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ON $\mathcal{N}$-NORMAL SUBSETS OF BN-ALGEBRAS

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Abstract

We introduce and investigate the notions of an $\mathcal{N}$-subalgebra and an $\mathcal{N}$-normal subset in BN-algebras.

1. Introduction

The study of BCK-algebras was initiated by Iseki [3] in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. BCK-algebra has been applied to many branches of mathematics, such as group theory, functional analysis and topology. Recently, Kim and Kim [6] introduced the notions of BN-algebras which is a generalization of BCK-algebras (see [2, 7]). Also, Jun et al. [5] and Jun and Kang [4] discussed about $\mathcal{N}$-structures in BCK-algebras. Fuzzy set theory in BN-algebras is discussed by some researchers (see [1, 8]). In this paper, we

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investigate an $\mathcal{N}$-subalgebra and an $\mathcal{N}$-normal subset in BN-algebras, and establish some of their related properties.

2. Preliminaries

Let us review some definitions and properties. The notion of a BN-algebra was introduced by Kim and Kim ([6]). An algebra $(X; *, \theta)$ of type $(2, 0)$ is said to be a *BN-algebra* if it satisfies: for all $x, y, z \in X$,

(K1) $x * x = \theta$,

(K2) $x * \theta = x$,

(K3) $(x * y) * z = (\theta * z) * (y * x)$.

Define a binary relation $\leq$ on a BN-algebra X by letting $x \leq y$ if and only if $x * y = \theta$. It is easy to see that, for any $x \in X$, if $x \leq \theta$, then $x = \theta$.

A non-empty subset S of a BN-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. S is said to be *normal* of X if $(x * a) * (y * b) \in S$, whenever $x * y, a * b \in S$.

Example 2.1 (See [1, 6]). Consider a BN-algebra $X = \{\theta, 1, 2, 3\}$ with the following Cayley table:

$*$	θ	1	2	3
θ	θ	1	2	3
1	1	θ	1	1
2	2	1	θ	1
3	3	1	1	θ

It is easy to check that $\{\theta, 1\}$ is a subalgebra of X, $\{\theta, 2, 3\}$ is not a subalgebra of X and $\{\theta, 3\}$ is a normal subset of X.

Example 2.2 (See [1, 6]). Consider a BN-algebra $X = \{\theta, 1, 2, 3\}$ with the following Cayley table:

$*$	θ	1	2	3
θ	θ	1	2	3
1	1	θ	3	θ
2	2	3	θ	2
3	3	θ	2	θ

It is easy to check that $\{\theta\}$ is a subalgebra and not normal of X.

3. Main Results

Denote by $\mathcal{F}(X, [-1, 0])$ the collection of functions from a non-empty set X to $[-1, 0]$. We say that an element of $\mathcal{F}(X, [-1, 0])$ is a *negative-valued function* from X to $[-1, 0]$ (briefly, $\mathcal{N}$*-function* on X). By an $\mathcal{N}$-*structure*, we mean an ordered pair (X, φ) of X and an $\mathcal{N}$-function φ on X. In what follows, let X denote a BN-algebra and φ be an $\mathcal{N}$-function on X unless otherwise specified.

Definition 3.1. By normal of X based on $\mathcal{N}$-function φ (briefly, $\mathcal{N}$-subalgebra of X), we mean an $\mathcal{N}$-structure (X, φ) in which φ satisfies the following assertion:

$$(\forall x, y \in X)(\varphi(x * y) \leq \max\{\varphi(x), \varphi(y)\}). \tag{1}$$

Example 3.2. Consider the BN-algebra $(X, *, \theta)$ given in Example 2.1. Define an $\mathcal{N}$-function φ by

$$\varphi(\theta) = -0.7, \quad \varphi(1) = -0.5, \quad \varphi(2) = -0.3, \quad \varphi(3) = -0.2.$$

It is easily verified that (X, φ) is an $\mathcal{N}$-subalgebra of X.

