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MULTIVARIATE SPATIAL ERROR THREE-STAGE LEAST SQUARES FIXED EFFECT PANEL: SIMULTANEOUS MODELS AND ESTIMATION OF THEIR PARAMETERS

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Abstract

Simultaneous equation models describe a two-way flow of influence among variables. Simultaneous equation models using panel data, especially for fixed effect where there are spatial errors with exact solutions, still require to be developed. This paper proposes feasible generalized least squares - three-stage least squares (FGLS-3SLS) to find all of the estimators with exact solution. The proposed estimators are proved to be consistent.

1. Introduction

There are two methods to find parameter estimators in simultaneous equation models, namely, single-equation methods and system methods. The former the methods which are applied to one equation of the system at a time and the latter the methods which are applied to all equations of the system simultaneously as described in [1]. The latter are the methods which are much more efficient than the former because they use much more information [1].

Solution techniques of system methods consist of two methods, namely, three-stage least squares (3SLS) and full information maximum likelihood (FIML). The estimators of the former are more robust than that of the latter [2]. Consequently, solution technique by means of 3SLS is much more advantageous than the one by FIML because it is both time saving and cost saving.

But sometimes, we have an obstacle to obtain the parameter estimators of simultaneous equation models because of the limitation of observations. However, these problems can be overcome by means of panel data. One advantage of panel data is their ability to increase the degree of freedom or the sample size [3-5].

If the model contains spatial influence and the spatial influence comes only through the error terms, we can use spatial error model. We refer to [14] for the use of first-order queen contiguity to find row-standardized spatial weight matrix and refer to [15-17] for examining spatial influence by means of Moran index.

There are some papers about estimation of parameters in simultaneous panel spatial regression models. However, they are limited only to random effects [6, 7, 11]. They used methods of spatial error component three-stage least squares (SEC-3SLS) [6] and five-step new estimation strategy [7, 11].

Based on above statements, we are motivated to develop simultaneous equation models for fixed effect panel data with one-way error component by means of 3SLS solutions. In this model, there are spatial influences and they come only through the error terms. At this time, we use fixed effect models with one-way error component. There are no lags for both endogenous explanatory variables and exogenous explanatory variables for this model. The objective of this paper is to obtain the closed-form and numerical approximation estimators of parameter models and to prove their consistency, especially for closed-form estimators.

2. Models Development

We refer to [5] with m simultaneous equations model in m endogenous variables, namely

$$\mathbf{y}_h = \mathbf{1}\mu_h + \mathbf{X}_h\boldsymbol{\alpha}_h + \mathbf{Y}_{-h}\boldsymbol{\beta}_{-h} + \mathbf{u}_h, \tag{2.1}$$

for $h = 1, 2, 3, \ldots, m,$ where $\mathbf{y}_h$ denotes the hth endogenous variable vector, $\mathbf{X}_h$ denotes the hth matrix of observations including (for example k_h) exogenous explanatory variables, $\mathbf{Y}_{-h}$ denotes the $-h$th matrix of observations including endogenous explanatory variables except the hth endogenous explanatory variables, μ_h denotes the hth mean parameter, $\boldsymbol{\alpha}_h$ denotes the hth parameter vector of exogenous explanatory variables, $\boldsymbol{\beta}_{-h}$ denotes the $-h$th parameter vector of endogenous explanatory variables, $\mathbf{u}_h$ denotes the hth random error vector assuming mean vector $\mathbf{0}$ and covariance matrix $\sigma_h^2\mathbf{I}_n$ (homoscedasticity) in which σ_h^2 denotes the unknown hth error variance and $\mathbf{I}_n$ denotes the $n \times n$ identity matrix, and $\mathbf{1}$ denotes the unit vector.

We suppose that (2.1) is over identified. Solution of (2.1) by 3SLS is premultiplying (2.1) by the matrix of observations on all the exogenous explanatory variables in the system. This shows that among all the exogenous explanatory variables and all the random errors are uncorrelated in the system, and we obtain

$$\mathbf{X}_{**}^{t}\mathbf{y} = \mathbf{X}_{**}^{t}\mathbf{G}\boldsymbol{\mu} + \mathbf{X}_{**}^{t}\mathbf{X}\boldsymbol{\alpha} + \mathbf{X}_{**}^{t}\mathbf{Y}_{-}\boldsymbol{\beta}_{-} + \mathbf{X}_{**}^{t}\mathbf{u}, \tag{2.2}$$

where $\mathbf{X}_{*}$ denotes $n \times \sum_{h=1}^{m} k_h$ matrix including all the exogenous explanatory variables in the system, $\mathbf{X}_{**}$ denotes $mn \times m\sum_{h=1}^{m} k_h$ diagonal matrix whose submain diagonal is $\mathbf{X}_{*}$, $\mathbf{y}$ denotes $mn \times 1$ vector including all of $n \times 1$ vectors, $\mathbf{y}_h$, $\mathbf{G}$ denotes $mn \times m$ diagonal matrix whose submain diagonal is $\mathbf{1}$, $\boldsymbol{\mu}$ denotes $m \times 1$ vector including all of μ_h, $\mathbf{X}$ denotes $mn \times \sum_{h=1}^{m} k_h$ diagonal matrix whose submain diagonal is $n \times k_h$ matrix, $\mathbf{X}_h$, $\boldsymbol{\alpha}$ denotes $\sum_{h=1}^{m} k_h \times 1$ vector including all of $k_h \times 1$ vectors, $\boldsymbol{\alpha}_h$, $\boldsymbol{\beta}_{-}$ denotes $m(m-1) \times 1$ vector including all of $(m-1) \times 1$ vectors, $\boldsymbol{\beta}_{-h}$, $\mathbf{Y}_{-}$ denotes $mn \times m(m-1)$ diagonal matrix whose submain diagonal is $n \times (m-1)$ matrix, $\mathbf{Y}_{-h}$, $\mathbf{u}$ denotes $mn \times 1$ vector including all of $n \times 1$ vectors, $\mathbf{u}_h$, and n denotes the sample size of observations.

The next model is fixed effect panel data regression model with one way error component [3, 4], namely

$$\mathbf{y}_j = \mathbf{1}\mu + \mathbf{X}_j\boldsymbol{\alpha} + \mathbf{1}\gamma_j + \mathbf{u}_j, \tag{2.3}$$

for $j = 1, 2, 3, ..., T$, where $\mathbf{y}_j$ denotes the endogenous variables vector of jth time period, $\mathbf{X}_j$ denotes the matrix of jth time period including (for example, k) exogenous explanatory variables, μ denotes the mean parameter,

$\boldsymbol{\alpha}$ denotes the parameters vector of exogenous explanatory variables, γ_j denotes the *j*th time period time specific effect parameter, $\mathbf{u}_j$ denotes the *j*th time period random error vector assuming mean vector $\mathbf{0}$ and covariance matrix $\sigma^2\mathbf{I}_n$, σ^2 denotes the unknown error variance. There is one restriction of (2.3), namely $\sum_{j=1}^{T}\gamma_j = 0$.

We refer to [8] for the properties of Kronecker products, [9] for reparametrization, [1, 10, 2] for 3SLS estimation, and [12] for GLS and FGLS. By (2.1), using panel data and then adopting models in (2.3), we obtain new form of equations:

$$\mathbf{y}_{hj} = \mathbf{1}\mu_h + \mathbf{X}_{hj}\boldsymbol{\alpha}_h + \mathbf{Y}_{-hj}\boldsymbol{\beta}_{-h} + \mathbf{1}\gamma_{hj} + \mathbf{u}_{hj} \tag{2.4}$$

for $h = 1, 2, 3, ..., m,$ $j = 1, 2, 3, ..., T,$ where $\mathbf{y}_{hj}$ denotes the *j*th time period *h*th endogenous variables vector, $\mathbf{X}_{hj}$ denotes the *j*th time period *h*th matrix including (for example, k_h) exogenous explanatory variables, $\mathbf{Y}_{-hj}$ denotes the *j*th time period –*h*th matrix including endogenous explanatory variables except the *j*th time period *h*th endogenous explanatory variables, γ_{hj} denotes the *j*th time period *h*th time specific effect parameter, $\mathbf{u}_{hj}$ denotes the *j*th time period *h*th random error vector assuming mean vector $\mathbf{0}$ and covariance matrix $\sigma_h^2\mathbf{I}_n$. There is one restriction, namely $\sum_{j=1}^{T}\gamma_j = 0$.

