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A SINGLE SERVER VACATION QUEUE WITH TYPE OF SERVICES AND WITH RESTRICTED ADMISSIBILITY

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Abstract

A single server queue with vacation has been considered. In addition, the admission to queue is based on a Bernoulli process and the server gives two types (type 1, type 2) of service and an optional service. The type 1 service is a two phase service and type 2 service is a single. We analyze single and multiple vacation models and derive the probability generating functions of the number of customers in the queue at different server's state. Then we obtain explicit expressions for various performance measures such as the mean queue size, the probabilities, the mean waiting time and the mean queue sizes when the server is busy and on vacation. We present some numerical results to show the effects of the parameters on some performance measures.

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1. Introduction

In a waiting line, the customers arrive at the facility and wait in the queue if the server is not available. If there are many customers in the queue, then they may suffer long delays which cause poor system performance. Thus, the arrival rate or the service rate may need to be controlled to reduce the delays. One finds quite a few papers in earlier literature on different control models of queueing systems, including control of servers, control of service rates, control of admission of customers and control of queue discipline. In control and design of waiting lines and networks, the restricted admissibility of customers has received great attention (e.g., [4, 10, 14, 16, 18]). However, the papers [4, 14, 16, 18] deal with control policies different from the one considered by [10]. For a few detailed examples of restricted admissibility models, refer to Madan and Choudhury [11]. Our policy is similar to the policy given in [3]. However, they considered bulk arrival and Bernoulli server vacation queue, but ours is a single arrival model but different types of services with vacation. In this restrictive policy, it is assumed that, not all arriving customers are allowed to join the system at all times. Such a policy may be applied in many real-life queueing situations, particularly in the over-crowded queues where arrivals occur faster than departures. One may encounter such situations in telecommunications, transportation, computer networks, traffic highways, dams, airports and there could be many more such situations.

The queueing system when the server becomes idle for a random period of time is not new. Miller [13] was the first to study such a model, where the server is unavailable during some random length of time (referred to as vacation) to the $M/G/1$ queueing system. The $M/G/1$ queueing models of similar nature have also been reported by a number of authors, since Levy and Yechiali [9] included several types of generalizations of the classical $M/G/1$ queueing system. These generalizations are useful in model building in many real-life situations such as digital communication, computer network and production/inventory systems (e.g., Doshi [2, 3] and Takagi [19]) and various vacation policies were defined on queues and analyzed by researchers (e.g., Takagi [19] and Tian and Zhang [20]). Some policies are single vacation policy, multiple vacation policy, Bernoulli vacation policy, compulsory vacation policy, etc. In a queue, an active server goes on servicing the waiting customers till the system becomes empty. As soon as the system becomes empty, the server goes on vacation in order to attend to certain preassigned tasks (such as machine repair, preventive maintenance, scanning for new work, etc), refer to as a generalized vacation. At the end of the vacation, it inspects the main system and then goes on serving the waiting units, if any, otherwise it takes a vacation again. This procedure is repeated till it finds at least one unit waiting in the waiting line for service. This vacation policy is called *multiple vacation policy*. On the other hand, at the end of the first vacation, it inspects the main system and then goes on serving the waiting units, if any, otherwise he waits idly for the new arrival. This vacation policy is called *single vacation policy*.

Madan and Choudhury [12] considered a batch arrival two stage heterogeneous service queue with a Bernoulli schedule server vacation based on a single vacation model under the restricted admissibility policy. Further, they assumed that at the end of each busy period, when a new customer or a batch of customers joins the system, the server enters into a random setup time process before actually starting service of the first customer of the renewed busy period. Madan and Choudhury [11] studied a batch arrival queue $M^X/(G_1, G_2)/1$ with restricted admissibility of arriving batches and modified server vacations under a single vacation policy.

In fact, various aspects of Bernoulli vacation models for single server queueing systems without restricted admissibility have been studied by a good number of authors. Considerable efforts have been devoted to study these types of models by Keilson and Servi [7, 8], Servi [17], Ramaswamy and Servi [15], Doshi [2, 3] and Takagi [19], among several others.

Kalyanaraman and Suvitha [5, 6] studied an $M/G/1$ queue with restricted admissibility, two general types of services and general vacation. In these two articles, Kalyanaraman and Suvitha [5, 6] dealt with compulsory vacation in one article and Bernoulli vacation in the other. In this article, we propose to study such a two-type and an optional service queue with vacations based on a single vacation model and multiple vacation model under the restricted admissibility policy. Such models with restricted admissibility and vacation have a good number of applications. For example, in computer communication systems, messages to be transmitted arrive in a random manner. Further, if the administrator of such a communication system feels that the massages are arriving faster than they can be transmitted, then she/he may adopt our policy of restricted admissibility of the arriving messages. This will help to prevent the system from becoming over-loaded. Moreover, the system may undergo routine maintenance from time to time. This is analogous to vacations considered in our model.

1.1. Model description

Consider a single server queue with the customers arrival follows, according to a Poisson process of intensity λ and the server provides two types of services, respectively, called *type* 1 *service* and *type* 2 *service*. At the beginning of a service, it is assumed that the customer has the choice of selecting type 1 service with probability p_1 and type 2 service with probability p_2 $(p_1 + p_2 = 1)$. The type 1 service is a two phase service and type 2 service is single. After completion of type 1 service, the customer leaves the system, whereas after completion of type 2 service, the customer leaves the system with probability $1 - p_3$ or chooses an optional service with probability p_3. After completion of optional service, the customer leaves the system. The service discipline is assumed to be first come, first served (FCFS). The service time distributions are general, the distribution functions are $B_{1,j}(x)$ for type 1 and *j*th $(j = 1, 2)$ phase of service, $B_2(x)$ for type 2 service, $B_3(x)$ for an optional service. The Laplace-Stieltjes transform (LST) for $B_{1,j}(x)$, $B_2(x)$, $B_3(x)$ are $B^*_{1,j}(\theta)$, $B^*_2(\theta)$, $B^*_3(\theta)$ and finite moments are $E(B^k_{1,j})$, $E(B^k_2)$, $E(B^k_3)$, $k \geq 1$. It may be noted that $B_{1,j}(x)$, $B_2(x)$, $B_3(x)$ and $V(x)$ $(B_{1,j}(\infty) = 1,$ $B_{1,j}(0) = 0,$ $B_2(\infty) = 1,$ $B_2(0) = 0,$ $B_3(\infty) = 1,$ $B_3(0) = 0,$ $V(\infty) = 1,$ $V(0) = 0)$ are continuous.

As soon as the system becomes empty, the server leaves the service area for a random period of time, called *vacation period*. This vacation period V is independently and identically distributed with distribution function $V(y)$, Laplace-Stieltjes transform (LST) $V^*(\theta)$ and finite moments $E(V^k)$, $k \geq 1$. Further, it is assumed that not all the arriving customers are allowed to join the system at all times. Let r $(0 < r < 1)$ be the probability that an arriving customer will be allowed to join the system while the server is busy with a customer, idle and let $p\,(0 < p < 1)$ be the probability that an arriving customer will be allowed to join the system while the server is on vacation.

For the analysis, the supplementary variable (the variable is elapsed service (vacation) time) technique has been used.

Let $N(t)$ be the queue size excluding one customer receiving service at time t, $\mu_{1,\,j}(t)$ be the elapsed type 1 service and jth phase of service time at t, $\mu_2(t)$ be the elapsed type 2 service time at t, $\mu_3(t)$ be the elapsed optional service time at t, $\gamma(t)$ be the elapsed vacation time at t, and $X(t)$ be the state of the server at time t. Then $\{(E(t),\, N(t)) : t \geq 0\}$ is a bivariate Markov process, where $E(t)$ is the elapsed (service or vacation) time at time t. For the mathematical definition of the models, we introduce the following notations:

$P_n^{(1,\,j)}(x,\, t) = Pr\,\{$at time t, there are n customers in the queue excluding one receiving the type 1 service and in the jth phase of service and the elapsed service time is $x\}$, $j = 1, 2$, $n \geq 0$,

$P_n^{(2)}(x,\, t) = Pr\,\{$at time t, there are n customers in the queue excluding one receiving the type 2 service and the elapsed service time is $x\}$, $n \geq 0$,

$P_n^{(3)}(x, t) = Pr\{$at time t, there are n customers in the queue excluding one receiving the optional service and the elapsed service time is $x\}$, $n \geq 0$,

$V_n(x, t) = Pr\{$at time t, the server is on vacation with elapsed vacation time x and the number of customers in the queue is $n\}$, $n \geq 0$ and

$Q(t) = Pr\{$at time t, there are no customers in the system and the server is idle$\}$.