Lemma 3.3. *Every* $\mathcal{N}$*-subalgebra* (X, φ) *of X satisfies the following inequality*:

$$(\forall x,\ y \in X)(\varphi(\theta) \leq \varphi(x)). \tag{2}$$

Proof. *Let* $x \in X$. *Then we have* $\varphi(\theta) = \varphi(x * x) \leq \max\{\varphi(x),\ \varphi(x)\} = \varphi(x)$. □

Theorem 3.4. *If every* $\mathcal{N}$*-subalgebra* $(X,\ \varphi)$ *of X satisfies the following inequality*:

$$(\forall x,\ y \in X)(\varphi(x * y) \leq \varphi(y)), \tag{3}$$

then φ *is a constant function*.

Proof. Let $x \in X$. Then we have $\varphi(x) = \varphi(x * \theta) \leq \varphi(\theta)$. It follows from Lemma 3.3 that $\varphi(x) = \varphi(\theta)$, and so φ is a constant function. □

For any $\mathcal{N}$-function φ on *X* and $t \in [-1,\ 0)$, the set

$$\mathcal{C}(\varphi;\ t) := \{x \in X \mid \varphi(x) \leq t\}$$

is called a *closed* $(\varphi,\ t)$*-cut* of φ, and the set

$$\mathcal{O}(\varphi;\ t) := \{x \in X \mid \varphi(x) < t\}$$

is called an *open* $(\varphi,\ t)$*-cut* of φ.

We provide a characterization of an $\mathcal{N}$-subalgebra.

Theorem 3.5. *Let* $(X,\ \varphi)$ *be an* $\mathcal{N}$*-structure of X and* φ. *Then* $(X,\ \varphi)$ *is an* $\mathcal{N}$*-subalgebra of X if and only if every non-empty closed* $(\varphi,\ t)$*-cut of* φ *is a subalgebra of X for all* $t \in [-1,\ 0)$.

Proof. The proof is straightforward. □

Definition 3.6. By normal of *X* based on $\mathcal{N}$-function φ (briefly, $\mathcal{N}$-normal of *X*), we mean an $\mathcal{N}$-structure $(X,\ \varphi)$ in which φ satisfies the following assertion:

$$(\forall x,\ y,\ a,\ b \in X)(\varphi((x * a) * (y * b)) \leq \max\{\varphi(x * y),\ \varphi(a * b)\}). \tag{4}$$

Example 3.7. Consider the $\mathcal{N}$ -structure (X, φ) which is described in Example 3.2. Then $\{\theta, 3\}$ is $\mathcal{N}$ -normal of *X*.

Theorem 3.8. *Every $\mathcal{N}$ -normal of X is an $\mathcal{N}$ -subalgebra of X.*

Proof. Let (X, φ) be an $\mathcal{N}$ -normal subset of *X* and let $x, y \in X$. Then we have

$$\begin{aligned}\varphi(x * y) &= \varphi((x * y) * (\theta * \theta)) \leq \max\{\varphi(x * \theta), \varphi(y * \theta)\} \\ &= \max\{\varphi(x), \varphi(y)\}.\end{aligned}$$

Therefore, (X, φ) is an $\mathcal{N}$ -subalgebra of *X*. □

The converse of Theorem 3.8 is not true in general, as seen from the following.

Example 3.9. The $\mathcal{N}$ -structure (X, φ) given in Example 3.2 shows that the converse of Theorem 3.8 does not hold since

$$\varphi((2 * 2) * (\theta * 3)) = \varphi(\theta * 3) = \varphi(3) \not\leq \varphi(1) = \max\{\varphi(2 * \theta), \varphi(2 * 3)\}.$$

By Lemma 3.3 and Theorem 3.8, we have the following result.

Corollary 3.10. *Every $\mathcal{N}$ -normal (X, φ) of X satisfies the following inequality*:

$$(\forall x, y \in X)(\varphi(\theta) \leq \varphi(x)).$$

By Lemma 3.3 and Theorem 3.8, we the have following result.