We refer to [13] for spatial error model, namely:

$$\mathbf{y} = \mathbf{1}\mu + \mathbf{X}\boldsymbol{\alpha} + \mathbf{u},$$

$$\mathbf{u} = \lambda\mathbf{W}\mathbf{u} + \varepsilon, \tag{2.5}$$

where $\mathbf{y}$ denotes the endogenous variables vector, $\mathbf{X}$ denotes the matrix of observations including (for example, *k*) exogenous explanatory variables, $\mathbf{u}$ denotes the spatial autocorrelation of random error vector, λ denotes the spatial autocorrelation parameter, $\mathbf{W}$ denotes the row-standardized spatial weight matrix, and ε denotes the random error vector assuming normal distribution with mean vector $\mathbf{0}$ and covariance matrix $\sigma^2\mathbf{I}_n$.

If (2.4) contains spatial influence and the spatial influence comes only through the error terms, then we can adopt model in (2.5) and we obtain new form of equations:

$$\mathbf{y}_{hj} = \mathbf{1}\mu_h + \mathbf{X}_{hj}\boldsymbol{\alpha}_h + \mathbf{Y}_{-hj}\boldsymbol{\beta}_{-h} + \mathbf{1}\gamma_{hj} + \mathbf{u}_{hj},$$

$$\mathbf{u}_{hj} = \lambda_h\mathbf{W}\mathbf{u}_{hj} + \boldsymbol{\varepsilon}_{hj}, \tag{2.6}$$

for $h = 1, 2, 3, \ldots, m,\ \ j = 1, 2, 3, \ldots, T,$ where $\mathbf{u}_{hj}$ denotes the jth time period hth spatial autocorrelation of random error vector, λ_h denotes the hth spatial autocorrelation parameter, and $\boldsymbol{\varepsilon}_{hj}$ denotes the jth time period hth random error vector assuming normal distribution with mean vector $\mathbf{0}$ and covariance matrix $\sigma_h^2\mathbf{I}_n$. There is one restriction, namely $\sum_{j=1}^{T}\gamma_j = 0.$

We refer to [14] for the use of first-order queen contiguity to find the row-standardized spatial weight matrix and to [15-17] for examining spatial influence by means of Moran index. Equation (2.6) can be rewritten as:

$$\mathbf{y}_{hj} = \mathbf{1}\mu_h + \mathbf{X}_{hj}\boldsymbol{\alpha}_h + \mathbf{Y}_{-hj}\boldsymbol{\beta}_{-h} + \mathbf{1}\gamma_{hj} + \mathbf{A}_h^{-1}\boldsymbol{\varepsilon}_{hj}. \tag{2.7}$$

Since $\mathbf{A}_h\mathbf{u}_{hj} = \boldsymbol{\varepsilon}_{hj}$, $\mathbf{u}_{hj} = \mathbf{A}_h^{-1}\boldsymbol{\varepsilon}_{hj}$, where $\mathbf{A}_h = \mathbf{I}_n - \lambda_h\mathbf{W}$.

For the solution of (2.7) by 3SLS, we obtain the following equation:

$$\mathbf{X}_{*j}^t\mathbf{y}_{hj} = \mathbf{X}_{*j}^t\mathbf{1}\mu_h + \mathbf{X}_{*j}^t\mathbf{X}_{hj}\boldsymbol{\alpha}_h + \mathbf{X}_{*j}^t\mathbf{Y}_{-hj}\boldsymbol{\beta}_{-h} + \mathbf{X}_{*j}^t\mathbf{1}\gamma_{hj} + \mathbf{X}_{*j}^t\mathbf{A}_h^{-1}\boldsymbol{\varepsilon}_{hj}, \tag{2.8}$$

but the restriction $\sum_{j=1}^{T}\gamma_j = 0$ will not be achieved. This is due to $\mathbf{X}_{*j}$ having in general, different values of the matrix of observations in every jth time period. This paper overcomes the restrictive problem by means of average value approach of the matrix of observations. We use this approach because the estimator of the mean is unbiased, consistent, and efficient as revealed by [1, 2, 5, 10].

As a consequence of this approach, we can write (2.8) as

$$\overline{\mathbf{X}}_{*}^{t}\mathbf{y}_{hj} = \overline{\mathbf{X}}_{*}^{t}\mathbf{1}\mu_{h} + \overline{\mathbf{X}}_{*}^{t}\mathbf{X}_{hj}\boldsymbol{\alpha}_{h} + \overline{\mathbf{X}}_{*}^{t}\mathbf{Y}_{-hj}\boldsymbol{\beta}_{-h} + \overline{\mathbf{X}}_{*}^{t}\mathbf{1}\gamma_{hj} + \overline{\mathbf{X}}_{*}^{t}\mathbf{A}_{h}^{-1}\boldsymbol{\varepsilon}_{hj}, \quad (2.9)$$

which can be rewritten to obtain new forms of vectors and matrices as follows:

$$\overline{\mathbf{X}}_{**}^{t}\mathbf{y}_{j} = \overline{\mathbf{X}}_{**}^{t}\mathbf{G}\boldsymbol{\mu} + \overline{\mathbf{X}}_{**}^{t}\mathbf{Z}_{j}\boldsymbol{\theta} + \overline{\mathbf{X}}_{**}^{t}\mathbf{G}\boldsymbol{\gamma}_{j} + \overline{\mathbf{X}}_{**}^{t}\mathbf{A}_{*}\boldsymbol{\varepsilon}_{j}, \quad (2.10)$$

where $\mathbf{Z}_{j} = [\mathbf{X}_{j} \vdots \mathbf{Y}_{-j}]$ and $\boldsymbol{\theta}^{t} = [\boldsymbol{\alpha} \vdots \boldsymbol{\beta}_{-}^{t}]$ having dimensions $mn \times \left(\sum_{h=1}^{m} k_{h} + m(m-1)\right)$ and $\left(\sum_{h=1}^{m} k_{h} + m(m-1)\right) \times 1$, respectively. $\mathbf{A}_{*}$ denotes $mn \times mn$ diagonal matrix whose submain diagonal is $\mathbf{A}_{h}^{-1}$, $\boldsymbol{\varepsilon}_{j}$ denotes $mn \times 1$ vector including all of $n \times 1$ vectors, $\boldsymbol{\varepsilon}_{hj}$. All others of the vectors and matrices are denoted like the ones in equation (2.2), we just add index j except for matrix of $\overline{\mathbf{X}}_{**}$. We change $\mathbf{X}_{*j}$ for $\overline{\mathbf{X}}_{*}$ with $\overline{\mathbf{X}}_{*} = \frac{1}{T}\sum_{j=1}^{T}\mathbf{X}_{*j}$ and $\mathbf{X}_{**}$ denotes $mn \times m\sum_{h=1}^{m} k_{h}$ diagonal matrix whose submain diagonal is $\overline{\mathbf{X}}_{*}$. For $j = 1, 2, 3, ..., T$, the restriction $\sum_{j=1}^{T}\gamma_{hj} = 0$ is changed to $\sum_{j=1}^{T}\boldsymbol{\gamma}_{j} = \mathbf{0}$.