Let $P_n^{(1, j)}(x)$, $P_n^{(2)}(x)$, $P_n^{(3)}(x)$, $V_n(x)$ and Q denote the corresponding steady state probabilities.

The probability generating functions for the probabilities $\{P_n^{(1, j)}(x)\}$, $\{P_n^{(2)}(x)\}$, $\{P_n^{(3)}(x)\}$ and $\{V_n(x)\}$ are, respectively, defined as

$$P^{(1, j)}(x, z) = \sum_{n=0}^{\infty} z^n P_n^{(1, j)}(x), \quad j = 1, 2,$$

$$P^{(2)}(x, z) = \sum_{n=0}^{\infty} z^n P_n^{(2)}(x), \quad P^{(3)}(x, z) = \sum_{n=0}^{\infty} z^n P_n^{(3)}(x)$$

and $V(x, z) = \sum_{n=0}^{\infty} z^n V_n(x).$

Further, it may be noted that $\mu_{1, j}(x)dx$ is the conditional probability of completion of the jth phase of type 1 service during the interval $(x, x + dx]$ given that the elapsed service time is x, $\mu_2(x)dx$ is the conditional probability of completion of the type 2 of service during the interval $(x, x + dx]$ given that the elapsed service time is x, $\mu_3(x)dx$ is the conditional probability of completion of the optional service during the interval $(x, x + dx]$ given that the elapsed service time is x and $\gamma(x)dx$ is the conditional probability of completion of the vacation during the interval $(x, x + dx]$ given that the elapsed vacation time is x so that

$$\mu_{1,j}(x)dx = \frac{b_{1,j}(x)}{1 - B_{1,j}(x)}, \quad \mu_2(x)dx = \frac{b_2(x)}{1 - B_2(x)},$$

$$\mu_3(x)dx = \frac{b_3(x)}{1 - B_3(x)} \quad \text{and} \quad \gamma(x)dx = \frac{v(x)}{1 - V(x)}$$

are the first order differential functions (hazard rates) of $B_{1,j}$, B_i $(i = 2, 3)$ and V, respectively.

We study two models. In the first one, upon termination of a single vacation, the server returns to the main queue and begins to serve those customers, if any, that have arrived during the vacation. If no customers have arrived, then the server waits for the first one to arrive when an ordinary $M/G/1$ busy period is initiated. In the second model, if the server finds the system empty at the end of a vacation, it immediately takes another vacation, and continues in this manner until finds at least one waiting unit upon return from a vacation. The diagram of our model is shown in Figure 1. In Section 2, Model 1 is analyzed and in Section 3, Model 2 is studied and in the last section, a numerical study is given.

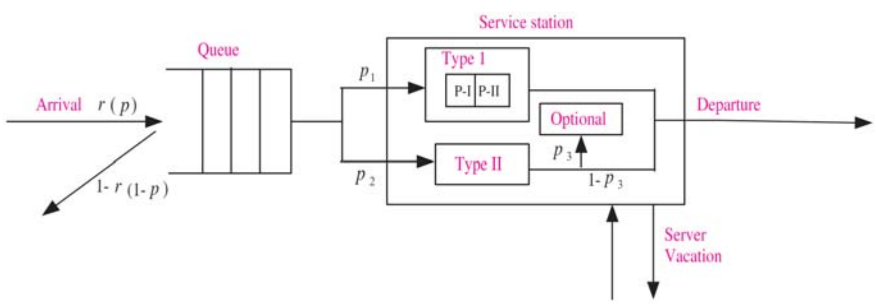


Figure 1. Our model.

2. Model 1: Single Vacation Model

The model analyzed in this section is, as defined earlier, a single server Poisson arrival queue with two types and an optional of generalized services, restricted admissibility and single vacation. For this model, at time *t*, we define the random variable $X(t)$ takes the value 0, when the server is idle, $X(t)$ takes the value 1, when the server is serving phase one of type 1 service, $X(t)$ takes the value 2, when the server is serving phase two of type 1 service, $X(t)$ takes the value 3, when the server is serving type 2 service, $X(t)$ takes the value 4, when the server is serving an optional service, $X(t)$ takes the value 5, when the server is on vacation.

Next, at time *t*, we define the random variable

$$E(t) = \begin{cases} 0; & \text{if } X(t) = 0, \\ \mu_{1,1}(t); & \text{if } X(t) = 1, \\ \mu_{1,2}(t); & \text{if } X(t) = 2, \\ \mu_2(t); & \text{if } X(t) = 3, \\ \mu_3(t); & \text{if } X(t) = 4, \\ \gamma(t); & \text{if } X(t) = 5. \end{cases}$$

We utilize the argument of Cox [1] and obtain the following Kolmogorov forward equations for our model under the steady state conditions:

$$\frac{d}{dx} P_0^{(1,1)}(x) + (\lambda + \mu_{1,1}(x)) P_0^{(1,1)}(x) = \lambda(1-r) P_0^{(1,1)}(x), \tag{1}$$

$$\frac{d}{dx} P_n^{(1,1)}(x) + (\lambda + \mu_{1,1}(x)) P_n^{(1,1)}(x) = \lambda(1-r) P_n^{(1,1)}(x) + \lambda r P_{n-1}^{(1,1)}(x), \quad n \geq 1, \tag{2}$$

$$\frac{d}{dx} P_0^{(1,2)}(x) + (\lambda + \mu_{1,2}(x)) P_0^{(1,2)}(x) = \lambda(1-r) P_0^{(1,2)}(x), \tag{3}$$

$$\frac{d}{dx}P_n^{(1,2)}(x)+(\lambda+\mu_{1,2}(x))P_n^{(1,2)}(x)=\lambda(1-r)P_n^{(1,2)}(x)+\lambda r P_{n-1}^{(1,2)}(x),\quad n\geq 1, \tag{4}$$

$$\frac{d}{dx}P_0^{(2)}(x)+(\lambda+\mu_2(x))P_0^{(2)}(x)=\lambda(1-r)P_0^{(2)}(x), \tag{5}$$

$$\frac{d}{dx}P_n^{(2)}(x)+(\lambda+\mu_2(x))P_n^{(2)}(x)=\lambda(1-r)P_n^{(2)}(x)+\lambda r P_{n-1}^{(2)}(x),\quad n\geq 1, \tag{6}$$

$$\frac{d}{dx}P_0^{(3)}(x)+(\lambda+\mu_3(x))P_0^{(3)}(x)=\lambda(1-r)P_0^{(3)}(x), \tag{7}$$

$$\frac{d}{dx}P_n^{(3)}(x)+(\lambda+\mu_3(x))P_n^{(3)}(x)=\lambda(1-r)P_n^{(3)}(x)+\lambda r P_{n-1}^{(3)}(x),\quad n\geq 1, \tag{8}$$

$$\frac{d}{dx}V_0(x)+(\lambda+\gamma(x))V_0(x)=\lambda(1-p)V_0(x), \tag{9}$$

$$\frac{d}{dx}V_n(x)+(\lambda+\gamma(x))V_n(x)=\lambda(1-p)V_n(x)+\lambda p V_{n-1}(x),\quad n\geq 1, \tag{10}$$