Corollary 3.11. *If every $\mathcal{N}$ -normal (X, φ) of X satisfies the following inequality*:

$$(\forall x, y \in X)(\varphi(x * y) \leq \varphi(y)),$$

then φ *is a constant function*.

For any element *w* of *X*, *we consider the set*

$$X_w := \{x \in X \mid \varphi(x) \leq \varphi(w)\}.$$

Obviously, X_w is a non-empty subset of *X* because $w \in X_w$.

Theorem 3.12. *Let w be an element of X. If* (X, φ) *is an* $\mathcal{N}$ *-normal subset of X, then the set* X_w *is a normal subset of X.*

Proof. Obviously, $\theta \in X_w$ by Corollary 3.10. Let $x, y, a, b \in X$ be such that $x * y \in X_w$ and $a * b \in X_w$. Then we have $\varphi(x * y) \leq \varphi(w)$ and $\varphi(a * b) \leq \varphi(w)$. Since (X, φ) is an $\mathcal{N}$ -normal subset of *X*, it follows from (4) that $\varphi((x * a) * (y * b)) \leq \max\{\varphi(x * y), \varphi(a * b)\} \leq \varphi(w)$ so that $(x * a) * (y * b) \in X_w$. Hence, X_w is a normal subset of *X*. □

Theorem 3.13. *For any normal subset U of X, there exists an* $\mathcal{N}$ *-function* φ *such that* (X, φ) *is an* $\mathcal{N}$ *-normal subset of X and* $\mathcal{C}(\varphi; t) = U$ *for some* $t \in [-1, 0)$.

Proof. Let *U* be a normal subset of *X* and let φ be an $\mathcal{N}$ -function on *X* defined by

$$\varphi(x) = \begin{cases} 0 & \text{if } x \notin U, \\ t & \text{if } x \in U, \end{cases}$$

where *t* is fixed in $t \in [-1, 0)$. Then (X, φ) is an $\mathcal{N}$ -structure of *X* and $\mathcal{C}(\varphi; t) = U$. □

We provide a characterization of an $\mathcal{N}$ -normal subset.

Theorem 3.14. *Let* (X, φ) *be an* $\mathcal{N}$ *-structure of X and* φ. *Then* (X, φ) *is an* $\mathcal{N}$ *-normal subset of X if and only if every non-empty closed* (φ, t)*-cut of* φ *is a normal subset of X for all* $t \in [-1, 0)$.

Proof. Assume that (X, φ) is an $\mathcal{N}$ -normal subset of *X* and let $t \in [-1, 0)$ be such that $\mathcal{C}(\varphi; t) \neq \emptyset$. Let $x * y, a * b \in \mathcal{C}(\varphi; t)$. Then we get $\varphi(x * y) \leq t$ and $\varphi(a * b) \leq t$. It follows from (4) that $\varphi((x * a) * (y * b)) \leq \max\{\varphi(x * y), \varphi(a * b)\} \leq t$ so that $(x * a) * (y * b) \in \mathcal{C}(\varphi; t)$. Hence, $\mathcal{C}(\varphi; t)$ is a normal subset of *X*.

Conversely, suppose that every non-empty closed (φ, t)-cut of φ is a normal subset of X for all $t \in [-1, 0)$. If (X, φ) is not an $\mathcal{N}$-normal subset of X, then $\varphi((x * a) * (y * b)) > t_0 \geq \max\{\varphi(x * y), \varphi(a * b)\}$ for some $x, y, a, b \in X$ and $t_0 \in [-1, 0)$. Hence, $x * y, a * b \in \mathcal{C}(\varphi; t_0)$ and $(x * a) * (y * b) \notin \mathcal{C}(\varphi; t_0)$. This is a contradiction. Thus, (X, φ) is an $\mathcal{N}$-normal subset of X. □

Corollary 3.15. *If (X, φ) is an $\mathcal{N}$-normal subset of X, then every non-empty open (φ, t)-cut of φ is a normal subset of X for all $t \in [-1, 0)$.*

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