3. Estimating the Parameters

Now, we estimate all of the parameter models from equation (2.10) and their estimators $\hat{\boldsymbol{\mu}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\gamma}}_{j}$. Estimation of (2.10) is done in three stages. During the first-stage, we estimate all the endogenous explanatory variables in the system in every time period as follows:

$$\mathbf{y}_{hj} = \mathbf{X}_j^* \boldsymbol{\alpha}_{hj} + \mathbf{v}_{hj}, \tag{3.1}$$

where $\mathbf{X}_j^*$ denotes the matrix of observations including intercept and all the exogenous explanatory variables in the system in every jth time period, $\boldsymbol{\alpha}_{hj}$ denotes the hth parameter vector of the exogenous explanatory variables in the system in every jth time period, and $\mathbf{v}_{hj}$ denotes the hth error random vector in every jth time period assuming mean vector $\mathbf{0}$ and covariance matrix $\sigma_{v_h}^2 \mathbf{I}_n$ in which $\sigma_{v_h}^2$ denotes the unknown v_hth error variance.

To obtain $\hat{\mathbf{y}}_{hj}$, we must find the estimator of $\boldsymbol{\alpha}_{hj}$. Estimator for $\boldsymbol{\alpha}_{hj}$ is obtained by minimizing residual sum of squares $(\mathbf{v}_{hj}^t \mathbf{v}_{hj})$ in least squares method. To minimize this residual sum of squares, we first differentiate with respect to $\boldsymbol{\alpha}_{hj}$, then by setting this derivative equal to zero, we obtain the estimator of $\boldsymbol{\alpha}_{hj}$ which is given by

$$\hat{\boldsymbol{\alpha}}_{hj} = (\mathbf{X}_j^{*t} \mathbf{X}_j^*)^{-1} \mathbf{X}_j^{*t} \mathbf{y}_{hj}. \tag{3.2}$$

We estimate $\mathbf{y}_{hj}$ by

$$\hat{\mathbf{y}}_{hj} = \mathbf{X}_j^* \hat{\boldsymbol{\alpha}}_{hj}, \tag{3.3}$$

and obtain

$$\hat{\mathbf{Y}}_{-1j} = [\hat{\mathbf{y}}_{2j}\ \hat{\mathbf{y}}_{3j}\ \hat{\mathbf{y}}_{4j} \cdots \hat{\mathbf{y}}_{mj}],\ \hat{\mathbf{Y}}_{-2j} = [\hat{\mathbf{y}}_{1j}\ \hat{\mathbf{y}}_{3j}\ \hat{\mathbf{y}}_{4j} \cdots \hat{\mathbf{y}}_{mj}],$$

$$\hat{\mathbf{Y}}_{-3j} = [\hat{\mathbf{y}}_{1j}\ \hat{\mathbf{y}}_{2j}\ \hat{\mathbf{y}}_{4j} \cdots \hat{\mathbf{y}}_{mj}],\ \hat{\mathbf{Y}}_{-mj} = [\hat{\mathbf{y}}_{1j}\ \hat{\mathbf{y}}_{2j}\ \hat{\mathbf{y}}_{3j} \cdots \hat{\mathbf{y}}_{m-1,\,j}].$$

During the second-stage, we first estimate parameter λ_h because this parameter is unknown. We assume that there is no spatial influence in (2.6). We substitute $\mathbf{Y}_{-hj}$ by $\hat{\mathbf{Y}}_{-hj}$ in (2.6), where $\mathbf{Y}_{-hj} = \hat{\mathbf{Y}}_{-hj} + \hat{\mathbf{V}}_{-hj}$ and obtain new equations as follow:

$$\mathbf{y}_{hj} = \mathbf{1}\mu_h + \mathbf{Z}_{hj}\boldsymbol{\theta}_h + \mathbf{1}\gamma_{hj} + \mathbf{u}^*_{hj}, \tag{3.4}$$

where $\mathbf{Z}_{hj} = [\mathbf{X}_{hj} \vdots \hat{\mathbf{Y}}_{-hj}]$ and $\boldsymbol{\theta}^t_h = [\boldsymbol{\alpha}^t_h \vdots \boldsymbol{\beta}^t_{-h}]$ having dimensions $n \times (k_h + m - 1)$ and $1 \times (k_h + m - 1)$, respectively, and $\mathbf{u}^*_{hj}$ denotes the composite random error with $\mathbf{u}^*_{hj} = \hat{\mathbf{V}}_{-hj}\boldsymbol{\beta}_{-h} + \mathbf{u}_{hj}$. By using the results of (3.3), we apply least squares method to find the parameter estimators of μ_h, $\boldsymbol{\theta}_h$, and γ_{hj}. Because the matrix in the right-hand side is less than full rank, to get the estimator of $\boldsymbol{\theta}_h$, we use $n \times n$ dimensional transformation matrix $\mathbf{Q}$ in which $\mathbf{Q1} = \mathbf{0}$. We note in passing that $\mathbf{Q} = \mathbf{I}_n - \frac{1}{n}\mathbf{11}^t$ is symmetrical and idempotent. Premultiplying (3.4) by **Q,** we have $\mathbf{Qy}_{hj} = \mathbf{QZ}_{hj}\boldsymbol{\theta}_h + \mathbf{Qu}^*_{hj}$ and by means of least squares method the estimator of $\boldsymbol{\theta}_h$ is as follows:

$$\hat{\boldsymbol{\theta}}_h = \left[\sum_{j=1}^{T} (\mathbf{QZ}_{hj})^t \mathbf{QZ}_{hj}\right]^{-1} \left[\sum_{j=1}^{T} (\mathbf{QZ}_{hj})^t \mathbf{Qy}_{hj}\right]$$

$$= \left[\sum_{j=1}^{T} \mathbf{Z}^t_{hj}\mathbf{QZ}_{hj}\right]^{-1} \left[\sum_{j=1}^{T} \mathbf{Z}^t_{hj}\mathbf{Qy}_{hj}\right]. \tag{3.5}$$

By (3.4), the estimators of μ_h and γ_{hj} are

$$\hat{\mu}_h = \frac{\mathbf{1}^t}{nT}\left[\sum_{j=1}^{T} \mathbf{y}_{hj} - \left(\sum_{j=1}^{T} \mathbf{Z}_{hj}\right)\hat{\boldsymbol{\theta}}_h\right], \tag{3.6}$$

$$\hat{\gamma}_{hj} = \frac{1}{n}(\mathbf{1}^t\mathbf{y}_{hj} - n\hat{\mu}_h - \mathbf{1}^t\mathbf{Z}_{hj}\hat{\boldsymbol{\theta}}_h), \tag{3.7}$$

respectively.

From (3.5)-(3.7), we can estimate $\mathbf{u}_{hj}^{*}$ as follows:

$$\hat{\mathbf{u}}_{hj}^{*} = \mathbf{y}_{hj} - \mathbf{1}(\hat{\mu}_h + \hat{\gamma}_{hj}) - \mathbf{Z}_{hj}\hat{\boldsymbol{\theta}}_h = \mathbf{y}_{hj} - \hat{\mathbf{y}}_{hj}. \tag{3.8}$$

We use $\hat{\mathbf{u}}_{hj}^{*}$ to estimate parameter λ_h by means of concentrated log-likelihood.

The likelihood function of $\boldsymbol{\varepsilon}_{hj}$, $i = 1, 2, ..., n,$ $j = 1, 2, 3, ..., T,$ denoted by L_h is as follows: $L_h = \prod_{j=1}^{T}(2\pi\sigma_h^2)^{-\frac{\pi}{2}} \exp\left(-\frac{1}{2\sigma_h^2}\boldsymbol{\varepsilon}_{hj}^{t}\boldsymbol{\varepsilon}_{hj}\right)$, and by Jacobian transformation, we obtain the natural logarithm of L_h as $\ln L_h = -\frac{nT}{2}\ln(2\pi\sigma_h^2) - \frac{1}{2\sigma_h^2}\sum_{j=1}^{T}\mathbf{u}_{hj}^{t}\mathbf{A}_h^{t}\mathbf{A}_h\mathbf{u}_{hj} + T\ln\|\mathbf{A}_h\|$, where $\|\mathbf{A}_h\|$ is the absolute of the determinant of $\mathbf{A}_h$.

