FRACTIONAL DIFFERENTIATION OF THE GENERALIZED LOMMEL-WRIGHT FUNCTION

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Abstract

We aim to establish certain image formulas of the generalized Lommel-Wright function for the fractional differentiation operators having Appell's function $F_3(\cdot)$ as a kernel. Also, we present some image formulas of the resulting identities for some integral transforms. The results presented here, being very general, are pointed out to be specialized to yield a number of results involving relatively simple fractional differentiation operators and simpler extensions of the Bessel function and the Struve function, including their numerous limiting cases.

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1. Introduction and Preliminaries

During the last four decades or so, the fractional calculus has found many applications in diverse research areas (see, e.g., [2, 3, 18, 19, 21, 26, 27]). The fractional differential operators involving various special functions have found significant importance and applications in diverse research fields of applicable mathematical analysis. Beginning with [15], a number of fractional differentiation operators involving various extensions of the hypergeometric function have been introduced and investigated (see, e.g., [4, 14, 16, 19, 21, 27]). A useful generalization of the hypergeometric fractional differentiation, including the Saigo operators [15, 16], has been introduced by Marichev [9] (for details, see [18, p. 194, eq. (10.47) and Section 10.3]) and later extended and studied by Saigo and Maeda [17, p. 393, Eq. (4.12) and Eq. (4.13)] in terms of any complex order whose kernel is the following Appell function $F_3(\cdot)$ of two variables (see, e.g., [5] and [24, p. 23]):

$$F_3(\alpha, \alpha', \beta, \beta'; \gamma; x; y)$$

$$= \sum_{m, n=0}^{\infty} \frac{(\alpha)_m (\alpha')_n (\beta)_m (\beta')_n}{(\gamma)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \, (\max\{|x|, |y|\} < 1). \qquad (1.1)$$

Let $\alpha, \alpha', \beta, \beta', \gamma \in \mathbb{C}$ with $\mathfrak{R}(\gamma) > 0$ and $x \in \mathbb{R}^+$. Then the generalized fractional integral operators involving the Appell hypergeometric function F_3 as a kernel are defined as follows (see [17]):

$$(I_{0, x}^{\alpha, \alpha', \beta, \beta', \gamma} f)(x)$$

$$= \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x (x - t)^{\gamma - 1} t^{-\alpha'} F_3\left(\alpha, \alpha', \beta, \beta'; \gamma; 1 - \frac{t}{x}, 1 - \frac{x}{t}\right) f(t) dt \qquad (1.2)$$

and

$$(I_{x, \infty}^{\alpha, \alpha', \beta, \beta', \gamma} f)(x)$$

$$= \frac{x^{-\alpha'}}{\Gamma(\gamma)} \int_x^{\infty} (t - x)^{\gamma - 1} t^{-\alpha} F_3\left(\alpha, \alpha', \beta, \beta'; \gamma; 1 - \frac{x}{t}, 1 - \frac{t}{x}\right) f(t) dt. \qquad (1.3)$$

Here and in the following, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{N}$ and $\mathbb{Z}_0^-$ denote the sets of complex numbers, real numbers, positive real numbers, positive integers and non-positive integers, respectively. Also, the generalized fractional derivative operators of a function $f(x)$ are defined as follows (see [17]):

$$(D_{0+}^{\alpha,\,\alpha',\,\beta,\,\beta',\,\gamma}f)(x) = (I_{0,\,x}^{-\alpha',\,-\alpha,\,-\beta',\,-\beta,\,-\gamma}f)(x)$$

$$= \left(\frac{d}{dx}\right)^k (I_{0,\,x}^{-\alpha',\,-\alpha,\,-\beta'+k,\,-\beta,\,-\gamma+k}f)(x) \qquad (1.4)$$

and

$$(D_{0-}^{\alpha,\,\alpha',\,\beta,\,\beta',\,\gamma}f)(x) = (I_{x,\,\infty}^{-\alpha',\,-\alpha,\,-\beta',\,-\beta,\,-\gamma}f)(x)$$

$$= \left(-\frac{d}{dx}\right)^k (I_{x,\,\infty}^{-\alpha',\,-\alpha,\,-\beta'+k,\,-\beta,\,-\gamma+k}f)(x), \qquad (1.5)$$

where $\mathfrak{R}(\gamma) > 0$ and $k := [\mathfrak{R}(\gamma)] + 1$. The Appell function (1.1) in (1.4) and (1.5) satisfies a system of two partial differential equations of the second order and reduces to the Gauss hypergeometric function $_2F_1$ as follows (see, e.g., [5] and [24]):

$$F_3(\alpha,\,\gamma-\alpha,\,\beta,\,\gamma-\beta;\,\gamma;\,x;\,y) = {}_2F_1(\alpha,\,\beta;\,\gamma;\,x+y-xy). \qquad (1.6)$$