$$\lambda Q=\lambda(1-r)Q+\int_0^\infty V_0(x)\gamma(x)dx. \tag{11}$$

The above set of equations is to be solved under the following boundary conditions at $x=0$:

$$\begin{aligned}P_0^{(1,1)}(0)&=\lambda r p_1 Q+p_1\int_0^\infty P_1^{(1,2)}(x)\mu_{1,2}(x)dx\\&+p_1(1-p_3)\int_0^\infty P_1^{(2)}(x)\mu_2(x)dx\\&+p_1\int_0^\infty P_1^{(3)}(x)\mu_3(x)dx+p_1\int_0^\infty V_1(x)\gamma(x)dx,\end{aligned} \tag{12}$$

$$P_n^{(1,1)}(0) = p_1\int_0^\infty P_{n+1}^{(1,2)}(x)\mu_{1,2}(x)dx$$

$$+ p_1(1-p_3)\int_0^\infty P_{n+1}^{(2)}(x)\mu_2(x)dx + p_1\int_0^\infty P_{n+1}^{(3)}(x)\mu_3(x)dx$$

$$+ p_1\int_0^\infty V_{n+1}(x)\gamma(x)dx,\ n \geq 1, \tag{13}$$

$$P_n^{(1,2)}(0) = \int_0^\infty P_n^{(1,1)}(x)\mu_{1,1}(x)dx,\ n \geq 0, \tag{14}$$

$$P_0^{(2)}(0) = \lambda r p_2 Q + p_2\int_0^\infty P_1^{(1,2)}(x)\mu_{1,2}(x)dx$$

$$+ p_2(1-p_3)\int_0^\infty P_1^{(2)}(x)\mu_2(x)dx$$

$$+ p_2\int_0^\infty P_1^{(3)}(x)\mu_3(x)dx + p_2\int_0^\infty V_1(x)\gamma(x)dx, \tag{15}$$

$$P_n^{(2)}(0) = p_2\int_0^\infty P_{n+1}^{(1,2)}(x)\mu_{1,2}(x)dx$$

$$+ p_2(1-p_3)\int_0^\infty P_{n+1}^{(2)}(x)\mu_2(x)dx + p_2\int_0^\infty P_{n+1}^{(3)}(x)\mu_3(x)dx$$

$$+ p_2\int_0^\infty V_{n+1}(x)\gamma(x)dx,\ n \geq 1, \tag{16}$$

$$P_n^{(3)}(0) = p_3\int_0^\infty P_n^{(2)}(x)\mu_2(x)dx,\ n \geq 0, \tag{17}$$

$$V_0(0) = \int_0^\infty P_0^{(1,2)}(x)\mu_{1,2}(x)dx$$

$$+ (1-p_3)\int_0^\infty P_0^{(2)}(x)\mu_2(x)dx + \int_0^\infty P_0^{(3)}(x)\mu_3(x)dx, \tag{18}$$

$$V_n(0) = 0,\ n \geq 1 \tag{19}$$

and the normalization condition

$$Q + \sum_{n=0}^{\infty}\int_0^{\infty}[P_n^{(1,1)}(x) + P_n^{(1,2)}(x) + P_n^{(2)}(x) + P_n^{(3)}(x) + V_n(x)]dx = 1. \quad (20)$$

Proceeding in the usual manner with equations (1)-(10), we obtain, for $x > 0$,

$$P^{(1,1)}(x, z) = P^{(1,1)}(0, z)(1 - B_{1,1}(x))e^{-Tx}, \quad (21)$$

$$P^{(1,2)}(x, z) = P^{(1,2)}(0, z)(1 - B_{1,2}(x))e^{-Tx}, \quad (22)$$

$$P^{(2)}(x, z) = P^{(2)}(0, z)(1 - B_2(x))e^{-Tx}, \quad (23)$$

$$P^{(3)}(x, z) = P^{(3)}(0, z)(1 - B_3(x))e^{-Tx}, \quad (24)$$

$$V(x, z) = V(0, z)(1 - V(x))e^{-Rx}, \quad (25)$$

where $T = \lambda r(1 - z)$ and $R = \lambda p(1 - z)$.

Next, we multiply equations (12)-(19) by appropriate powers of z and then take the summation over all possible values of n and use (11), (18), (21)-(25), respectively. Thus

$$zP^{(1,1)}(0, z) = \lambda r p_1(z - 1)Q + p_1 B_{1,2}^*(T)P^{(1,2)}(0, z)$$
$$+ p_1(1 - p_3)B_2^*(T)P^{(2)}(0, z) + p_1 B_3^*(T)P^{(3)}(0, z)$$
$$+ p_1 V^*(R)V(0, z) - p_1 V_0(0), \quad (26)$$

$$p^{(1,2)}(0, z) = B_{1,1}^*(T)P^{(1,1)}(0, z), \quad (27)$$

$$[z - p_2(1 - p_3)B_2^*(T)]P^{(2)}(0, z)$$
$$= \lambda r p_2(z - 1)Q + p_2 B_{1,2}^*(T)P^{(1,2)}(0, z) + p_2 B_3^*(T)\, P^{(3)}(0, z)$$
$$+ p_2 V^*(R)V(0, z) - p_2 V_0(0), \quad (28)$$

$$P^{(3)}(0,\ z) = p_3 B_2^*(T) P^{(2)}(0,\ z), \tag{29}$$

$$V(0,\ z) = V_0(0). \tag{30}$$

Using equations (27), (29) and (30) in (26) and (28), we get

$$\begin{aligned}
&[z - p_1 B_{1,1}^*(T) B_{1,2}^*(T)] P^{(1,1)}(0,\ z) \\
&= \lambda r p_1 (z-1) Q + p_1 B_2^*(T)(1 - p_3 + p_3 B_3^*(T)) P^{(2)}(0,\ z) \\
&\quad + p_1 (V^*(R) - 1) V_0(0),
\end{aligned} \tag{31}$$

$$\begin{aligned}
&[z - p_2 B_2^*(T)(1 - p_3 + p_3 B_3^*(T))] P^{(2)}(0,\ z) \\
&= \lambda r p_2 (z-1) Q + p_2 B_{1,1}^*(T) B_{1,2}^*(T) P^{(1,1)}(0,\ z) \\
&\quad + p_2 (V^*(R) - 1) V_0(0).
\end{aligned} \tag{32}$$

From (31) and (32), we get

$$P^{(1,1)}(0,\ z) = \frac{\lambda r p_1 (z-1) Q + p_1 (V^*(R) - 1) V_0(0)}{D(z)}, \tag{33}$$

$$P^{(2)}(0,\ z) = \frac{\lambda r p_2 (z-1) Q + p_2 (V^*(R) - 1) V_0(0)}{D(z)}, \tag{34}$$

where $D(z) = z - p_1 B_{1,1}^*(T) B_{1,2}^*(T) - p_2 B_2^*(T)(1 - p_3 + p_3 B_3^*(T))$.