We take derivative of σ_h^2. Setting this derivative equal to zero, we obtain the estimator of σ_h^2, namely

$$\hat{\sigma}_h^2 = \frac{1}{nT}\sum_{j=1}^{T}\mathbf{u}_{hj}^{t}\mathbf{A}_h^{t}\mathbf{A}_h\mathbf{u}_{hj}. \tag{3.9}$$

By (3.9), we obtain the concentrated log-likelihood as follows:

$$\ln L_h^{con} = C - \frac{nT}{2}\ln\left(\frac{1}{nT}\sum_{j=1}^{T}\mathbf{u}_{hj}^{t}\mathbf{A}_h^{t}\mathbf{A}_h\mathbf{u}_{hj}\right) + T\ln\|\mathbf{A}_h\|, \tag{3.10}$$

where $C = -\frac{nT}{2}\log(2\pi) - \frac{nT}{2}$.

Let $\mathbf{W}$ have eigenvalues $\omega_1, \omega_2, ..., \omega_n$. The acceptable spatial autocorrelation parameter is $\frac{1}{\omega_{\text{minimum}}} < \lambda_h < 1$ [18]. We use numerical method for $\ln L_h^{con}$ to find estimator of λ_h, namely method of forming sequence of λ_h by means of R program. Its procedure is as follows:

(1) We make sequence values of λ_h, where λ_h = seq(start value, end value, increasing).

(2) For every $\mathbf{u}_{hj}$, $h = 1, 2, ..., m$, we insert values of λ_h in (3.10) and find the largest $\ln L_h^{con}$. Because the values of $\mathbf{u}_{hj}$ are unknown, we use residual error from (3.8).

(3) Finding the value of λ_h, we obtain the largest $\ln L_h^{con}$.

By substituting $\mathbf{A}_h^{-1} = \hat{\mathbf{A}}_h^{-1}$ with $\hat{\mathbf{A}}_h = \mathbf{I}_n - \hat{\lambda}_h \mathbf{W}$ and $\mathbf{Y}_{-hj} = \hat{\mathbf{Y}}_{-hj} + \hat{\mathbf{V}}_{-hj}$, where $\hat{\mathbf{V}}_{-hj}$ is residual error matrix from (3.1), into (2.7), we obtain

$$\mathbf{y}_{hj} = \mathbf{1}\mu_h + \mathbf{Z}_{hj}\boldsymbol{\theta}_h + \mathbf{1}\gamma_{hj} + \mathbf{u}_{hj}^{**}, \tag{3.11}$$

where $\mathbf{Z}_{hj} = [\mathbf{X}_{hj} \vdots \hat{\mathbf{Y}}_{-hj}]$, $\boldsymbol{\theta}_h^t = [\boldsymbol{\alpha}_h^t \vdots \boldsymbol{\beta}_{-h}^t]$ and $\mathbf{u}_{hj}^{**} = \hat{\mathbf{V}}_{-hj}\boldsymbol{\beta}_{-h} + \hat{\mathbf{A}}_h^{-1}\boldsymbol{\varepsilon}_{hj}$ having dimensions $n \times (k_h + m - 1)$, $1 \times (k_h + m - 1)$ and $n \times 1$, respectively. $\hat{\mathbf{V}}_{-hj}\boldsymbol{\beta}_{-h} = (\mathbf{Y}_{-hj} - \hat{\mathbf{Y}}_{-hj})\boldsymbol{\beta}_{-h}$ can be absorbed into error $\hat{\mathbf{A}}_h^{-1}\boldsymbol{\varepsilon}_{hj}$ because $\mathbf{X}_{hj}$ and $(\mathbf{Y}_{-hj} - \hat{\mathbf{Y}}_{-hj})$ as well as $\hat{\mathbf{Y}}_{-hj}$ and $(\mathbf{Y}_{-hj} - \hat{\mathbf{Y}}_{-hj})$ are independent [2, 10].

Since the error variance in equation (3.11) is not constant and the matrix in the right-hand side is less than full rank, we overcome these problems by means of reparametrization and generalized least squares (GLS). The estimators are as follows:

$$\hat{\boldsymbol{\theta}}_h = \left[\sum_{j=1}^{T} \mathbf{Z}_{hj}^t \hat{\mathbf{A}}_h^t \hat{\mathbf{A}}_h [\mathbf{1b}_h^t - \mathbf{I}_n] \mathbf{Z}_{hj}\right]^{-1} \sum_{j=1}^{T} \mathbf{Z}_{hj}^t \hat{\mathbf{A}}_h^t \hat{\mathbf{A}}_h [\mathbf{1b}_h^t - \mathbf{I}_n] \mathbf{y}_{hj}, \tag{3.12}$$

$$\hat{\mu}_h = \frac{1}{T}\mathbf{b}_h^t \sum_{j=1}^{T} (\mathbf{y}_{hj} - \mathbf{Z}_{hj}\hat{\boldsymbol{\theta}}_h), \tag{3.13}$$

$$\hat{\gamma}_{hj} = \mathbf{b}_h^t(\mathbf{y}_{hj} - \mathbf{1}\hat{\mu}_h - \mathbf{Z}_{hj}\hat{\boldsymbol{\theta}}_h), \tag{3.14}$$

where $\mathbf{b}_h^t = (\mathbf{1}^t \hat{\mathbf{A}}_h^t \hat{\mathbf{A}}_h \mathbf{1})^{-1}\mathbf{1}^t \hat{\mathbf{A}}_h^t \hat{\mathbf{A}}_h$ which having $1 \times n$ dimension. By (3.12) to (3.14), we can estimate $\mathbf{u}_{hj}^{**}$ as follows:

$$\hat{\mathbf{u}}_{hj}^{**} = \mathbf{y}_{hj} - \mathbf{1}(\hat{\mu}_h + \hat{\gamma}_{hj}) - \mathbf{Z}_{hj}\hat{\boldsymbol{\theta}}_h. \tag{3.15}$$

We then use (3.15) and (3.9) to find the estimated covariance matrix of the estimator $\hat{\mathbf{u}}_{hj}^{**}$, namely

$$\hat{\boldsymbol{\Sigma}} = \begin{bmatrix} \hat{\sigma}_1^2 & \hat{\sigma}_{12} & \hat{\sigma}_{13} & \cdots & \hat{\sigma}_{1m} \\ \hat{\sigma}_{21} & \hat{\sigma}_2^2 & \hat{\sigma}_{23} & \cdots & \hat{\sigma}_{2m} \\ \hat{\sigma}_{31} & \hat{\sigma}_{32} & \hat{\sigma}_3^2 & \cdots & \hat{\sigma}_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{\sigma}_{m1} & \hat{\sigma}_{m2} & \hat{\sigma}_{m3} & \cdots & \hat{\sigma}_m^2 \end{bmatrix}, \hat{\sigma}_h^2 = \hat{\sigma}_{hh^*} \text{ if } h = h^* \text{ with}$$

$$\hat{\sigma}_{hh^*} = \frac{1}{nT}\sum_{j=1}^{T} \hat{\mathbf{u}}_{hj}^{**t} \hat{\mathbf{A}}_h^t \hat{\mathbf{A}}_{h^*} \hat{\mathbf{u}}_{h^*j}^{**},$$

where $\hat{\sigma}_h^2$ denotes the hth estimated error variance, $\hat{\sigma}_{hh^*}$ denotes the h^*th and the hth estimated error covariance, and $\hat{\boldsymbol{\Sigma}}$ denotes $m \times m$ estimated covariance matrix.