Further, we have

$$F_3(\alpha,\,0,\,\beta,\,\beta';\,\gamma;\,x;\,y) = {}_2F_1(\alpha,\,\beta;\,\gamma;\,x) \qquad (1.7)$$

and

$$F_3(0,\,\alpha',\,\beta,\,\beta';\,\gamma;\,x;\,y) = {}_2F_1(\alpha',\,\beta';\,\gamma;\,y). \qquad (1.8)$$

In view of (1.7), the general operators (1.2) and (1.3) reduce to the Saigo operators

$$(I_{0,\,x}^{\alpha,\,\beta,\,\gamma}f)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta,\,-\gamma;\,\alpha;\,1-\frac{t}{x}\right)f(t)\,dt \qquad (1.9)$$

and

$$(I_{x,\infty}^{\alpha,\beta,\gamma}f)(x) = \frac{1}{\Gamma(\alpha)}\int_x^\infty (x-t)^{\alpha-1}t^{-\alpha-\beta}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;1-\frac{x}{t}\right)f(t)\,dt. \quad (1.10)$$

Also, the left- and right-sided Saigo fractional derivative operators can be defined as (see, e.g., [15] and [25]):

$$(D_{0+}^{\alpha,\beta,\gamma}f)(x) = (I_{0,x}^{-\alpha,-\beta,\alpha+\gamma}f) = \left(\frac{d}{dx}\right)^n \{(I_{0,x}^{-\alpha+\gamma,-\beta-\gamma,\alpha+\gamma-n}f)(x)\} \quad (1.11)$$

and

$$(D_{0-}^{\alpha,\beta,\gamma}f)(x) = (I_{x,\infty}^{-\alpha,-\beta,\alpha+\gamma}f) = \left(-\frac{d}{dx}\right)^n \{(I_{x,\infty}^{-\alpha+\gamma,-\beta-\gamma,\alpha+\gamma-n}f)(x)\}, \quad (1.12)$$

where $\mathfrak{R}(\alpha) \geq 0$ and $n = [\mathfrak{R}(\alpha)] + 1$. Taking $\alpha = 0$, the operators (1.4) and (1.5) reduce to the Saigo fractional derivative operators (1.11) and (1.12), respectively.

Setting $\beta = -\alpha$, the operators (1.11) and (1.12) reduce to the Riemann-Liouville fractional derivative operator and the Weyl fractional derivative operator as follows (see, e.g., [6]):

$$(D_{0+}^{\alpha,-\alpha,\gamma}f)(x) = (^{RL}D_{0+}^{\alpha}f)(x)$$

$$= \left(\frac{d}{dx}\right)^n \left\{\frac{1}{\Gamma(n-\alpha)}\int_0^x \frac{f(t)}{(x-t)^{\alpha-n+1}}\,dt\right\} \quad (1.13)$$

and

$$(D_{0-}^{\alpha,-\alpha,\gamma}f)(x) = (^{W}D_{0-}^{\alpha}f)(x)$$

$$= \left(-\frac{d}{dx}\right)^n \left\{\frac{1}{\Gamma(n-\alpha)}\int_x^\infty \frac{f(t)}{(t-x)^{\alpha-n+1}}\,dt\right\}, \quad (1.14)$$

where $x \in \mathbb{R}^+$, $\mathfrak{R}(\alpha) \geq 0$ and $n := [\mathfrak{R}(\alpha)] + 1$.

Taking $\beta = 0$, the operators (1.11) and (1.12) reduce to left- and right-sided Erdélyi-Kober fractional differential operators, respectively (see, e.g., [6])

$$(D_{0+}^{\alpha,0,\gamma} f)(x) = (\,^{EK} D_{0+}^{\alpha,\gamma} f)(x)$$

$$= x^{\gamma} \left(\frac{d}{dx}\right)^n \left\{ \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{t^{\alpha+\gamma} f(t)}{(x-t)^{\alpha-n+1}} dt \right\} \qquad (1.15)$$

and

$$(D_{0-}^{\alpha,0,\gamma} f)(x) = (\,^{EK} D_{0-}^{\alpha,\gamma} f)(x)$$

$$= x^{\alpha+\gamma} \left(-\frac{d}{dx}\right)^n \left\{ \frac{1}{\Gamma(n-\alpha)} \int_x^{\infty} \frac{t^{-\gamma} f(t)}{(t-x)^{\alpha-n+1}} dt \right\}, \qquad (1.16)$$

where $x \in \mathbb{R}^+$, $\mathfrak{R}(\alpha) \geq 0$ and $n := [\mathfrak{R}(\alpha)] + 1$.