The unknowns $P_0^{(1,2)}(x)$, $P_0^{(2)}(x)$ and $P_0^{(3)}(x)$ are obtained by using the probability generating functions, we get

$$P_0^{(1,2)}(x) = \frac{p_1 B_{1,1}^*(\lambda r)[\lambda r Q + (1 - V^*(\lambda p)) V_0(0)][1 - B_{1,2}(x)] e^{-\lambda r x}}{p_1 B_{1,1}^*(\lambda r) B_{1,2}^*(\lambda r) + p_2 B_2^*(\lambda r)(1 - p_3 + p_3 B_3^*(\lambda r))}, \tag{35}$$

$$P_0^{(2)}(x) = \frac{p_2 [\lambda r Q + (1 - V^*(\lambda p)) V_0(0)][1 - B_2(x)] e^{-\lambda r x}}{p_1 B_{1,1}^*(\lambda r) B_{1,2}^*(\lambda r) + p_2 B_2^*(\lambda r)(1 - p_3 + p_3 B_3^*(\lambda r))}, \tag{36}$$

$$P_0^{(3)}(x) = \frac{p_2 p_3 B_2^*(\lambda r)[\lambda rQ + (1 - V^*(\lambda p))V_0(0)][1 - B_3(x)]e^{-\lambda rx}}{p_1 B_{1,1}^*(\lambda r) B_{1,2}^*(\lambda r) + p_2 B_2^*(\lambda r)(1 - p_3 + p_3 B_3^*(\lambda r))}. \quad (37)$$

Using equations (35)-(37) in (18), we get

$$V^*(\lambda p)V_0(0) = \lambda rQ. \quad (38)$$

Next, using equation (38) on (30), (33) and (34), we get

$$V(0,\ z) = \frac{\lambda rQ}{V^*(\lambda p)}, \quad (39)$$

$$P^{(1,1)}(0,\ z) = \frac{\lambda rp_1[(z-1)V^*(\lambda p) + V^*(R) - 1]Q}{V^*(\lambda p)D(z)}, \quad (40)$$

$$P^{(2)}(0,\ z) = \frac{\lambda rp_2[(z-1)V^*(\lambda p) + V^*(R) - 1]Q}{V^*(\lambda p)D(z)}. \quad (41)$$

Using equations (40), (41) in (27), (29), we get

$$P^{(1,2)}(0,\ z) = \frac{\lambda rp_1 B_{1,1}^*(T)[(z-1)V^*(\lambda p) + V^*(R) - 1]Q}{V^*(\lambda p)D(z)}, \quad (42)$$

$$P^{(3)}(0,\ z) = \frac{\lambda rp_2 p_3 B_2^*(T)[(z-1)V^*(\lambda p) + V^*(R) - 1]Q}{V^*(\lambda p)D(z)}. \quad (43)$$

Integrating (21)-(25) with respect to x and using (39)-(43), we get

$$P^{(1,1)}(z) = \frac{\lambda rp_1[(z-1)V^*(\lambda p) + V^*(R) - 1](1 - B_{1,1}^*(T))Q}{TV^*(\lambda p)D(z)}, \quad (44)$$

$$P^{(1,2)}(z) = \frac{\lambda rp_1 B_{1,1}^*(T)[(z-1)V^*(\lambda p) + V^*(R) - 1](1 - B_{1,2}^*(T))Q}{TV^*(\lambda p)D(z)}, \quad (45)$$

$$P^{(2)}(z) = \frac{\lambda rp_2[(z-1)V^*(\lambda p) + V^*(R) - 1](1 - B_2^*(T))Q}{TV^*(\lambda p)D(z)}, \quad (46)$$

$$P^{(3)}(z) = \frac{\lambda r p_2 p_3 B_2^*(T)\,[(z-1)V^*(\lambda p) + V^*(R) - 1]\,(1 - B_3^*(T))Q}{TV^*(\lambda p)D(z)}, \tag{47}$$

$$V(z) = \frac{\lambda r(1 - V^*(R))Q}{RV^*(\lambda p)}. \tag{48}$$

The unknown idle probability Q is obtained using the normalizing condition (equation (20)) as

$$Q = \frac{V^*(\lambda p)(1-\rho)}{V^*(\lambda p) + \lambda E(V)[\rho(p-r) + r]}. \tag{49}$$

Equations (44)-(48) together with equation (49) are the probability generating functions of the number of customers in the queue when the server is serving type 1 service and is in the jth $(j = 1, 2)$ phase of service, serving type 2 service and an optional service, respectively, when the server is on vacation.

Here $Q > 0$ guarantees the existence of the probability generating functions in equations (44)-(48) and therefore the stability condition for the system is $\rho < 1,$ where

$$\rho = \lambda r[p_1(E(B_{1,1}) + E(B_{1,2})) + p_2 E(B_2) + p_2 p_3 E(B_3)].$$

The probability generating function that the number of customers in the queue irrespective of the server state is

$$\begin{aligned} U(z) &= Q + P^{(1,1)}(z) + P^{(1,2)}(z) + P^{(2)}(z) + P^{(3)}(z) + V(z) \\ &= \frac{(1-\rho)}{RD(z)[V^*(\lambda p) + \lambda E(V)(\rho(p-r) + r)]} \\ &\quad \times \{R(z-1)V^*(\lambda p) - \lambda(p-r)(V^*(R) - 1) \\ &\quad \times [p_1 B_{1,1}^*(T) B_{1,2}^*(T) + p_2 B_2^*(T)(1 - p_3 + p_3 B_3^*(T))] \\ &\quad + \lambda(p - zr)(V^*(R) - 1)\}. \end{aligned} \tag{50}$$

2.1. The performance measures

Using straightforward calculations, the following performance measures have been obtained.

(i) The mean number of customers in the queue is

$$L_q^s = \lim_{z\to 1}\frac{dU(z)}{dz}$$

$$= \frac{\lambda^2 pE(V^2)[\rho(p-r)+r]}{2[V^*(\lambda p)+\lambda E(V)(\rho(p-r)+r)]}$$

$$+ \frac{\lambda^2 r^2[V^*(\lambda p)+\lambda pE(V)]}{2(1-\rho)[V^*(\lambda p)+\lambda E(V)(\rho(p-r)+r)]}$$

$$\times\{p_1[E(B_{1,1}^2)+2E(B_{1,1})E(B_{1,2})+E(B_{1,2}^2)]$$

$$+ p_2[E(B_2^2)+2p_3E(B_2)E(B_3)+p_3E(B_3^2)]\},$$

where $U(z)$ is given in equation (50).

(ii) The expected waiting time in the queue is

$$W_q^s = \frac{L_q^s}{\lambda'} = \frac{\lambda pE(V^2)[\rho(p-r)+r]}{2r[V^*(\lambda p)+\lambda pE(V)]} + \frac{\lambda r}{2(1-\rho)}$$

$$\times\{p_1[E(B_{1,1}^2)+2E(B_{1,1})E(B_{1,2})+E(B_{1,2}^2)]$$

$$+ p_2[E(B_2^2)+2p_3E(B_2)E(B_3)+p_3E(B_3^2)]\},$$

where

$$\lambda' = \text{actual arrival rate}$$

$$= \lambda r[P^{(1,1)}(1)+P^{(1,2)}(1)+P^{(2)}(1)+P^{(3)}(1)+Q]+\lambda pV(1)$$

$$= \frac{\lambda r[V^*(\lambda p)+\lambda pE(V)]}{V^*(\lambda p)+\lambda E(V)(\rho(p-r)+r)}.$$

(iii) The probability that the server is busy

$$P_b^s = \frac{\rho[V^*(\lambda p) + \lambda p E(V)]}{V^*(\lambda p) + \lambda E(V)(\rho(p-r)+r)}.$$

(iv) The probability that the server is on vacation

$$P_v^s = \frac{\lambda r E(V)(1-\rho)}{V^*(\lambda p) + \lambda E(V)(\rho(p-r)+r)}.$$

(v) The idle probability

$$Q^s = \frac{V^*(\lambda p)(1-\rho)}{V^*(\lambda p) + \lambda E(V)(\rho(p-r)+r)}.$$

(vi) The mean number of the customers in the queue when the server is busy is given by

$$L_{qb}^s = \lim_{z\to 1}\frac{d}{dz}\left(\frac{\lambda r N(z) Q}{TV^*(\lambda p) D(z)}\right)$$

$$= \frac{\lambda^2 p^2 \rho E(V^2)}{2[V^*(\lambda p) + \lambda E(V)(\rho(p-r)+r)]}$$

$$+ \frac{\lambda^2 r^2 [V^*(\lambda p) + \lambda p E(V)]}{2(1-\rho)[V^*(\lambda p) + \lambda E(V)(\rho(p-r)+r)]}$$

$$\times \{p_1[(B_{1,1}^2) + 2E(B_{1,1})E(B_{1,2}) + E(B_{1,2}^2)]$$

$$+ p_2[E(B_2^2) + 2p_3 E(p_2)E(B_3) + p_3 E(B_3^2)]\},$$

where

$$N(z) = [1 - p_1 B_{1,1}^*(T) B_{1,2}^*(T) - p_2 B_2^*(T)(1 - p_3$$

$$+ p_3 B_3^*(T))][V^*(\lambda p)(z-1) + V^*(R) - 1].$$

(vii) The mean number of customers in the queue when the server is on vacation is

$$L_{qv}^{s} = \lim_{z\to 1}\frac{d}{dz}\left(\frac{\lambda r[1-V^{*}(R)]Q}{RV^{*}(\lambda p)}\right) = \frac{\lambda^{2} prE(V^{2})(1-\rho)}{2[V^{*}(\lambda p)+\lambda E(V)(\rho(p-r)+r)]}.$$

