \oplus (Z_8 \oplus Z_8)$ with addition and multiplication defined as:

$$((a_1, b_1), (a_2, b_2)) + ((x_1, y_1), (x_2, y_2))$$

$$= ((a_1 + x_1, b_1 + y_1), (a_2 + x_2, b_2 + y_2)),$$

$$((a_1, b_1), (a_2, b_2)) \cdot (x, y) = ((a_1 x, b_1 y), (a_2 x, b_2 y)).$$

The submodule $L = \{(0, 0), (0, 0), (0, 4), (0, 0)\}$ is a weakly primary submodule but not a weakly prime submodule because $((0, 2), (0, 0)) \cdot (0, 2) \in L$ and neither $(0, 2) \in (L : M)$ nor $((0, 2), (0, 0)) \in L$.

(b) Every primary submodule is a weakly primary submodule over R. The converse is not true as in the following example:

Let $R = Z_4 \oplus Z_4$ and M be an R-module as $M = (Z_4 \oplus Z_4) \oplus (Z_4 \oplus Z_4)$ with addition and multiplication defined as:

$$((a_1, b_1), (a_2, b_2)) + ((x_1, y_1), (x_2, y_2))$$

$$= ((a_1 + x_1, b_1 + y_1), (a_2 + x_2, b_2 + y_2)),$$

$$((a_1, b_1), (a_2, b_2)) \cdot (x, y) = ((a_1 x, b_1 y), (a_2 x, b_2 y)).$$

The submodule $L = \{(0, 0), (0, 0), (0, 2), (0, 0)\}$ is a weakly primary submodule but not a primary submodule because $((0, 1), (0, 0)) \cdot (1, 0) = 0 \in L$ and neither $(1, 0) \in \sqrt{(L : M)}$ nor $((0, 1), (0, 0)) \in L$.

The following theorem gives the condition that makes the weakly primary submodule primary:

Theorem 2.6*. *Let M be an R-module. Let N be a weakly primary submodule of M. If* $\sqrt{(N : M)} \cdot N \neq 0,$ *then N is a primary submodule of M.*

Proof. Let $r \in R$ and $m \in M$ with $eRm \subseteq N$. If $rRm \neq 0,$ then N is a weakly primary submodule giving $m \in N$ or $r \in \sqrt{(N : M)}$. So assume that $rRm = 0$. If $0 \neq rN$, then $\exists x \in N$ such that $rx \neq 0$. Now, $0 \neq rRx = rR(m + x) \subseteq N$, so N is a weakly primary submodule giving $(m + x) \in N$ or $r \in \sqrt{(N : M)}$, thus $m \in N$ or $r \in \sqrt{(N : M)}$.

Now we assume that $rN = 0$.

Case 1. If $\sqrt{(N : M)}m \neq 0$, then $\exists k \in \sqrt{(N : M)}$ such that $km \neq 0$. So $0 \neq kRm = (r + k)Rm \subseteq N$. Also, $m \in N$ or $(r + k) \in \sqrt{(N : M)}$. Since $k \in \sqrt{(N : M)}$, we have $m \in N$ or $r \in \sqrt{(N : M)}$.

Case 2. $\sqrt{(N : M)}m = 0$. Since $\sqrt{(N : M)} \cdot N \neq 0$, $\exists s \in \sqrt{(N : M)}$ and $t \in N$ such that $st \neq 0$. Then $0 \neq sRt = (r + s)R(m + t) \subseteq N$ so $(m + t) \in N$ or $(r + s) \in \sqrt{(N : M)}$, thus $m \in N$ or $(r + s) \in \sqrt{(N : M)}$, thus $m \in N$ or $r \in \sqrt{(N : M)}$. □

Corollary 2.7. *Let M be an R-module. Let N be a weakly primary submodule of M. If N is not primary submodule, then for any ideal I of R such that* $I \subseteq \sqrt{(N : M)}$, *we have* $IN = 0$. *In particular,* $\sqrt{(N : M)} \cdot N = 0$.

Now, we give a characterization of the weakly primary submodule:

Theorem 2.8*. *Let M be an R-module. A proper submodule N of M is weakly primary submodule of M if and only if for any ideal I of R and for any submodule K of M with* $0 \neq IK \subset N$ *either* $I \subseteq \sqrt{(N : M)}$ *or* $K \subseteq N$.