From (2.10), we have error covariance matrix $\text{var}(\overline{\mathbf{X}}_{**}^t \hat{\mathbf{A}}_* \boldsymbol{\varepsilon}_j) = \overline{\mathbf{X}}_{**}^t \hat{\mathbf{A}}_* \text{var}(\boldsymbol{\varepsilon}_j) \hat{\mathbf{A}}_*^t \overline{\mathbf{X}}_{**}$. This covariance shows that random errors are heteroscedastic, where $\text{var}(\boldsymbol{\varepsilon}_j) = E(\boldsymbol{\varepsilon}_j \boldsymbol{\varepsilon}_j^t)$ for $h = h^* = 1, 2, 3, \ldots, m$, $\boldsymbol{\varepsilon}_j^t = [\boldsymbol{\varepsilon}_{1j}\ \boldsymbol{\varepsilon}_{2j} \cdots \boldsymbol{\varepsilon}_{mj}]$, $\boldsymbol{\varepsilon}_{hj}^t = [\boldsymbol{\varepsilon}_{h1j}\ \boldsymbol{\varepsilon}_{h2j} \cdots \boldsymbol{\varepsilon}_{hnj}]$, in which we assumed that

$$E(\varepsilon_{hij}\varepsilon_{h^*i^*j}) = \begin{cases} \sigma_{hh^*} & \text{if } i = i^* \\ 0 & \text{if } i \neq i^* \end{cases}$$

so that $E(\boldsymbol{\varepsilon}_{hj}\boldsymbol{\varepsilon}^t_{h^*j}) = \sigma_{hh^*}\mathbf{I}_n$. We obtain $\text{var}(\boldsymbol{\varepsilon}_j) = \boldsymbol{\Sigma} \otimes \mathbf{1}_n$ with $mn \times mn$ as its dimension. Consequently, $\text{var}(\overline{\mathbf{X}}^t_{**}\hat{\mathbf{A}}_*\boldsymbol{\varepsilon}_j) = \overline{\mathbf{X}}^t_{**}\hat{\mathbf{A}}_*(\boldsymbol{\Sigma} \otimes \mathbf{I}_n)\hat{\mathbf{A}}^t_*\overline{\mathbf{X}}_{**} = \boldsymbol{\Sigma}_\#$ which is $m\sum_{h=1}^{m} k_h \times m\sum_{h=1}^{m} k_h$ symmetrical matrix.

We use the estimator of $\boldsymbol{\Sigma}_\#$ because $\boldsymbol{\Sigma}$ is unknown. The estimator of $\boldsymbol{\Sigma}_\#$ is as follows:

$$\hat{\boldsymbol{\Sigma}}_\# = \begin{bmatrix} \hat{\sigma}^2_{1\#} & \hat{\sigma}_{12\#} & \hat{\sigma}_{13\#} & \cdots & \hat{\sigma}_{1m\#} \\ \hat{\sigma}_{21\#} & \hat{\sigma}^2_{2\#} & \hat{\sigma}_{23\#} & \cdots & \hat{\sigma}_{2m\#} \\ \hat{\sigma}_{31\#} & \hat{\sigma}_{32\#} & \hat{\sigma}^2_{3\#} & \cdots & \hat{\sigma}_{3m\#} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{\sigma}_{m1\#} & \hat{\sigma}_{m2\#} & \hat{\sigma}_{m3\#} & \cdots & \hat{\sigma}^2_{m\#} \end{bmatrix}, \ \hat{\sigma}^2_{h\#} = \hat{\sigma}_{hh^*\#} \text{ if } h = h^* \text{ with}$$

$\hat{\sigma}_{hh^*\#} = \hat{\sigma}_{hh^*}\overline{\mathbf{X}}^t_*\hat{\mathbf{A}}^{-1}_h(\hat{\mathbf{A}}^{-1}_{h^*})^t\overline{\mathbf{X}}_*$.

In the above results, we see that the error variance in equation (2.10) is not constant and the matrix in the right-hand side is less than full rank. For the last-stage, we overcome those problems again by means of reparametrization and GLS. The estimators are as follows:

$$\hat{\boldsymbol{\theta}} = \left[\sum_{j=1}^{T}\mathbf{Z}^t_j\hat{\mathbf{H}}^*\hat{\mathbf{M}}^*\mathbf{Z}_j\right]^{-1}\sum_{j=1}^{T}\mathbf{Z}^t_j\hat{\mathbf{H}}^*\hat{\mathbf{M}}^*\mathbf{y}_j, \tag{3.16}$$

$$\hat{\boldsymbol{\mu}} = [T\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}]^{-1}\mathbf{G}^t\hat{\mathbf{H}}^*\sum_{j=1}^{T}(\mathbf{y}_j - \mathbf{Z}_j\hat{\boldsymbol{\theta}}), \tag{3.17}$$

$$\hat{\boldsymbol{\gamma}}_j = [\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}]^{-1}\mathbf{G}^t\hat{\mathbf{H}}^*(\mathbf{y}_j - \mathbf{G}\hat{\boldsymbol{\mu}} - \mathbf{Z}_j\hat{\boldsymbol{\theta}}), \tag{3.18}$$

where $\hat{\mathbf{H}}^* = \overline{\mathbf{X}}_{**}\hat{\mathbf{\Sigma}}_{\#}^{-1}\overline{\mathbf{X}}_{**}^t$ and $\hat{\mathbf{M}}^* = \mathbf{G}[\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}]^{-1}\mathbf{G}^t\hat{\mathbf{H}}^* - \mathbf{I}_{mn}$. $\hat{\mathbf{H}}^*$, $\hat{\mathbf{M}}^*$, and $\mathbf{I}_{mn}$ have dimension $mn \times mn$.

In this paper, the estimators of $\boldsymbol{\theta}$, $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}_j$ are called the *estimators* of feasible generalized least squares-multivariate spatial error three-stage least squares fixed effect panel simultaneous (FGLS-MSE3SLSFEPS).

4. Properties of Estimators

Theorem (Consistency). *If* $\overline{\mathbf{X}}_{**}^t\mathbf{y}_j = \overline{\mathbf{X}}_{**}^t\mathbf{G}\boldsymbol{\mu} + \overline{\mathbf{X}}_{**}^t\mathbf{Z}_j\boldsymbol{\theta} + \overline{\mathbf{X}}_{**}^t\mathbf{G}\boldsymbol{\gamma}_j + \overline{\mathbf{X}}_{**}^t\mathbf{A}_*\boldsymbol{\varepsilon}_j$ *as defined in* (2.10), *then* $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\mu}}$ *and* $\hat{\boldsymbol{\gamma}}_j$ *are consistent estimators*.

Proof. Recall (2.10). This can be rewritten as $\mathbf{y}_j = \mathbf{G}\boldsymbol{\mu} + \mathbf{Z}_j\boldsymbol{\theta} + \mathbf{G}\boldsymbol{\gamma}_j + \mathbf{A}_*\boldsymbol{\varepsilon}_j$. Estimators of equation (2.10) are as follows:

$$\hat{\boldsymbol{\theta}} = \left[\sum_{j=1}^{T}\mathbf{Z}_j^t\hat{\mathbf{H}}^*\hat{\mathbf{M}}^*\mathbf{Z}_j\right]^{-1}\sum_{j=1}^{T}\mathbf{Z}_j^t\hat{\mathbf{H}}^*\hat{\mathbf{M}}^*\mathbf{y}_j$$

$$= \boldsymbol{\theta} + \left[\sum_{j=1}^{T}\mathbf{Z}_j^t\hat{\mathbf{H}}^*\hat{\mathbf{M}}^*\mathbf{Z}_j\right]^{-1}\left[\sum_{j=1}^{T}\mathbf{Z}_j^t\hat{\mathbf{H}}^*\hat{\mathbf{M}}^*\mathbf{A}_*\boldsymbol{\varepsilon}_j\right], \ \hat{\mathbf{M}}^*\mathbf{G} = \mathbf{0},$$