The generalized fractional integral operators (1.2) and (1.3) of a power function are given, respectively, as follows (see, e.g., [17, 19]):

$$(I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1})(x)$$

$$= \frac{\Gamma(\rho)\Gamma(\rho + \gamma - \alpha - \alpha' - \beta)\Gamma(\rho + \beta' - \alpha')}{\Gamma(\rho + \beta')\Gamma(\rho + \gamma - \alpha - \alpha')\Gamma(\rho + \gamma - \alpha' - \beta)} x^{\rho-\alpha-\alpha'+\gamma-1}$$

$$(\mathfrak{R}(\gamma) > 0, \; \mathfrak{R}(\rho) > \max\{0, \; \mathfrak{R}(\alpha + \alpha' + \beta - \gamma), \; \mathfrak{R}(\alpha' - \beta')\}) \qquad (1.17)$$

and

$$(I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1})(x)$$

$$= \frac{\Gamma(1 - \rho - \gamma + \alpha + \alpha')\Gamma(1 - \rho + \alpha + \beta' - \gamma)\Gamma(1 - \rho - \beta)}{\Gamma(1 - \rho)\Gamma(1 - \rho + \alpha + \alpha' + \beta' - \gamma)\Gamma(1 - \rho + \alpha - \beta)} x^{\rho-\alpha-\alpha'+\gamma-1}$$

$$(\mathfrak{R}(\gamma) > 0, \; \mathfrak{R}(\rho) < 1 + \min\{\mathfrak{R}(-\beta), \; \mathfrak{R}(\alpha + \alpha' - \gamma), \; \mathfrak{R}(\alpha + \beta' - \gamma)\}).$$

$$(1.18)$$

The Beta transform of complex valued function $f(z)$ of real variable z is defined as follows (see, e.g., [20]):

$$B\{f(z) : a, b\} = \int_0^1 z^{a-1}(1 - z)^{b-1} f(z)dz \quad (\min\{\Re(a), \Re(b)\} > 0), \quad (1.19)$$

whose particular case when $f(z) = 1$ reduces to the familiar Beta function (see, e.g., [23, p. 8, equation (43)])

$$B(a, b) = \begin{cases} \int_0^1 z^{a-1}(1 - z)^{b-1}dz & (\Re(a) > 0; \Re(b) > 0) \\ \dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)} & (a, b \in \mathbb{C}\backslash\mathbb{Z}_0^-). \end{cases} \quad (1.20)$$

Then Beta transform of power function $z^{\rho-1}$ is given as

$$B\{z^{\rho-1}; a, b\} = \int_0^1 z^{a+\rho-2}(1 - z)^{b-1}dz = \frac{\Gamma(a + \rho - 1)\Gamma(b)}{\Gamma(a + \rho + b - 1)}$$

$$(\Re(a + \rho) > 1, \Re(b) > 0). \quad (1.21)$$

The P_δ-transform of a complex valued function $f(z)$ of a real variable z denoted by $P_\delta[f(z); s]$ is a function $F(s)$ of a complex variable s, valid under certain conditions on $f(z)$ (see Lemma 1.1 below), is defined as follows (see [8]):

$$P_\delta[f(z); s] = F(s) = \int_0^\infty [1 + (\delta - 1)s]^{-\frac{z}{\delta-1}} f(z)dz \quad (\delta > 1). \quad (1.22)$$

Lemma 1.1. *Let $f(z)$ be integrable over any finite interval (a, b) with $0 < a < z < b$ and there exists $c \in \mathbb{R}$ such that*

(i) for any $b \in \mathbb{R}^+$, $\int_b^\varrho e^{-cz} f(z)dz$ tends to a finite number as $\varrho \to \infty$,

and

(ii) for any $a \in \mathbb{R}^+$, $\int_\nu^a |f(z)|dz$ tends to a finite number as $\nu \to 0+$.

Then the P_δ-transform $P_\delta[f(z); s]$ exists for $\Re\left(\dfrac{\ln[1 + (\delta - 1)s]}{\delta - 1}\right) > c.$

The P_δ-transform of $z^{\rho-1}$ is given by

$$P_\delta[z^{\rho-1};\ s] = \left\{\frac{\delta-1}{\ln[1+(\delta-1)s]}\right\}^\rho \Gamma(\rho) \quad (\Re(\rho) > 0,\ \delta > 1). \qquad (1.23)$$

Recently, Agarwal et al. [1] have found the solution of fractional Volterra integral equation and non-homogeneous time fractional heat equation using integral transform of pathway type and P_δ-transform. Also, Srivastava et al. [22] have presented some results involving generalized hypergeometric function by using P_δ-transform.

Taking the limit in (1.22) as $\delta \to 1+$, we find that the P_δ-transform reduces to the Laplace transform (see, e.g., [20]):

$$L[f(z);\ s] = \int_0^\infty e^{-zs} f(z)\,dz \quad (\Re(s) > 0). \qquad (1.24)$$

The following relationship holds true between the P_δ-transform in (1.22) and the Laplace transform in (1.24):

$$P_\delta[f(t):s] = L\left[f(t):\frac{\ln[1+(\delta-1)s]}{\delta-1}\right] \quad (\delta > 1), \qquad (1.25)$$

or, equivalently,

$$L[f(t):s] = P_\delta\left[f(t):\frac{e^{(\delta-1)s}-1}{\delta-1}\right] \quad (\delta > 1). \qquad (1.26)$$

The relations (1.25) and (1.26) can be applied to convert the table of Laplace transforms into the corresponding table of P_δ-transforms and vice versa.