3. Model 2: Multiple Vacation Model

The model analyzed in this section is, as defined earlier, a single server Poisson arrival queue with two types and an optional of generalized services, restricted admissibility and multiple vacation. For this model, at time *t*, we define the random variable $X(t)$ takes the value 1, when the server is serving phase one of type 1 service, $X(t)$ takes the value 2, when the server is serving phase two of type 1 service, $X(t)$ takes the value 3, when the server is serving type 2 service, $X(t)$ takes the value 4, when the server is serving an optional service, $X(t)$ takes the value 5, when the server is on vacation. Next, at time *t*, we define the random variable

$$E(t) = \begin{cases} \mu_{1,1}(t); & \text{if } X(t) = 1, \\ \mu_{1,2}(t); & \text{if } X(t) = 2, \\ \mu_{2}(t); & \text{if } X(t) = 3, \\ \mu_{3}(t); & \text{if } X(t) = 4, \\ \gamma(t); & \text{if } X(t) = 5. \end{cases}$$

Equations (1)-(10) is to be solved under the following boundary conditions at $x = 0$:

$$P_{n}^{(1,1)}(0) = p_1\int_0^\infty P_{n+1}^{(1,2)}(x)\mu_{1,2}(x)dx + p_1(1-p_3)$$

$$\times\int_0^\infty P_{n+1}^{(2)}(x)\mu_2(x)dx + p_1\int_0^\infty P_{n+1}^{(3)}(x)\mu_3(x)dx$$

$$+ p_1\int_0^\infty V_{n+1}(x)\gamma(x)dx, \quad n \geq 0, \tag{51}$$

$$P_n^{(1,2)}(0) = \int_0^\infty P_n^{(1,1)}(x)\mu_{1,1}(x)dx, \quad n \geq 0, \tag{52}$$

$$P_n^{(2)}(0) = p_2\int_0^\infty P_{n+1}^{(1,2)}(x)\mu_{1,2}(x)dx + p_2(1-p_3)$$

$$\times \int_0^\infty P_{n+1}^{(2)}(x)\mu_2(x)dx + p_2\int_0^\infty P_{n+1}^{(3)}(x)\mu_3(x)dx$$

$$+ p_2\int_0^\infty V_{n+1}(x)\gamma(x)dx, \quad n \geq 0, \tag{53}$$

$$P_n^{(3)}(0) = p_3\int_0^\infty P_n^{(2)}(x)\mu_2(x)dx, \quad n \geq 0, \tag{54}$$

$$V_0(0) = \int_0^\infty P_0^{(1,2)}(x)\mu_{1,2}(x)dx + (1-p_3)$$

$$\times \int_0^\infty P_0^{(2)}(x)\mu_2(x)dx + \int_0^\infty P_0^{(3)}(x)\mu_3(x)dx$$

$$+ \int_0^\infty V_0(0)\gamma(x)dx, \tag{55}$$

$$V_n(0) = 0, \quad n \geq 1, \tag{56}$$

and the normalization condition

$$\sum_{n=0}^\infty \int_0^\infty [P_n^{(1,1)}(x) + P_n^{(1,2)}(x) + P_n^{(2)}(x) + P_n^{(3)}(x) + V_n(x)]dx = 1.$$

We multiply equations (51)-(56) by appropriate powers of z and then take the summation over all possible values of n and use (21)-(25), respectively. Thus

$$zP^{(1,1)}(0, z) = p_1V^*(R)V(0, z) + p_1B_{1,2}^*(T)P^{(1,2)}(0, z)$$

$$+ p_1(1-p_3)B_2^*(T)P^{(2)}(0, z)$$

$$+ p_1B_3^*(T)P^{(3)}(0, z) - p_1V_0(0), \tag{57}$$

$$P^{(1,2)}(0,\ z) = B_{1,1}^{*}(T)P^{(1,1)}(0,\ z), \tag{58}$$

$$\begin{aligned} zP^{(2)}(0,\ z) = {} & p_2 V^{*}(R)V(0,\ z) + p_2 B_{1,2}^{*}(T)P^{(1,2)}(0,\ z) \\ & + p_2(1 - p_3)B_2^{*}(T)P^{(2)}(0,\ z) \\ & + p_2 B_3^{*}(T)P^{(3)}(0,\ z) - p_2 V_0(0), \end{aligned} \tag{59}$$

$$P^{(3)}(0,\ z) = p_3 B_2^{*}(T)P^{(2)}(0,\ z), \tag{60}$$

$$V(0,\ z) = V_0(0). \tag{61}$$

Using equations (58), (60), (61) in (57), (59),

$$\begin{aligned} & [z - p_1 B_{1,1}^{*}(T)B_{1,2}^{*}(T)]P^{(1,1)}(0,\ z) \\ & = p_1 B_2^{*}(T)(1 - p_3 + p_3 B_3^{*}(T))P^{(2)}(0,\ z) + p_1(V^{*}(R) - 1)V_0(0), \end{aligned} \tag{62}$$

$$\begin{aligned} & [z - p_2 B_2^{*}(T)(1 - p_3 + p_3 B_3^{*}(T))]P^{(2)}(0,\ z) \\ & = p_2 B_{1,1}^{*}(T)B_{1,2}^{*}(T)P^{(1,1)}(0,\ z) + p_2(V^{*}(R) - 1)V_0(0). \end{aligned} \tag{63}$$

From equations (62) and (63),

$$P^{(1,1)}(0,\ z) = \frac{p_1[V^{*}(R) - 1]V_0(0)}{D(z)}, \tag{64}$$

$$P^{(2)}(0,\ z) = \frac{p_2[V^{*}(R) - 1]V_0(0)}{D(z)}. \tag{65}$$

Using equations (64) and (65) in (58), (60),

$$P^{(1,2)}(0,\ z) = \frac{p_1 B_{1,1}^{*}(T)[V^{*}(R) - 1]V_0(0)}{D(z)}, \tag{66}$$

$$P^{(3)}(0,\ z) = \frac{p_2 p_3 B_2^{*}(T)[V^{*}(R) - 1]V_0(0)}{D(z)}. \tag{67}$$

Integrating (21)-(25) with respect to x and using (61), (64)-(67), we get

$$P^{(1,1)}(z) = \frac{p_1(V^*(R)-1)(1-B_{1,1}^*(T))V_0(0)}{TD(z)}, \tag{68}$$

$$P^{(1,2)}(z) = \frac{p_1 B_{1,1}^*(T)(V^*(R)-1)(1-B_{1,2}^*(T))V_0(0)}{TD(z)}, \tag{69}$$

$$P^{(2)}(z) = \frac{p_2(V^*(R)-1)(1-B_2^*(T))V_0(0)}{TD(z)}, \tag{70}$$

$$P^{(3)}(z) = \frac{p_2 p_3 B_2^*(T)(V^*(R)-1)(1-B_3^*(T))V_0(0)}{TD(z)}, \tag{71}$$

$$V(z) = \frac{(1-V^*(R))V_0(0)}{R}. \tag{72}$$

The unknown probability $V_0(0)$ is obtained using the normalizing condition

$$P^{(1,1)}(1) + P^{(1,2)}(1) + P^{(2)}(1) + P^{(3)}(1) + V(1) = 1,$$

as

$$V_0(0) = \frac{r(1-\rho)}{E(V)[r+\rho(p-r)]}. \tag{73}$$

Equations (68)-(72) together with equation (73) are, respectively, the probability generating functions of the number of customers in the queue when the server is serving type 1 service and is in the jth $(j = 1, 2)$ phase of service, serving type 2 service and an optional service, respectively, when the server is on vacation.