$$\hat{\boldsymbol{\mu}} = [T\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}]^{-1}\mathbf{G}^t\hat{\mathbf{H}}^*\sum_{j=1}^{T}(\mathbf{y}_j - \mathbf{Z}_j\hat{\boldsymbol{\theta}})$$

$$= \boldsymbol{\mu} + [T\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}]^{-1}\left(\sum_{j=1}^{T}\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{Z}_j(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}) + \sum_{j=1}^{T}\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{A}_*\boldsymbol{\varepsilon}_j\right), \ \sum_{j=1}^{T}\boldsymbol{\gamma}_j = \mathbf{0},$$

$$\hat{\boldsymbol{\gamma}}_j = [\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}]^{-1}\mathbf{G}^t\hat{\mathbf{H}}^*(\mathbf{y}_j - \mathbf{G}\hat{\boldsymbol{\mu}} - \mathbf{Z}_j\hat{\boldsymbol{\theta}})$$

$$= (\boldsymbol{\mu} - \hat{\boldsymbol{\mu}}) + [\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}]^{-1}\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{Z}_j(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}) + \boldsymbol{\gamma}_j + [\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}]^{-1}\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{A}_*\boldsymbol{\varepsilon}_j.$$

We refer to [1, 2, 10, 19-21]. Asymptotic expectation and variance of $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\gamma}}_j$ are as follows:

$$\overline{E}\{\hat{\boldsymbol{\theta}}\} = \lim_{\substack{n\to\infty \\ T\to\infty}} E\{\hat{\boldsymbol{\theta}}\} = \boldsymbol{\theta} + \left[\lim_{\substack{n\to\infty \\ T\to\infty}} \frac{1}{nT} \sum_{j=1}^{T} \mathbf{Z}_j^t \hat{\mathbf{H}}^* \hat{\mathbf{M}}^* \mathbf{Z}_j\right]^{-1}$$

$$\cdot \left[\lim_{\substack{n\to\infty \\ T\to\infty}} \frac{1}{nT} \sum_{j=1}^{T} \mathbf{Z}_j^t \hat{\mathbf{H}}^* \hat{\mathbf{M}}^* \mathbf{A}_* E\{\boldsymbol{\varepsilon}_j\}\right]$$

$$= \boldsymbol{\theta} + \left[\lim_{\substack{n\to\infty \\ T\to\infty}} \frac{1}{nT} \mathbf{K}\right]^{-1} \left[\lim_{\substack{n\to\infty \\ T\to\infty}} \frac{1}{nT} \sum_{j=1}^{T} \mathbf{Z}_j^t \hat{\mathbf{H}}^* \hat{\mathbf{M}}^* \mathbf{A}_* \times \mathbf{0}\right]$$

$$= \boldsymbol{\theta} + \left[\lim_{\substack{n\to\infty \\ T\to\infty}} \overline{\mathbf{K}}\right]^{-1} \times \mathbf{0} = \boldsymbol{\theta} + [\overline{\mathbf{K}}]^{-1} \times \mathbf{0} = \boldsymbol{\theta},$$

where $\mathbf{K}$ and $\overline{\mathbf{K}}$ are constant nonsingular matrices. We have

$$\operatorname{var}\{\hat{\boldsymbol{\theta}}\} = \text{asy.var}\left\{\left[\sum_{j=1}^{T} \mathbf{Z}_j^t \hat{\mathbf{H}}^* \hat{\mathbf{M}}^* \mathbf{Z}_j\right]^{-1} \left[\sum_{j=1}^{T} \mathbf{Z}_j^t \hat{\mathbf{H}}^* \hat{\mathbf{M}}^* \mathbf{A}_* \boldsymbol{\varepsilon}_j\right]\right\}$$

$$= \left[\sum_{j=1}^{T} \mathbf{Z}_j^t \hat{\mathbf{H}}^* \hat{\mathbf{M}}^* \mathbf{Z}_j\right]^{-1} \left[\sum_{j=1}^{T} \mathbf{Z}_j^t \hat{\mathbf{H}}^* \hat{\mathbf{M}}^* \mathbf{A}_* (\boldsymbol{\Sigma} \otimes \mathbf{I}_n) \mathbf{A}_*^t \hat{\mathbf{H}}^* \hat{\mathbf{M}}^* \mathbf{Z}_j\right]$$

$$\cdot \left[\sum_{j=1}^{T} \mathbf{Z}_j^t \hat{\mathbf{H}}^* \hat{\mathbf{M}}^* \mathbf{Z}_j\right]^{-1},$$

where $\hat{\mathbf{H}}^*$ and $\hat{\mathbf{H}}^* \hat{\mathbf{M}}^*$ are symmetrical. Now,

$$\lim_{\substack{n\to\infty \\ T\to\infty}} \text{asy.var}\{\hat{\boldsymbol{\theta}}\} = \left[\sum_{j=1}^{T} \mathbf{Z}_j^t \hat{\mathbf{H}}^* \hat{\mathbf{M}}^* \mathbf{Z}_j\right]^{-1}$$

$$\cdot \left[\sum_{j=1}^{T} \mathbf{Z}_j^t \hat{\mathbf{H}}^* \hat{\mathbf{M}}^* \mathbf{A}_* \left\{\lim_{\substack{n\to\infty \\ T\to\infty}} \frac{1}{nT} (\boldsymbol{\Sigma} \otimes \mathbf{I}_n)\right\} \mathbf{A}_*^t \hat{\mathbf{H}}^* \hat{\mathbf{M}}^* \mathbf{Z}_j\right]$$

$$\times \left[\lim_{\substack{n\to\infty \\ T\to\infty}} \frac{1}{nT} \sum_{j=1}^{T} \mathbf{Z}_j^t \hat{\mathbf{H}}^* \hat{\mathbf{M}}^* \mathbf{Z}_j \right]^{-1}$$

$$= [\mathbf{K}]^{-1} \left[\sum_{j=1}^{T} \mathbf{Z}_j^t \hat{\mathbf{H}}^* \hat{\mathbf{M}}^* \mathbf{A}_* \times \mathbf{0} \times \mathbf{A}_*^t \hat{\mathbf{H}}^* \hat{\mathbf{M}}^* \mathbf{Z}_j \right] [\overline{\mathbf{K}}]^{-1}$$

$$= [\mathbf{K}]^{-1} \times \mathbf{0} \times [\overline{\mathbf{K}}]^{-1} = \mathbf{0}.$$

This shows that $\hat{\boldsymbol{\theta}}$ is asymptotically unbiased estimator. If $n \to \infty$ or $T \to \infty$ or both of $n \to \infty$ and $T \to \infty$, then $\text{asy.var}\{\hat{\boldsymbol{\theta}}\} \to \mathbf{0}$. Therefore, $\hat{\boldsymbol{\theta}}$ is a consistent estimator. Next,

$$\overline{E}\{\hat{\boldsymbol{\mu}}\} = \lim_{\substack{n\to\infty \\ T\to\infty}} E\{\hat{\boldsymbol{\mu}}\}$$

$$= \boldsymbol{\mu} + \left(\lim_{\substack{n\to\infty \\ T\to\infty}} \frac{1}{nT} \left[\frac{1}{n} \mathbf{G}^t \hat{\mathbf{H}}^* \mathbf{G} \right]^{-1} \right)$$

$$\left(\sum_{j=1}^{T} \mathbf{G}^t \hat{\mathbf{H}}^* \mathbf{Z}_j \left(\boldsymbol{\theta} - \lim_{\substack{n\to\infty \\ T\to\infty}} \{\hat{\boldsymbol{\theta}}\} \right) + \lim_{\substack{n\to\infty \\ T\to\infty}} \sum_{j=1}^{T} \mathbf{G}^t \hat{\mathbf{H}}^* \mathbf{A}_* E\{\boldsymbol{\varepsilon}_j\} \right)$$