We need to recall the following integral formula involving the Whittaker function (see, e.g., [10, p. 56]):

$$\int_0^\infty t^{\xi-1} e^{-\frac{t}{2}} W_{\sigma,\eta}(t)\,dt = \frac{\Gamma\left(\xi+\eta+\frac{1}{2}\right)\Gamma\left(\xi-\eta+\frac{1}{2}\right)}{\Gamma\left(\xi-\sigma+\frac{1}{2}\right)}$$

$$(\sigma \in \mathbb{C},\ \Re(\xi \pm \eta) > -1/2). \qquad (1.27)$$

Here $W_{\sigma, \eta}(z)$ is the Whittaker function defined by (see, e.g., [10, p. 22])

$$W_{\sigma, \eta}(z) = \frac{\Gamma(-2\eta)}{\Gamma\left(\frac{1}{2} - \sigma - \eta\right)} M_{\sigma, \eta}(z) + \frac{\Gamma(2\eta)}{\Gamma\left(\frac{1}{2} - \sigma + \eta\right)} M_{\sigma, -\eta}(z) = W_{\sigma, -\eta}(z)$$

$$(\sigma \in \mathbb{C}, \ \Re(1/2 + \eta \pm \sigma) > 0), \tag{1.28}$$

where

$$M_{\sigma, \eta}(z) = z^{\eta + \frac{1}{2}} e^{-\frac{z}{2}} {}_1F_1\left(\frac{1}{2} - \sigma + \eta; \ 2\eta + 1; \ z\right)$$

$$(\Re(1/2 + \eta \pm \sigma) > 0, \ |\arg z| < \pi). \tag{1.29}$$

Oteiza et al. [11] introduced and investigated the generalized Lommel-Wright function $J_{\nu, \lambda}^{\mu, m}(z)$, as a further (4-indices) generalization of the Bessel and Bessel-Maitland (Wright) functions, as follows:

$$J_{\nu, \lambda}^{\mu, m}(z) = (z/2)^{\nu + 2\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{\{\Gamma(\lambda + k + 1)\}^m \Gamma(\nu + k\mu + \lambda + 1)}$$

$$= (z/2)^{\nu + 2\lambda} {}_1\Psi_{m+1}[(1, 1); \underbrace{(\lambda + 1, 1)}_{m\text{-times}}, (\nu + \lambda + 1, \mu); -z^2/4]$$

$$(z \in \mathbb{C}\backslash(-\infty, 0], \ \mu \in \mathbb{R}^+, \ m \in \mathbb{N}, \ \nu, \lambda \in \mathbb{C}), \tag{1.30}$$

where ${}_p\Psi_q$ denotes the Fox-Wright generalized hypergeometric function defined by (see, e.g., [24, p. 21]; see also [6, p. 56 et seq.])

$$_p\Psi_q\left[\begin{matrix}(\alpha_1, A_1), \ldots, (\alpha_p, A_p); \\ (\beta_1, B_1), \ldots, (\beta_q, B_q);\end{matrix} z\right] = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j + A_j k)}{\prod_{j=1}^{q} \Gamma(\beta_j + B_j k)} \frac{z^k}{k!}, \tag{1.31}$$

where the coefficients $A_1, \ldots, A_p \in \mathbb{R}^+$ and $B_1, \ldots, B_q \in \mathbb{R}^+$ such that

$$1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geq 0. \tag{1.32}$$

The special case $m = 1$ of the generalized Lommel-Wright function (1.30) reduces to the generalization of the Bessel function introduced by Pathak [13] (see also [7, p. 353]):

$$J^{\mu}_{\nu,\lambda}(z) = J^{\mu,1}_{\nu,\lambda}(z) = \left(\frac{z}{2}\right)^{\nu+2\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k}}{\Gamma(\lambda + k + 1)\Gamma(\nu + k\mu + \lambda + 1)}$$

$$(z \in \mathbb{C}\backslash(-\infty, 0], \mu \in \mathbb{R}^+, \nu, \lambda \in \mathbb{C}). \tag{1.33}$$

Setting $m = 1$, $\mu = 1$ and $\lambda = \frac{1}{2}$ in (1.30), we obtain the Struve function $H_\nu(\cdot)$ (see, e.g., [10, p. 28, eq. (1.170)])

$$H_\nu(z) = J^{1,1}_{\nu,1/2} = \left(\frac{z}{2}\right)^{\nu+1} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k}}{\Gamma\left(k + \frac{3}{2}\right)\Gamma\left(k + \nu + \frac{3}{2}\right)}$$

$$(z \in \mathbb{C}\backslash(-\infty, 0], \nu \in \mathbb{C}). \tag{1.34}$$

Taking $m = 1$, $\mu = 1$ and $\lambda = 0$ in (1.30), we get the Bessel function (see, e.g., [10, p. 27, eq. (1.161)])

$$J_\nu(z) = J^{1,1}_{\nu,0}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{\nu+2k}}{\Gamma(\nu + k + 1)k!}$$

$$(z \in \mathbb{C}\backslash(-\infty, 0], \Re(\nu) > -1). \tag{1.35}$$

The generalized Lommel-Wright function $J^{\mu,m}_{\nu,\lambda}(z)$ and their special cases $J^{\mu}_\nu(z)$, $J^{\mu}_{\nu,\lambda}(z)$, depending on the arbitrary fractional parameter $\mu \in \mathbb{R}^+$, present fractional order extensions of the Bessel function $J_\nu(z)$ and as such, are closely related to fractional order analogues of the Bessel operators and fractional order equations and systems modeling numerous real world phenomena arising in applied science (see, e.g., [12, 14]).