Proof. Suppose that N is a weakly primary submodule of M. If N is primary, then for any ideal I of R and for any submodule K of M with

$0 \neq IK \subset N$ either $I \subseteq \sqrt{(N:M)}$ or $K \subseteq N$ is trivial. So assume N is not primary submodule of M. Let $0 \neq IK \subset N$ with $x \in K - N$. Now let $r \in I$. If $0 \neq rRx$, since $rRx \subseteq N$ and N is a weakly primary submodule, so $r \in \sqrt{(N:M)}$, thus $I \subseteq \sqrt{(N:M)}$. Now, if $0 = rRx$, assume that $rK \neq 0$, say $rk \neq 0$ for some $k \in K$. Now $0 \neq rRk \subseteq N$, then $r \in \sqrt{(N:M)}$. If $k \notin N$, then $r \in \sqrt{(N:M)}$. If $k \in N$, then $0 \neq rRk = rR(k+x) \subseteq N$, so $k + x \in N$ or $r \in \sqrt{(N:M)}$. Since $x \notin N$, $r \in \sqrt{(N:M)}$. So we can assume that $rK = 0$. Suppose that $Ix \neq 0$, and let $ix \neq 0$, where $i \in I$. Now, $0 \neq iRx \subseteq N$, then N is a weakly primary submodule giving that $i \in \sqrt{(N:M)}$. As $0 \neq iRx = (r+i)Rx \subseteq N$, so $r \in \sqrt{(N:M)}$. Thus, we can assume $Ix = 0$ since $IK \neq 0$, $\exists i \in I$ and $k \in K$ such that $ik \neq 0$, now $0 \neq iRk \subseteq N$.

By Corollary 2.7, we have $\sqrt{(N:M)} \cdot N = 0$ and $0 \neq iRk = iR(k+x) \subseteq N$. If $i \in \sqrt{(N:M)}$ and $k + x \notin N$, since $0 \neq (r+i)R(k+x) = iRk \subseteq N$ we have $(r+i) \in \sqrt{(N:M)}$, and so $r \in \sqrt{(N:M)}$. Now if $i \notin \sqrt{(N:M)}$ and $k + x \in N$, since $0 \neq iRk \subseteq N$, we have $k \in N$, so $x \in N$, which is a contradiction. Therefore, $r \in \sqrt{(N:M)}$ thus $I \subseteq \sqrt{(N:M)}$.

Now, suppose for any ideal I of R and for any submodule K of M with $0 \neq IK \subset N$ either $I \in \sqrt{(N:M)}$ or $K \subseteq N$. To prove that N is a weakly primary submodule, assume that $sRm \subseteq N$, where $s \in R$ and $m \in M$. Let $I = Rs$ and $K = Rm$. So $0 \neq IK = RsRm \subseteq N$, so either $I \subseteq \sqrt{(N:M)}$ or $K \subseteq N$. Thus, $s \in \sqrt{(N:M)}$ or $m \in N$. □

Theorem 2.9*. *Let M_1 and M_2 be R-modules, $M = M_1 \oplus M_2$ and let $N \subseteq M_1 \oplus M_2$. Then if $N = K \oplus M_2$ (or $N = M_1 \oplus K$) is a weakly primary submodule of M for some submodule K of M_1, then K is a weakly primary submodule of M_1 (resp. K is a weakly primary submodule of M_2).*

Proof. Let $N = K \oplus M_2$ be a weakly primary submodule of $M = M_1 \oplus M_2$, let $0 \neq rRm_1 \subseteq K$, where $r \in R$ and $m_1 \in M$ such that $m_1 \notin K$, then $(m_1, 0) \notin K \oplus M_2$, $0 \neq rR(m_1, 0) \subseteq K \oplus M_2$. Since $N = K \oplus M_2$ is a weakly primary submodule, there exists a positive integer n such that $r^n(M_1 \oplus M_2) \subseteq K \oplus M_2$, hence $r^n M_1 \subseteq K$ for some positive integer n. Thus, $r \in \sqrt{(K : M_1)}$. Thus, K is a weakly primary submodule of M_1. □

3. Weakly *p*-radical Submodule over Non-commutative Ring

The weakly prime radical submodules over non-commutative ring have been introduced by Behboodi [6].