Here $V_0(0) > 0$ guarantees the existence of the probability generating functions in equations (68)-(72) and therefore the stability condition for the system is $\rho < 1,$ where

$$\rho = \lambda r[p_1(E(B_{1,1}) + E(B_{1,2})) + p_2E(B_2) + p_2p_3E(B_3)].$$

The probability generating function that the number of customers in the queue irrespective of the server state is

$$U(z) = P^{(1,1)}(z) + P^{(1,2)}(z) + P^{(2)}(z) + P^{(3)}(z) + V(z)$$

$$= \frac{(1 - V^*(R))(1 - \rho)}{RD(z)E(V)[r + \rho(p - r)]}\{zr - p + (p - r)$$

$$\times [p_1B^*_{1,1}(T)B^*_{1,2}(T) + p_2B^*_2(T) \times (1 - p_3 + p_3B^*_3(T))]\}. \quad (74)$$

3.1. The performance measures

Using straightforward calculations, the following performance measures have been obtained.

(i) Mean number of customers in the queue is

$$L_q^m = \lim_{z \to 1} \frac{dU(z)}{dz}$$

$$= \frac{\lambda pE(V^2)}{2E(V)} + \frac{\lambda^2 r^2 p}{2(1 - \rho)(\rho(p - r) + r)}$$

$$\times \{p_1[E(B^2_{1,1}) + 2E(B_{1,1})E(B_{1,2}) + E(B^2_{1,2})]$$

$$+ p_2[E(B_2^2) + 2p_3E(B_2)E(B_3) + p_3E(B_3^2)]\},$$

where $U(z)$ is given in equation (74).

(ii) The expected waiting time in the queue is

$$W_q^m = \frac{L_q^m}{\lambda'} = \frac{E(V^2)[r + \rho(p - r)]}{2rE(V)} + \frac{\lambda r}{2(1 - \rho)}$$

$$\times \{p_1[E(B^2_{1,1}) + 2E(B_{1,1})E(B_{1,2}) + E(B^2_{1,2})]$$

$$+ p_2[E(B_2^2) + 2p_3E(B_2)E(B_3) + p_3E(B_3^2)]\},$$

where

$$\lambda' = \text{ actual arrival rate}$$

$$= \lambda r[P^{(1,1)}(1) + P^{(1,2)}(1) + P^{(2)}(1) + P^{(3)}(1)] + \lambda pV(1)$$

$$= \frac{\lambda pr}{r + \rho(p - r)}.$$

(iii) The probability that the server is busy

$$P_b^m = \frac{\rho p}{r + \rho(p - r)}.$$

(iv) The probability that the server is on vacation

$$P_v^m = \frac{r(1 - \rho)}{r + \rho(p - r)}.$$

(v) The mean number of customers in the queue when the server busy is

$$L_{qb}^m = \lim_{z \to 1} \frac{d}{dz}\left(\frac{N_1(z)V_0(0)}{TD(z)}\right)$$

$$= \frac{\lambda p^2 \rho E(V^2)}{2E(V)[r + \rho(p - r)]} + \frac{\lambda^2 r^2 p}{2(1 - \rho)(\rho(p - r) + r)}\{p_1[E(B_{1,1}^2)$$

$$+ 2E(B_{1,1})E(B_{1,2}) + E(B_{1,2}^2)]$$

$$+ p_2[E(B_2^2) + 2p_3E(B_2)E(B_3) + p_3E(B_3^2)]\},$$

where

$$N_1(z) = [V^*(R) - 1][1 - p_1B_{1,1}^*(T)B_{1,2}^*(T) - p_2B_2^*(T)(1 - p_3 + p_3B_3^*(T))].$$

(vi) The mean number of customers in the queue when the server is on vacation is

$$L_{qv}^m = \lim_{z \to 1} \frac{d}{dz}\left(\frac{[1 - V^*(R)]V_0(0)}{R}\right) = \frac{\lambda prE(V^2)(1 - \rho)}{2E(V)[r + \rho(p - r)]}.$$

Special cases

Cases	Single vacation	Multiple vacation
$p_1 = 0$, $p_2 = p = r = 1$	$\rho = \lambda[E(B_2) + p_3 E(B_3)]$ $P_b^s = \rho$ $P_v^s = \dfrac{\lambda E(V)(1-\rho)}{V^*(\lambda) + \lambda E(V)}$ $Q = \dfrac{V^*(\lambda)(1-\rho)}{V^*(\lambda) + \lambda E(V)}$ $L_q^s = \dfrac{\lambda^2 E(V^2)}{2[V^*(\lambda) + \lambda E(V)]} + \dfrac{\lambda^2 C_1}{2(1-\rho)}$ $L_{qb}^s = \dfrac{\lambda^2 \rho E(V^2)}{2[V^*(\lambda) + \lambda E(V)]} + \dfrac{\lambda^2 C_1}{2(1-\rho)}$ $L_{qv}^s = \dfrac{\lambda^2 E(V^2)(1-\rho)}{2[V^*(\lambda) + \lambda E(V)]}$	$\rho = \lambda[E(B_2) + p_3 E(B_3)]$ $P_b^m = \rho$ $P_v^m = 1 - \rho$ — $L_q^m = \dfrac{\lambda E(V^2)}{2E(V)} + \dfrac{\lambda^2 C_1}{2(1-\rho)}$ $L_{qb}^m = \dfrac{\lambda \rho E(V^2)}{2E(V)} + \dfrac{\lambda^2 C_1}{2(1-\rho)}$ $L_{qv}^m = \dfrac{\lambda E(V^2)(1-\rho)}{2E(V)}$
In the above case $p_3 = 0$	$\rho = \lambda E(B_2)$ $P_b^s = \rho$ $P_v^s = \dfrac{\lambda E(V)(1-\rho)}{V^*(\lambda) + \lambda E(V)}$ $Q = \dfrac{V^*(\lambda)(1-\rho)}{V^*(\lambda) + \lambda E(V)}$ $L_q^s = \dfrac{\lambda^2 E(V^2)}{2[V^*(\lambda) + \lambda E(V)]} + \dfrac{\lambda^2 E(B_2^2)}{2(1-\rho)}$ $L_{qb}^s = \dfrac{\lambda^2 \rho E(V^2)}{2[V^*(\lambda) + \lambda E(V)]} + \dfrac{\lambda^2 E(B_2^2)}{2(1-\rho)}$ $L_{qv}^s = \dfrac{\lambda^2 E(V^2)(1-\rho)}{2[V^*(\lambda) + \lambda E(V)]}$	$\rho = \lambda E(B_2)$ $P_b^m = \rho$ $P_v^m = 1 - \rho$ — $L_q^m = \dfrac{\lambda E(V^2)}{2E(V)} + \dfrac{\lambda^2 E(B_2^2)}{2(1-\rho)}$ $L_{qb}^m = \dfrac{\lambda \rho E(V^2)}{2E(V)} + \dfrac{\lambda^2 E(B_2^2)}{2(1-\rho)}$ $L_{qv}^m = \dfrac{\lambda E(V^2)(1-\rho)}{2E(V)}$