In this paper, we aim to present certain image formulas of the generalized Lommel-Wright function for the fractional differentiation operators having Appell's function $F_3(\cdot)$ as a kernel. Also, we give some image formulas of the resulting identities for some integral transforms.

2. Image Formulas Associated with Fractional Derivative Operators

Here, we establish image formulas of the generalized Lommel-Wright function for the Saigo-Meada fractional derivative operators (1.4) and (1.5), which are expressed in terms of the Fox-Wright function.

Theorem 2.1. *Let* $x, \mu \in \mathbb{R}^+$, $m \in \mathbb{N}$, *and* $z \in \mathbb{C}\backslash(-\infty, 0]$. *Also, let* $\alpha, \alpha', \beta, \beta', \gamma, \rho, \nu, \lambda \in \mathbb{C}$ *be such that* $\mathfrak{R}(\gamma) > 0$, $\mathfrak{R}(\nu) > -1$, *and*

$$\mathfrak{R}(\rho + \nu) > \max\{0, \mathfrak{R}(\gamma - \alpha - \alpha' - \beta'), \mathfrak{R}(\beta - \alpha)\}. \tag{2.1}$$

Then

$$[D_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} t^{\rho-1} J_{\nu, \lambda}^{\mu, m}(zt)](x)$$

$$= x^{A+\alpha+\alpha'-\gamma-1}\left(\frac{z}{2}\right)^{\nu+2\lambda}$$

$$\times {}_4\Psi_{4+m}\left[\begin{array}{l} (A, 2), (A - \gamma + \alpha + \alpha' + \beta', 2), \\ (A - \beta, 2), (A - \gamma + \alpha + \alpha', 2), (A - \gamma + \alpha + \beta', 2), \end{array}\right.$$

$$\left.\begin{array}{l} (A - B + \alpha, 2), (1, 1); \\ (\nu + \lambda + 1, \mu), \underbrace{(\lambda + 1, 1)}_{m-\text{times}}; \end{array} -\frac{(zx)^2}{4}\right], \tag{2.2}$$

where

$$A := \rho + \nu + 2\lambda. \tag{2.3}$$

Proof. Applying the fractional derivative (1.4) to the function (1.30) and changing the order of integration and summation, which is justified under the

conditions here, we get

$$[I_{0,\,x}^{-\alpha',\,-\alpha,\,-\beta',\,-\beta,\,-\gamma}t^{\rho-1}J_{\nu,\,\lambda}^{\mu,\,m}(zt)](x)$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k\left(\dfrac{z}{2}\right)^{\nu+2\lambda+2k}\Gamma(k+1)}{(\Gamma(\lambda+k+1))^m\Gamma(\nu+k\mu+\lambda+1)k!}$$

$$\times\,(I_{0,\,x}^{-\alpha',\,-\alpha,\,-\beta',\,-\beta,\,-\gamma}t^{\nu+2\lambda+2k+\rho-1})(x). \tag{2.4}$$

Using (1.17) with ρ replaced by $\rho+\nu+2\lambda+2k,$ we find

$$[I_{0,\,x}^{-\alpha',\,-\alpha,\,-\beta',\,-\beta,\,-\gamma}t^{\rho-1}J_{\nu,\,\lambda}^{\mu,\,m}(zt)](x)$$

$$= x^{A+\alpha+\alpha'-\gamma-1}\left(\frac{z}{2}\right)^{\nu+2\lambda}$$

$$\times \sum_{k=0}^{\infty}\frac{(-1)^k\,\Gamma(A+2k)\Gamma(k+1)}{\Gamma(A-\beta+2k)(\Gamma(\lambda+1+k))^m}$$

$$\times\frac{\Gamma(A-\gamma+\alpha+\alpha'+\beta'+2k)\Gamma(A-\beta+\alpha+2k)}{\Gamma(A-\gamma+\alpha+\alpha'+2k)\Gamma(\nu+\lambda+1+\mu k)\Gamma(A-\gamma+\alpha+\beta'+2k)}$$

$$\times\frac{(zx)^{2k}}{4^k\,k!}, \tag{2.5}$$

where A is given in (2.3). Expressing the summation in the right side of (2.5) in view of (1.31), we obtain the result (2.2).