Definition 3.1. Let N be a submodule of an R-module M. If there exist weakly prime submodules containing N, then the intersection of all weakly prime submodules containing N is the weakly prime radical submodule of N, denoted by $wrad(N)$. If there is no weakly prime submodule containing N, then $wrad(N) = M$. In particular, $wrad(M) = M$. We say that a submodule N is *weakly radical submodule* if $wrad(N) = N$.

Now we introduce the concept of the weakly p-radical submodule.

Definition 3.2. Let N be a submodule of an R-module M. If there exist weakly primary submodules containing N, then the intersection of all weakly primary submodules containing N is the weakly primary radical submodule of N, denoted by $wprad(N)$. If there is no weakly primary submodule containing N, then $wprad(N) = M$. In particular, $wprad(M) = M$.

We say that a submodule N is *weakly p-radical submodule* if $wprad(N) = N$.

Proposition 3.3. *It is clear that every weakly primary submodule is a weakly p-radical submodule.*

Proof. Trivial. □

Proposition 3.4. *Let N and L be submodules of an R-module M. Then*:

(1) $N \subseteq wprad(N)$.

(2) *If* $N \subseteq L$, *then* $wprad(N) \subseteq wprad(L)$.

(3) $wprad(N)$ *is a weakly p-radical submodule, i.e.*, $wprad(wprad(N)) = wprad(N)$.

(4) $wprad(N \cap L) \subseteq wprad(N) \cap wprad(L)$.

(5) $wprad(N + L) = wprad(wprad(N) + wprad(L))$.

Proof. (1) Trivial.

(2) Let $N \subseteq L$ and let K be weakly primary submodule containing L. Then $N \subseteq K$, hence $wprad(N) \subseteq wprad(L)$.

(3) By (1), $N \subseteq wprad(N)$ and by (2), $wprad(N) \subseteq wprad(wprad(N))$. Let K be weakly primary submodule containing N. Then $wprad(N) \subseteq wprad(K) = K$. So every weakly primary submodule containing N also contains $wprad(N)$. Thus, $wprad(wprad(N)) \subseteq wprad(N)$, therefore $wprad(wprad(N)) = wprad(N)$.

(4) Since $N \cap L \subseteq N$ and $N \cap L \subseteq L$, by (2), $wprad(N \cap L) \subseteq wprad(N)$ and $wprad(N \cap L) \subseteq wprad(L)$, therefore $wprad(N \cap L) \subseteq wprad(N) \cap wprad(L)$.

(5) By (1), $N + L \subseteq wprad(N) + wprad(L)$ and by (2), $wprad(N + L) \subseteq wprad(wprad(N) + wprad(L))$. Now let K be a weakly primary submodule containing $N + L$, then $N \subseteq K$ and $L \subseteq K$. Hence, $wprad(N) \subseteq wprad(K) = K$ and $wprad(L) \subseteq wprad(K) = K$. Thus, $wprad(N) + wprad(L) \subseteq K$. So every weakly primary submodule containing $N + L$ also contains $wprad(N) + wprad(L)$. Therefore,

$$wprad(wprad(N) + wprad(L)) \subseteq wprad(N + L).$$

So $wprad(N + L) = wprad(wprad(N) + wprad(L)).$ □

Proposition 3.5. *Let N and L be submodules of an R-module M such that whenever* $N \cap L \subseteq K$ *we have* $N \subseteq K$ *or* $L \subseteq K$ *for any weakly primary submodule K of M. Then* $wprad(N \cap L) = wprad(N) \cap wprad(L).$

Proof. By (4) of Proposition 3.4, $wprad(N \cap L) \subseteq wprad(N) \cap wprad(L).$ Now if $wprad(N \cap L) = M,$ thus $wprad(N) = wrad(L) = M$ so $wprad(N \cap L) = wprad(N) \cap wprad(L).$ If $wprad(N \cap L) \neq M,$ then there exists a weakly primary submodule K such that $N \cap L \subseteq K.$ By hypotheses, $N \subseteq K$ or $L \subseteq K$ so that $wprad(N) \subseteq K$ or $wprad(L) \subseteq K.$

This is true for all weakly primary submodules containing $N \cap L.$ Then $wprad(N) \cap wprad(L) \subseteq wprad(N \cap L).$ Thus, $wprad(N \cap L) = wprad(N) \cap wprad(L).$ □

The following theorem gives a characterization of the weakly *p*-radical submodule of an *R*-module which satisfies the ACC on weakly *p*-radical submodules.