$$= \boldsymbol{\mu} + \left(\lim_{\substack{n\to\infty \\ T\to\infty}} \frac{1}{nT} \left[\frac{1}{n} \mathbf{K}_1 \right]^{-1} \right) \left(\sum_{j=1}^{T} \mathbf{G}^t \hat{\mathbf{H}}^* \mathbf{Z}_j (\boldsymbol{\theta} - \boldsymbol{\theta}) + \lim_{\substack{n\to\infty \\ T\to\infty}} \sum_{j=1}^{T} \mathbf{G}^t \hat{\mathbf{H}}^* \mathbf{A}_* \times \mathbf{0} \right)$$

$$= \boldsymbol{\mu} + \left(\lim_{\substack{n\to\infty \\ T\to\infty}} \frac{1}{nT} [\overline{\mathbf{K}}_1]^{-1} \right) \times \mathbf{0} = \boldsymbol{\mu},$$

where $\mathbf{K}_1$ and $\overline{\mathbf{K}}_1$ are constant nonsingular matrices. We have

$$\text{asy. var}\{\hat{\boldsymbol{\mu}}\} = \text{asy.var}\left\{ [T\mathbf{G}^t \hat{\mathbf{H}}^* \mathbf{G}]^{-1} \sum_{j=1}^{T} \mathbf{G}^t \hat{\mathbf{H}}^* \mathbf{Z}_j \hat{\boldsymbol{\theta}} \right\}$$

$$+ \text{asy.var}\left\{ [T\mathbf{G}^t \hat{\mathbf{H}}^* \mathbf{G}]^{-1} \sum_{j=1}^{T} \mathbf{G}^t \hat{\mathbf{H}}^* \mathbf{A}_* \boldsymbol{\varepsilon}_j \right\},$$

$$\text{asy.var}\left\{[T\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}]^{-1}\sum_{j=1}^{T}\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{Z}_j\hat{\boldsymbol{\theta}}\right\}$$

$$= [T\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}]^{-1}\left[\sum_{j=1}^{T}\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{Z}_j\text{asy.var}\{\hat{\boldsymbol{\theta}}\}\mathbf{Z}_j^t\hat{\mathbf{H}}^*\mathbf{G}\right][T\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}]^{-1},$$

$$\text{asy.var}\left\{[T\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}]^{-1}\sum_{j=1}^{T}\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{A}_*\boldsymbol{\varepsilon}_j\right\}$$

$$= [\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}]^{-1}[\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{A}_*(\boldsymbol{\Sigma}\oplus\mathbf{I}_n)\mathbf{A}_*^t\hat{\mathbf{H}}^*\mathbf{G}][T\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}]^{-1},$$

$$\lim_{\substack{n\to\infty\\T\to\infty}}\text{asy.var}\left\{[T\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}]^{-1}\sum_{j=1}^{T}\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{Z}_j\hat{\boldsymbol{\theta}}\right\}$$

$$= \lim_{\substack{n\to\infty\\T\to\infty}}\frac{1}{nT}\left[\frac{1}{n}\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}\right]^{-1}$$

$$\cdot\left[\sum_{j=1}^{T}\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{Z}_j\left(\lim_{\substack{n\to\infty\\T\to\infty}}\text{asy.var}\{\hat{\boldsymbol{\theta}}\}\right)\mathbf{Z}_j^t\hat{\mathbf{H}}^*\mathbf{G}\right]\lim_{\substack{n\to\infty\\T\to\infty}}\frac{1}{nT}\left[\frac{1}{n}\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}\right]^{-1}$$

$$= \lim_{\substack{n\to\infty\\T\to\infty}}\frac{1}{nT}[\overline{\mathbf{K}}_1]^{-1}\left[\sum_{j=1}^{T}\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{Z}_j\times\mathbf{0}\times\mathbf{Z}_j^t\hat{\mathbf{H}}^*\mathbf{G}\right]\lim_{\substack{n\to\infty\\T\to\infty}}\frac{1}{nT}[\overline{\mathbf{K}}_1]^{-1} = \mathbf{0},$$

$$\lim_{\substack{n\to\infty\\T\to\infty}}\text{asy.var}\left\{[T\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}]^{-1}\sum_{j=1}^{T}\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{A}_*\boldsymbol{\varepsilon}_j\right\}$$

$$= [\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}]^{-1}\left[\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{A}_*\left\{\lim_{\substack{n\to\infty\\T\to\infty}}\frac{1}{nT}(\boldsymbol{\Sigma}\oplus\mathbf{I}_n)\right\}\mathbf{A}_*^t\hat{\mathbf{H}}^*\mathbf{G}\right]\left[\lim_{\substack{n\to\infty\\T\to\infty}}\frac{1}{n}\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}\right]^{-1}$$

$$= [\mathbf{K}_1]^{-1}[\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{A}_* \times \mathbf{0} \times \mathbf{A}_*^t\hat{\mathbf{H}}^*\mathbf{G}]\left[\lim_{\substack{n\to\infty \\ T\to\infty}} \overline{\mathbf{K}}_1\right]^{-1}$$

$$= [\mathbf{K}_1]^{-1} \times \mathbf{0} \times [\overline{\mathbf{K}}_1]^{-1} = \mathbf{0}.$$

Consequently,

$$\lim_{\substack{n\to\infty \\ T\to\infty}} \text{asy.var}\{\hat{\boldsymbol{\mu}}\} = \lim_{\substack{n\to\infty \\ T\to\infty}} \text{asy.var}\left\{[T\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}]^{-1}\sum_{j=1}^{T}\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{Z}_j\hat{\boldsymbol{\theta}}\right\}$$

$$+ \lim_{\substack{n\to\infty \\ T\to\infty}} \text{asy.var}\left\{[T\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}]^{-1}\sum_{j=1}^{T}\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{A}_*\boldsymbol{\varepsilon}_j\right\}$$

$$= \mathbf{0}.$$

This shows that $\hat{\boldsymbol{\mu}}$ is asymptotically unbiased estimator. If $n \to \infty$ or $T \to \infty$ or both of $n \to \infty$ and $T \to \infty$, then $\text{asy.var}\{\hat{\boldsymbol{\mu}}\} \to \mathbf{0}$. Therefore, $\hat{\boldsymbol{\mu}}$ is a consistent estimator. Now,

$$\overline{E}\{\hat{\boldsymbol{\gamma}}_j\} = \lim_{\substack{n\to\infty \\ T\to\infty}} E\{\hat{\boldsymbol{\gamma}}_j\}$$

$$= \left(\boldsymbol{\mu} - \lim_{\substack{n\to\infty \\ T\to\infty}} E\{\hat{\boldsymbol{\mu}}\}\right) + [\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}]^{-1}\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{Z}_j\left(\boldsymbol{\theta} - \lim_{\substack{n\to\infty \\ T\to\infty}} E\{\hat{\boldsymbol{\theta}}\}\right) + \boldsymbol{\gamma}_j$$

$$+ \lim_{\substack{n\to\infty \\ T\to\infty}} [\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}]^{-1}\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{A}_*E\{\boldsymbol{\varepsilon}_j\}$$

$$= (\boldsymbol{\mu} - \boldsymbol{\mu}) + [\mathbf{K}_1]^{-1}\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{Z}_j(\boldsymbol{\theta} - \boldsymbol{\theta}) + \boldsymbol{\gamma}_j + \lim_{\substack{n\to\infty \\ T\to\infty}} [\mathbf{K}_1]^{-1}\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{A}_* \times \mathbf{0} = \boldsymbol{\gamma}_j,$$

$$\text{asy.var}\{\hat{\boldsymbol{\gamma}}_j\} = \text{asy.var}\{\hat{\boldsymbol{\mu}}\} + \text{asy.var}\{[\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}]^{-1}\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{Z}_j\hat{\boldsymbol{\theta}}\}$$