Theorem 2.2. *Let* $x,\,\mu\in\mathbb{R}^+,\ \ m\in\mathbb{N},\ \ and\ \ z\in\mathbb{C}\backslash(-\infty,\,0].$ *Also, let* $\alpha,\,\alpha',\,\beta,\,\beta',\,\gamma,\,\rho,\,\nu,\,\lambda\in\mathbb{C}$ *be such that* $\mathfrak{R}(\gamma)>0,\ \mathfrak{R}(\nu)>-1,$ *and*

$$\mathfrak{R}(\rho-\nu)<1+\min\{0,\,\mathfrak{R}(\beta'),\,\mathfrak{R}(\gamma-\alpha-\alpha'),\,\mathfrak{R}(\gamma-\alpha'-\beta)\}. \tag{2.6}$$

Then

$$[D_{0-}^{\alpha,\,\alpha',\,\beta,\,\beta',\,\gamma}t^{\rho-1}J_{\nu,\,\lambda}^{\mu,\,m}(z/t)](x)$$

$$= x^{\alpha+\alpha'-\gamma-B}\left(\frac{z}{2}\right)^{\nu+2\lambda}$$

$$\times {}_4\Psi_{4+m}\left[\begin{array}{l}(B+\gamma-\alpha-\alpha',\,2),\,(B-\alpha'-\beta+\gamma,\,2),\\ (B,\,2),\,(B-\alpha-\alpha'-\beta+\gamma,\,2),\,(B-\alpha'+\beta',\,2),\end{array}\right.$$

$$\left.\begin{array}{l}(B+\beta',\,2),\,(1,\,1);\\ (\nu+\lambda+1,\,\mu),\,\underbrace{(\lambda+1,\,1)}_{m-\text{times}};\end{array}-\frac{z^2}{4x^2}\right], \tag{2.7}$$

where

$$B := 1 - \rho + \nu + 2\lambda. \tag{2.8}$$

Proof. Here, by applying (1.5), as in the proof of Theorem 2.1, we can get the result (2.7). We omit the details. $\qquad\square$

3. Image Formulas Associated with Integral Transforms

Here, we present Beta transform and P_δ-transform formulas of the fractional derivatives in Theorems 2.1 and 2.2, which are asserted in Theorems 3.1, 3.2, 3.3 and 3.4. We also establish certain integral formulas for Whittaker transforms of the fractional derivatives in Theorems 2.1 and 2.2, which are given in Theorems 3.5 and 3.6. Similarly, as in Section 2, by choosing to use appropriate operators and formulas, we can prove the results in this section. So we omit the involved proofs.

Theorem 3.1. *Let* x, $\mu \in \mathbb{R}^+$ *and* $m \in \mathbb{N}$. *Also, let* a, b, α, α', β, β', γ, ρ, ν, λ, $z \in \mathbb{C}$ *be such that* $\mathfrak{R}(\nu) > -1$, $\min\{\mathfrak{R}(a), \mathfrak{R}(b), \mathfrak{R}(\gamma), \mathfrak{R}(z)\} > 0$, *and*

$$\mathfrak{R}(\rho + \nu) > \max\{0,\, \mathfrak{R}(\gamma - \alpha - \alpha' - \beta'),\, \mathfrak{R}(\beta - \alpha)\}. \tag{3.1}$$

Then

$$B\{[D_{0+}^{\alpha,\,\alpha',\,\beta,\,\beta',\,\gamma}t^{\rho-1}J_{\nu,\,\lambda}^{\mu,\,m}(zt)](x) : a,\,b\}$$

$$= \frac{x^{A+\alpha+\alpha'-\gamma-1}\Gamma(b)}{2^{\nu+2\lambda}}$$

$$\times {}_5\Psi_{5+m}\left[\begin{array}{l}(A,\,2),\,(A-\gamma+\alpha+\alpha'+\beta',\,2),\\[4pt](A-\beta,\,2),\,(A-\gamma+\alpha+\alpha',\,2),\,(A-\gamma+\alpha+\beta',\,2),\end{array}\right.$$

$$\left.\begin{array}{l}(A-\beta+\alpha,\,2),\,(C-b,\,2),\,(1,\,1);\\[4pt](\nu+\lambda+1,\,\mu),\,(C,\,2),\,\underbrace{(\lambda+1,\,1)}_{m-\text{times}};\end{array} -\frac{x^2}{4}\right], \tag{3.2}$$

where A is given in (2.3) *and*

$$C := a + b + \nu + 2\lambda. \tag{3.3}$$

Theorem 3.2. *Let* $x,\,\mu \in \mathbb{R}^+$ *and* $m \in \mathbb{N}$. *Also, let* $a,\,b,\,\alpha,\,\alpha',\,\beta,\,\beta',\,\gamma$, $\rho,\,\nu,\,\lambda,\,z \in \mathbb{C}$ *be such that* $\mathfrak{R}(\nu) > -1$, $\min\{\mathfrak{R}(a),\,\mathfrak{R}(b),\,\mathfrak{R}(\gamma),\,\mathfrak{R}(z)\} > 0,$ *and*

$$\mathfrak{R}(\rho - \nu) < 1 + \min\{0,\,\mathfrak{R}(\beta'),\,\mathfrak{R}(\gamma-\alpha-\alpha'),\,\mathfrak{R}(\gamma-\alpha'-\beta)\}. \tag{3.4}$$