Theorem 3.6. *Let M be an R-module. If M satisfies the ACC on weakly p-radical submodules, then every weakly p-radical submodule is the weakly p-radical of a finitely generated submodule.*

Proof. Assume that there exists a weakly *p*-radical submodule *N* which is not the weakly *p*-radical of a finitely generated submodule. Let $n_1 \in N$ and let $N_1 = wprad(n_1R).$ Then $N_1 \subsetneqq N$ so there exists $n_2 \in N - N_1.$ Let $N_2 = wprad(n_1R + n_2R),$ then $N_1 \subsetneqq N_2 \subsetneqq N.$ So there exists $n_3 \in N - N_2$ etc. This gives an ascending chain of weakly *p*-radical submodules $N_1 \subsetneqq N_2 \subsetneqq N_3 \subsetneqq \cdots,$ which is a contradiction. □

The following theorem comes directly from Proposition 3.4.

Theorem 3.7. *If every weakly p-radical submodule is the weakly p-radical of a finitely generated submodule. Then every weakly primary submodule is the weakly p-radical of a finitely generated submodule.*

Theorem 3.8. *Let M be an R-module. The following statements are equivalent*:

(1) *For each proper submodule N of M, there exists* $m \in N$ *such that* $wprad(N) = wprad(Rm)$.

(2) *For each proper submodule N of M, if* $N \subseteq \bigcup_{\alpha \in \lambda} N_\alpha$, *where* $\{N_\alpha : \alpha \in \lambda\}$ *is a family of submodules of M, then* $N \subseteq wprad(N_{\alpha'})$ *for some* $\alpha' \in \lambda$.

(3) *For each proper submodule N of M, if* $N \subseteq \bigcup_{\alpha \in \lambda} N_\alpha$, *where* $\{N_\alpha : \alpha \in \lambda\}$ *is a family of weakly primary radical submodules of M, then* $N \subseteq N_{\alpha'}$ *some for* $\alpha' \in \lambda$.

Proof. (1) $\Rightarrow$ (2) Let N be a proper submodule of M. If $N \subseteq \bigcup_{\alpha \in \lambda} N_\alpha$, where $\{N_\alpha : \alpha \in \lambda\}$ is a family of submodules of M. By (1), there exists $m \in N$ such that $wprad(N) = wprad(Rm)$. So $m \in \bigcup_{\alpha \in \lambda} N_\alpha$ and hence $m \in N_{\alpha'}$ some for $\alpha' \in \lambda$. Therefore, $N \subseteq wprad(N) = wprad(Rm) \subseteq wprad(N_{\alpha'})$ for some $\alpha' \in \lambda$.

(2) $\Rightarrow$ (3) Let N be a proper submodule of M. If $N \subseteq \bigcup_{\alpha \in \lambda} N_\alpha$, where $\{N_\alpha : \alpha \in \lambda\}$ is a family of weakly primary radical submodules of M. By (2), $N \subseteq wprad(N_{\alpha'})$ for some $\alpha' \in \lambda$. Since $N_{\alpha'}$ is weakly primary radical submodule, $N \subseteq N_{\alpha'}$ some for $\alpha' \in \lambda$.

(3) $\Rightarrow$ (1) Let N be a proper submodule of M. It is clear that for each $m \in N$, $wprad(Rm) \subseteq wprad(N)$. Now suppose that $wprad(N) \nsubseteq wprad(Rm)$ for each $m \in N$. Then for each $m \in N$, there exists a weakly primary radical submodule N_m for which $Rm \subseteq N_m$ and $N \not\subset N_m$. So $N = \bigcup_{m \in N} Rm \subseteq \bigcup_{m \in N} N_m$ which is a contradiction to (3). □

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