$p_2 = 0$, $p_1 = p = r = 1$	$\rho = \lambda[E(B_{1,1}) + E(B_{1,2})]$ $P_b^s = \rho$ $P_v^s = \dfrac{\lambda E(V)(1-\rho)}{V^*(\lambda) + \lambda E(V)}$ $Q = \dfrac{V^*(\lambda)(1-\rho)}{V^*(\lambda) + \lambda E(V)}$ $L_q^s = \dfrac{\lambda^2 E(V^2)}{2[V^*(\lambda) + \lambda E(V)]} + \dfrac{\lambda^2 C_2}{2(1-\rho)}$ $L_{qb}^s = \dfrac{\lambda^2 \rho E(V^2)}{2[V^*(\lambda) + \lambda E(V)]} + \dfrac{\lambda^2 C_2}{2(1-\rho)}$ $L_{qv}^s = \dfrac{\lambda^2 E(V^2)(1-\rho)}{2[V^*(\lambda) + \lambda E(V)]}$	$\rho = \lambda[E(B_{1,1}) + E(B_{1,2})]$ $P_b^m = \rho$ $P_v^m = 1-\rho$ — $L_q^m = \dfrac{\lambda E(V^2)}{2E(V)} + \dfrac{\lambda^2 C_2}{2(1-\rho)}$ $L_{qb}^m = \dfrac{\lambda \rho E(V^2)}{2E(V)} + \dfrac{\lambda^2 C_2}{2(1-\rho)}$ $L_{qv}^m = \dfrac{\lambda E(V^2)(1-\rho)}{2E(V)}$

where

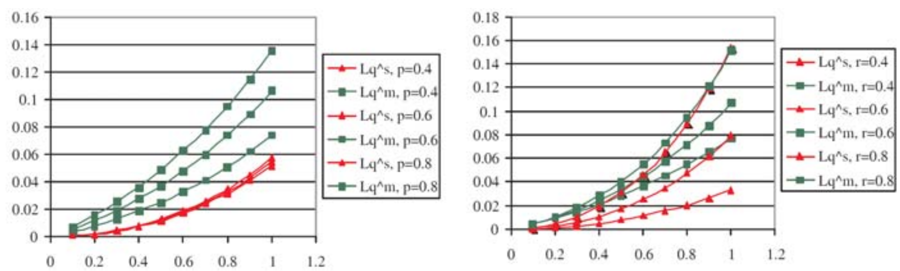
$$C_1 = E(B_2^2) + 2p_3 E(B_2)E(B_3) + E(B_3^2)$$

and

$$C_2 = E(B_{1,1}^2) + 2E(B_{1,1})E(B_{1,2}) + E(B_{1,2}^2).$$

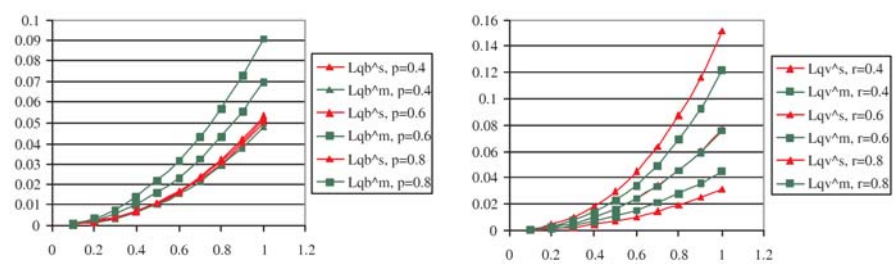
4. The Numerical Study

In this section, we present some numerical examples to show the effect of varying the parameters. We consider the performance measures: the mean number of customers in the queue (L_q), the mean number of customers in the queue when the server is busy (L_{qb}), the mean number of customers in the queue when the server is on vacation (L_{qv}), the expected waiting time in the queue (W_q) and Q, P_b, P_v are the probabilities of the server being idle, the busy and on vacation, respectively. Moreover, for the purpose of numerical illustrations, we assume that the arrival process is Poisson with parameter varying from 0.1 to 1.0, the service times distribution function is negative exponential with parameters $\mu_{1,1} = 4.0$, $\mu_{1,2} = 5.0$, $\mu_2 = 6.0$, $\mu_3 = 3.0$, the vacation time follows an Erlang distribution with parameter $\theta = 7.0$. Also, here the parameters $p_1 = 0.6$, $p_2 = 0.4$, $p_3 = 0.8$, $k = 8$ are fixed. The parametric values are chosen to satisfy the stability condition. Figures 2-5 represent the functions of L_q, L_{qb}, L_{qv}, W_q with respect to the arrival rate λ and the probabilities p, r. All functions are found to be increasing functions. Tables 1-6 present the probability that the server is idle, busy and on vacation with respect to λ, p, r. The table shows that as λ increases, the probability corresponding to busy vacation increases for both single and multiple vacation models, but the idle probability decreases.



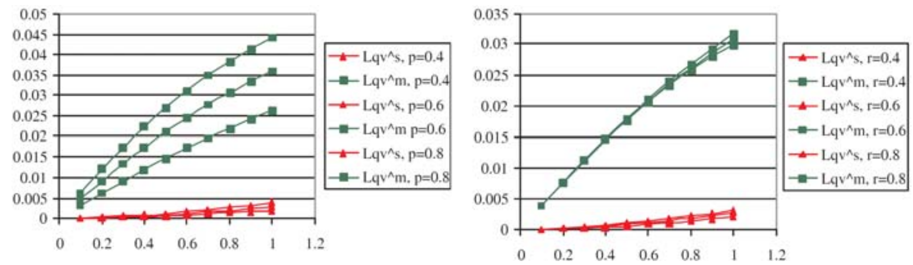
(a) Fixed $r = 0.5$ and varying p (b) Fixed $p = 0.5$ and varying r

Figure 2. Arrival rate versus L_q.



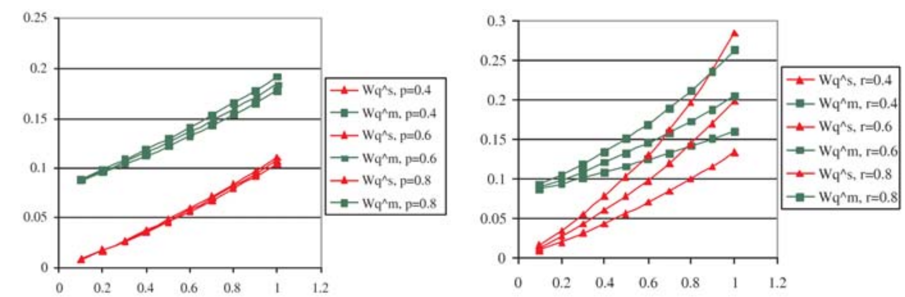
(a) Fixed $r = 0.5$ and varying p (b) Fixed $p = 0.5$ and varying r

Figure 3. Arrival rate versus L_{qb}.



(a) Fixed $r = 0.5$ and varying p (b) Fixed $p = 0.5$ and varying r

Figure 4. Arrival rate versus L_{qv}.



(a) Fixed $r = 0.5$ and varying p (b) Fixed $p = 0.5$ and varying r

Figure 5. Arrival rate versus W_q.

Table 1. The idle probability for single vacation model

λ	$p = 0.2$	$p = 0.4$	$p = 0.6$	$p = 0.8$	$p = 1.0$
0.1	0.9709	0.9708	0.9708	0.9707	0.9706
0.2	0.9422	0.9419	0.9417	0.9415	0.9412
0.3	0.9139	0.9133	0.9128	0.9123	0.9118
0.4	0.8860	0.8850	0.8841	0.8832	0.8823
0.5	0.8584	0.8570	0.8556	0.8542	0.8528
0.6	0.8313	0.8293	0.8274	0.8254	0.8235
0.7	0.8045	0.8019	0.7993	0.7967	0.7942
0.8	0.7782	0.7748	0.7715	0.7683	0.7650
0.9	0.7521	0.7480	0.7440	0.7400	0.7360
1.0	0.7265	0.7216	0.7167	0.7120	0.7072