Then

$$B\{(D_{0-}^{\alpha,\,\alpha',\,\beta,\,\beta',\,\gamma}t^{\rho-1}J_{\nu,\,\lambda}^{\mu,\,m}(z/t))(x) : a,\,b\}$$

$$= \frac{x^{\alpha+\alpha'-\gamma-B}\Gamma(b)}{2^{\nu+2\lambda}}$$

$$\times {}_5\Psi_{5+m}\left[\begin{array}{l}(B+\gamma-\alpha-\alpha',\,2),\,(B-\alpha'-\beta+\gamma,\,2),\\[4pt](B,\,2),\,(B-\alpha-\alpha'-\beta+\gamma,\,2),\,(B-\alpha'+\beta',\,2),\end{array}\right.$$

$$\left.\begin{array}{l}(B+\beta',\,2),\,(C-b,\,2),\,(1,\,1);\\[4pt](\nu+\lambda+1,\,\mu),\,(C,\,2),\,\underbrace{(\lambda+1,\,1)}_{m-\text{times}};\end{array} -\frac{1}{4x^2}\right], \tag{3.5}$$

where B and C are given in (2.8) *and* (3.3), *respectively.*

Theorem 3.3. *Let* $x, \mu \in \mathbb{R}^+$, $m \in \mathbb{N}$, $\delta > 1$. *Also, let* $\alpha, \alpha', \beta, \beta', \gamma,$ $\rho, \nu, \lambda \in \mathbb{C}$ *be such that* $\mathfrak{R}(\nu) > -1$, $\min\{\mathfrak{R}(\gamma), \mathfrak{R}(s)\} > 0$, *and*

$$\mathfrak{R}(\rho + \nu) > \max\{0, \mathfrak{R}(\gamma - \alpha - \alpha' - \beta'), \mathfrak{R}(\beta - \alpha)\}. \tag{3.6}$$

Then

$$P_\delta\{z^{l-1}(D_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} t^{\rho-1} J_{\nu, \lambda}^{\mu, m}(zt))(x) : s\}$$

$$= \{\Lambda(\delta, s)\}^{l+\nu+2\lambda} \frac{x^{A+\alpha+\alpha'-\gamma-1}}{2^{\nu+2\lambda}}$$

$$\times {}_5\Psi_{4+m} \left[\begin{array}{l} (A, 2), (A - \gamma + \alpha + \alpha' + \beta', 2), (A - \beta + \alpha, 2), \\ (A - \beta, 2), (A - \gamma + \alpha + \alpha', 2), (A - \gamma + \alpha + \beta', 2), \end{array} \right.$$

$$\left. \begin{array}{l} (l + \nu + 2\lambda, 2), (1, 1); \\ (\nu + \lambda + 1, \mu), \underbrace{(\lambda + 1, 1)}_{m-\text{times}}; \end{array} -\frac{\{\Lambda(\delta, s)x\}^2}{4} \right], \tag{3.7}$$

where A is given in (2.3) *and*

$$\Lambda(\delta, s) := \frac{\delta - 1}{\ln[1 + (\delta - 1)s]}. \tag{3.8}$$

Theorem 3.4. *Let* $x, \delta, \mu \in \mathbb{R}^+$, $m \in \mathbb{N}$. *Also, let* $\alpha, \alpha', \beta, \beta', \gamma, \rho,$ $\nu, \lambda, l \in \mathbb{C}$ *be such that* $\mathfrak{R}(\nu) > -1$, $\min\{\mathfrak{R}(\gamma), \mathfrak{R}(s), \mathfrak{R}(l)\} > 0$, *and*

$$\mathfrak{R}(\rho - \nu) < 1 + \min\{0, \mathfrak{R}(\beta'), \mathfrak{R}(\gamma - \alpha - \alpha'), \mathfrak{R}(\gamma - \alpha' - \beta)\}. \tag{3.9}$$

Then

$$P_\delta\{z^{l-1}[D_{0-}^{\alpha, \alpha', \beta, \beta', \gamma} t^{\rho-1} J_{\nu, \lambda}^{\mu, m}(z/t)](x) : s\}$$

$$= \{\Lambda(\delta, s)\}^{l+\nu+2\lambda} \frac{x^{\rho-\nu-2\lambda+\alpha'+\alpha-\gamma-1}}{2^{\nu+2\lambda}}$$

$$\times\ _5\Psi_{4+m}\left[\begin{array}{l}(B + \gamma - \alpha' - \alpha,\ 2),\ (B - \alpha' - \beta + \gamma,\ 2),\ (B + \beta',\ 2),\\[4pt] (B,\ 2),\ (B - \alpha' - \alpha - \beta + \gamma,\ 2),\ (B - \alpha' + \beta',\ 2),\end{array}\right.$$

$$\left.\begin{array}{l}(l + \nu + 2\lambda,\ 2),\ (1,\ 1);\\[4pt] (\nu + \lambda + 1,\ \mu),\ \underbrace{(\lambda + 1,\ 1)}_{m-\text{times}};\ -\dfrac{\{\Lambda(\delta,\ s)\}^2}{4x^2}\end{array}\right],\qquad(3.10)$$

where B and $\Lambda(\delta,\ s)$ *are given in* (2.8) *and* (3.8), *respectively.*