Table 2. The idle probability for single vacation model

λ	$r = 0.2$	$r = 0.4$	$r = 0.6$	$r = 0.8$	$r = 1.0$
0.1	0.9882	0.9766	0.9650	0.9536	0.9423
0.2	0.9764	0.9532	0.9305	0.9083	0.8864
0.3	0.9644	0.9299	0.8965	0.8639	0.8322
0.4	0.9524	0.9067	0.8628	0.8205	0.7797
0.5	0.9403	0.8836	0.8297	0.7781	0.7286
0.6	0.9281	0.8606	0.7970	0.7366	0.6790
0.7	0.9158	0.8377	0.7647	0.6960	0.6308
0.8	0.9035	0.8149	0.7329	0.6562	0.5838
0.9	0.8911	0.7923	0.7016	0.6173	0.5381
1.0	0.8786	0.7697	0.6707	0.5792	0.4935

Table 3. The probability that the server is busy

$r = 0.5$						
λ	Prob.	$p = 0.2$	$p = 0.4$	$p = 0.6$	$p = 0.8$	$p = 1.0$
0.1	P_b^s	0.0221	0.0221	0.0222	0.0223	0.0223
	P_b^m	0.0090	0.0178	0.0265	0.0350	0.0434
0.2	P_b^s	0.0440	0.0442	0.0445	0.0447	0.0449
	P_b^m	0.0182	0.0358	0.0527	0.0691	0.0849
0.3	P_b^s	0.0657	0.0662	0.0668	0.0673	0.0679
	P_b^m	0.0277	0.0539	0.0788	0.1023	0.1247
0.4	P_b^s	0.0872	0.0882	0.0891	0.0901	0.0910
	P_b^m	0.0375	0.0722	0.1045	0.1347	0.1629
0.5	P_b^s	0.1086	0.1101	0.1116	0.1130	0.1145
	P_b^m	0.0475	0.0907	0.1301	0.1663	0.1995

0.6	P_b^s	0.1299	0.1320	0.1340	0.1361	0.1381
	P_b^m	0.0578	0.1093	0.1555	0.1971	0.2348
0.7	P_b^s	0.1510	0.1538	0.1565	0.1593	0.1620
	P_b^m	0.0684	0.1281	0.1806	0.2271	0.2686
0.8	P_b^s	0.1720	0.1756	0.1791	0.1826	0.1860
	P_b^m	0.0794	0.1471	0.2055	0.2564	0.3012
0.9	P_b^s	0.1929	0.1973	0.2017	0.2060	0.2102
	P_b^m	0.0907	0.1662	0.2302	0.2851	0.3326
1.0	P_b^s	0.2137	0.2190	0.2243	0.2295	0.2346
	P_b^m	0.1023	0.1856	0.2547	0.3130	0.3629

Table 4. The probability that the server is busy

$p = 0.5$						
λ	Prob.	$r = 0.2$	$r = 0.4$	$r = 0.6$	$r = 0.8$	$r = 1.0$
0.1	P_b^s	0.0089	0.0178	0.0266	0.0353	0.0440
	P_b^m	0.0219	0.0221	0.0223	0.0225	0.0227
0.2	P_b^s	0.0179	0.0356	0.0531	0.0704	0.0875
	P_b^m	0.0432	0.0439	0.0447	0.0455	0.0464
0.3	P_b^s	0.0269	0.0534	0.0795	0.1052	0.1305
	P_b^m	0.0639	0.0656	0.0674	0.0693	0.0712
0.4	P_b^s	0.0361	0.0713	0.1058	0.1398	0.1732
	P_b^m	0.0842	0.0871	0.0903	0.0936	0.0973
0.5	P_b^s	0.0453	0.0893	0.1322	0.1742	0.2155
	P_b^m	0.1039	0.1084	0.1133	0.1187	0.1246

0.6	P_b^s	0.0546	0.1073	0.1584	0.2084	0.2576
	P_b^m	0.1232	0.1296	0.1366	0.1445	0.1534
0.7	P_b^s	0.0640	0.1253	0.1846	0.2425	0.2995
	P_b^m	0.1420	0.1505	0.1601	0.1711	0.1837
0.8	P_b^s	0.0734	0.1433	0.2108	0.2766	0.3414
	P_b^m	0.1603	0.1713	0.1839	0.1984	0.2156
0.9	P_b^s	0.0829	0.1614	0.2369	0.3106	0.3833
	P_b^m	0.1782	0.1918	0.2078	0.2266	0.2492
1.0	P_b^s	0.0925	0.1796	0.2631	0.3445	0.4253
	P_b^m	0.1956	0.2123	0.2319	0.2557	0.2848

Table 5. The probability that the server is on vacation

$r = 0.5$						
λ	Prob.	$p = 0.2$	$p = 0.4$	$p = 0.6$	$p = 0.8$	$p = 1.0$
0.1	P_v^s	0.0070	0.0070	0.0070	0.0070	0.0070
	P_v^m	0.9910	0.9822	0.9735	0.9650	0.9566
0.2	P_v^s	0.0138	0.0138	0.0138	0.0138	0.0138
	P_v^m	0.9818	0.9642	0.9473	0.9309	0.9151
0.3	P_v^s	0.0204	0.0204	0.0204	0.0204	0.0204
	P_v^m	0.9723	0.9461	0.9212	0.8977	0.8753
0.4	P_v^s	0.0268	0.0268	0.0267	0.0267	0.0267
	P_v^m	0.9625	0.9278	0.8955	0.8653	0.8371
0.5	P_v^s	0.0329	0.0329	0.0328	0.0328	0.0327
	P_v^m	0.9525	0.9093	0.8699	0.8337	0.8005

0.6	P_v^s	0.0388	0.0387	0.0386	0.0385	0.0384
	P_v^m	0.9422	0.8907	0.8445	0.8029	0.7652
0.7	P_v^s	0.0444	0.0443	0.0441	0.0440	0.0439
	P_v^m	0.9316	0.8719	0.8194	0.7729	0.7314
0.8	P_v^s	0.0498	0.0496	0.0494	0.0492	0.0490
	P_v^m	0.9206	0.8529	0.7945	0.7436	0.6988
0.9	P_v^s	0.0549	0.0546	0.0543	0.0540	0.0538
	P_v^m	0.9093	0.8338	0.7698	0.7149	0.6674
1.0	P_v^s	0.0598	0.0594	0.0590	0.0586	0.0582
	P_v^m	0.8977	0.8144	0.7453	0.6870	0.6371

Table 6. The probability that the server is on vacation

$p = 0.5$						
λ	Prob.	$r = 0.2$	$r = 0.4$	$r = 0.6$	$r = 0.8$	$r = 1.0$
0.1	P_v^s	0.0029	0.0057	0.0084	0.0111	0.0137
	P_v^m	0.9781	0.9779	0.9777	0.9775	0.9773
0.2	P_v^s	0.0057	0.0112	0.0164	0.0214	0.0261
	P_v^m	0.9568	0.9561	0.9553	0.9545	0.9536
0.3	P_v^s	0.0086	0.0166	0.0241	0.0309	0.0372
	P_v^m	0.9361	0.9344	0.9326	0.9307	0.9288
0.4	P_v^s	0.0115	0.0219	0.0313	0.0397	0.0472
	P_v^m	0.9158	0.9129	0.9097	0.9064	0.9027
0.5	P_v^s	0.0144	0.0271	0.0382	0.0477	0.0559
	P_v^m	0.8961	0.8916	0.8867	0.8813	0.8754

0.6	P_v^s	0.0173	0.0321	0.0446	0.0550	0.0634
	P_v^m	0.8786	0.8704	0.8634	0.8555	0.8466
0.7	P_v^s	0.0202	0.0370	0.0507	0.0615	0.0697
	P_v^m	0.8586	0.8495	0.8399	0.8289	0.8163
0.8	P_v^s	0.0231	0.0417	0.0563	0.0672	0.0747
	P_v^m	0.8397	0.8287	0.8161	0.8016	0.7844
0.9	P_v^s	0.0260	0.0463	0.0615	0.0721	0.0786
	P_v^m	0.8218	0.8082	0.7922	0.7734	0.7508
1.0	P_v^s	0.0289	0.0507	0.0662	0.0763	0.0812
	P_v^m	0.8044	0.7877	0.7681	0.7443	0.7152

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