Theorem 3.5. *Let* $x,\ \mu \in \mathbb{R}^+$, $m \in \mathbb{N}$. *Also, let* $\alpha,\ \alpha',\ \beta,\ \beta',\ \gamma,\ \eta,\ \lambda,\ \nu,$ $\xi,\ \rho,\ \sigma \in \mathbb{C}$ *be such that* $\mathfrak{R}(\nu) > -1$, $\mathfrak{R}(\gamma) > 0$, $\mathfrak{R}(\xi \pm \eta) > -1/2$, *and*

$$\mathfrak{R}(\rho + \nu) > \max\{0,\ \mathfrak{R}(\gamma - \alpha - \alpha' - \beta'),\ \mathfrak{R}(\beta - \alpha)\}.\qquad(3.11)$$

Then

$$\int_0^\infty z^{\sigma-1}e^{-z/2}W_{\sigma,\eta}(z)\{(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}J_{\nu,\lambda}^{\mu,m}(zt))\}\,dz$$

$$=\frac{x^{A+\alpha+\alpha'-\gamma-1}}{2^{\nu+2\lambda}}$$

$$\times\ _6\Psi_{5+m}\left[\begin{array}{l}(A,\ 2),\ (A - \gamma + \alpha + \alpha' + \beta',\ 2),\ (A - \beta + \alpha,\ 2),\\[4pt] (A - \beta,\ 2),\ (A - \gamma + \alpha + \alpha',\ 2),\ (A - \gamma + \alpha + \beta',\ 2),\end{array}\right.$$

$$\left.\begin{array}{l}(E + \eta,\ 2),\ (E - \eta,\ 2),\ (1,\ 1);\\[4pt] (\nu + \lambda + 1,\ \mu),\ (E - \sigma,\ 2),\ \underbrace{(\lambda + 1,\ 1)}_{m-\text{times}};\ -\dfrac{x^2}{4}\end{array}\right],\qquad(3.12)$$

where A is given in (2.3) *and*

$$E := \xi + \nu + 2\lambda + 1/2.\qquad(3.13)$$

Theorem 3.6. *Let* $x,\ \mu \in \mathbb{R}^+$, $m \in \mathbb{N}$. *Also, let* $\alpha,\ \alpha',\ \beta,\ \beta',\ \gamma,\ \eta,\ \lambda,\ \nu,$ $\xi,\ \rho,\ \sigma \in \mathbb{C}$ *be such that* $\mathfrak{R}(\nu) > -1$, $\mathfrak{R}(\nu) > 0$, $\mathfrak{R}(\xi \pm \eta) > -1/2$, *and*

$$\mathfrak{R}(\rho - \nu) < 1 + \min\{0, \mathfrak{R}(\beta'), \mathfrak{R}(\gamma - \alpha - \alpha'), \mathfrak{R}(\gamma - \alpha' - \beta)\}. \quad (3.14)$$

Then

$$\int_0^\infty z^{\sigma-1} e^{-z/2} W_{\sigma,\eta}(z) \{(D_{0-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} J_{\nu,\lambda}^{\mu,m}(zt))\} dz$$

$$= \frac{x^{\rho-\nu-2\lambda+\alpha+\alpha'-\gamma-1}}{2^{\nu+2\lambda}}$$

$$\times {}_6\Psi_{5+m}\left[\begin{matrix} (B + \gamma - \alpha - \alpha', 2), (B - \alpha' - \beta + \gamma, 2), (\beta + \beta', 2), \\ (B, 2), (B - \alpha - \alpha' - \beta + \gamma, 2), (\beta - \alpha' + \beta', 2), \end{matrix} \right.$$

$$\left. \begin{matrix} (E + \eta, 2), (E - \eta, 2), (1, 1); \\ (\nu + \lambda + 1, \mu), (E - \sigma, 2), \underbrace{(\lambda + 1)}_{m-\text{times}}; \end{matrix} -\frac{1}{4x^2} \right], \quad (3.15)$$

where B and E are given in (2.8) *and* (3.13), *respectively.*

4. Concluding Remarks

The results presented here, being very general in both the generalized fractional differential operator and the generalized Lommel-Wright function, can be specialized to yield a number of identities involving relatively simple fractional differential operators and simpler extensions of the Bessel function and the Struve function. We can also consider numerous limiting cases of the results presented here.

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