$$+ \text{asy.var}\{[\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}]^{-1}\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{A}_*\boldsymbol{\varepsilon}_j\},$$

$$\text{asy.var}\{[\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}]^{-1}\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{Z}_j\hat{\boldsymbol{\theta}}\}$$

$$= [\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}]^{-1}\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{Z}_j\text{asy.var}\{\hat{\boldsymbol{\theta}}\}\mathbf{Z}_j^t\hat{\mathbf{H}}^*\mathbf{G}[\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}]^{-1},$$

$$\text{asy.var}\{[\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}]^{-1}\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{A}_*\boldsymbol{\varepsilon}_j\}$$

$$= [\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}]^{-1}\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{A}_*(\boldsymbol{\Sigma}\otimes\mathbf{I}_n)\mathbf{A}_*^t\hat{\mathbf{H}}^*\mathbf{G}[\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}]^{-1},$$

$$\lim_{n\to\infty}\text{asy.var}\{[\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}]^{-1}\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{Z}_j\hat{\boldsymbol{\theta}}\}$$

$$= [\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}]^{-1}\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{Z}_j(\lim_{n\to\infty}\text{asy. var}\{\hat{\boldsymbol{\theta}}\})\mathbf{Z}_j^t\hat{\mathbf{H}}^*\mathbf{G}[\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}]^{-1}$$

$$= [\mathbf{K}_1]^{-1}\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{Z}_j\times\mathbf{0}\times\mathbf{Z}_j^t\hat{\mathbf{H}}^*\mathbf{G}[\mathbf{K}_1]^{-1} = \mathbf{0},$$

$$\lim_{n\to\infty}\text{asy.var}\{[\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}]^{-1}\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{A}_*\boldsymbol{\varepsilon}_j\}$$

$$= [\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}]^{-1}\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{A}_*\left\{\lim_{n\to\infty}\frac{1}{n}(\boldsymbol{\Sigma}\otimes\mathbf{I}_n)\right\}\mathbf{A}_*^t\hat{\mathbf{H}}^*\mathbf{G}\left[\lim_{n\to\infty}\frac{1}{n}\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}\right]^{-1}$$

$$= [\mathbf{K}_1]^{-1}\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{A}_*\times\mathbf{0}\times\mathbf{A}_*^t\hat{\mathbf{H}}^*\mathbf{G}[\overline{\mathbf{K}}_1]^{-1} = \mathbf{0}.$$

Consequently,

$$\begin{aligned}\lim_{\substack{n\to\infty\\T\to\infty}}\text{asy.var}\{\hat{\boldsymbol{\gamma}}_j\} &= \lim_{\substack{n\to\infty\\T\to\infty}}\text{asy.var}\{\hat{\boldsymbol{\mu}}\} + \lim_{\substack{n\to\infty\\T\to\infty}}\text{asy.var}\{[\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}]^{-1}\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{Z}_j\hat{\boldsymbol{\theta}}\}\\ &\quad + \lim_{\substack{n\to\infty\\T\to\infty}}\text{asy.var}\{[\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{G}]^{-1}\mathbf{G}^t\hat{\mathbf{H}}^*\mathbf{A}_*\boldsymbol{\varepsilon}_j\}\\ &= \mathbf{0}.\end{aligned}$$

This shows that $\hat{\boldsymbol{\gamma}}_j$ is asymptotically unbiased estimator. If $n\to\infty,$ then $\text{asy.var}\{\hat{\boldsymbol{\gamma}}_j\}\to\mathbf{0}.$ Therefore, $\hat{\boldsymbol{\gamma}}_j$ is a consistent estimator.

5. Illustration

Suppose there are three dependent variables y_1, y_2, y_3 and six exogenous explanatory variables x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32} observed for two time periods and the number of observations being 10 locations with equation models as follows:

$$
\begin{aligned}
y_{1ij} &= \mu_1 + \alpha_{11}x_{11ij} + \alpha_{12}x_{12ij} + \beta_{12}y_{2ij} + \beta_{13}y_{3ij} + \gamma_{1j} + u_{1ij},\\
y_{2ij} &= \mu_2 + \alpha_{21}x_{21ij} + \alpha_{22}x_{22ij} + \beta_{21}y_{1ij} + \beta_{23}y_{3ij} + \gamma_{2j} + u_{2ij},\\
y_{3ij} &= \mu_3 + \alpha_{31}x_{31ij} + \alpha_{32}x_{32ij} + \beta_{31}y_{1ij} + \beta_{32}y_{2ij} + \gamma_{3j} + u_{3ij},\\
u_{1ij} &= \lambda_1\mathbf{w}_i^t\mathbf{u}_{1j} + \varepsilon_{1ij},\ \varepsilon_{1ij} \sim N(0, \sigma_1^2),\\
u_{2ij} &= \lambda_2\mathbf{w}_i^t\mathbf{u}_{2j} + \varepsilon_{2ij},\ \varepsilon_{2ij} \sim N(0, \sigma_2^2),\\
u_{3ij} &= \lambda_3\mathbf{w}_i^t\mathbf{u}_{3j} + \varepsilon_{3ij},\ \varepsilon_{3ij} \sim N(0, \sigma_3^2),
\end{aligned}
\tag{5.1}
$$

where

$$
\mathbf{W} = \begin{bmatrix} w_{11} & w_{12} & w_{13} & \cdots & w_{1,10} \\ w_{21} & w_{22} & w_{23} & \cdots & w_{2,10} \\ w_{31} & w_{32} & w_{33} & \cdots & w_{3,10} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{10,1} & w_{10,2} & w_{10,3} & \cdots & w_{10,10} \end{bmatrix} = \begin{bmatrix} \mathbf{w}_1^t \\ \mathbf{w}_2^t \\ \mathbf{w}_3^t \\ \vdots \\ \mathbf{w}_{10}^t \end{bmatrix} = [\mathbf{w}_i^t],
$$

$i = 1, 2, 3, \ldots, 10$

and

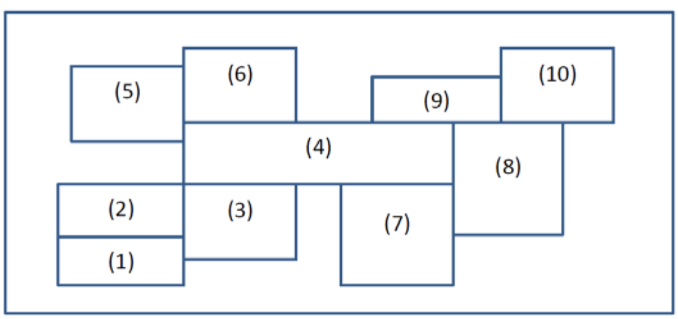

Figure 5.1. Illustration of the 10 neighboring locations.

Then from Figure 5.1, we obtain row-standardized spatial weight matrix as follows:

$$\mathbf{W} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{7} & \frac{1}{7} & 0 & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}.$$

The formulation of Moran index is as follows:

$$I_{hj} = \frac{\sum_{i=1}^{10} \sum_{i^*=1}^{10} w_{ii^*}(y_{hij} - \bar{y}_{hj})(y_{hi^*j} - \bar{y}_{hj})}{\sum_{i=1}^{10} (y_{hij} - \bar{y}_{hj})^2}$$

$$= \frac{\mathbf{y}_{hj}^{*t} \mathbf{W} \mathbf{y}_{hj}^{*}}{\mathbf{y}_{hj}^{*t} \mathbf{y}_{hj}^{*}}, \quad h = 1, 2, 3, \quad j = 1, 2,$$

where $\bar{y}_{hj} = \frac{1}{10} \sum_{i=1}^{10} y_{hij}$ and $\mathbf{y}_{hj}^{*} = \mathbf{y}_{hj} - \bar{\mathbf{y}}_{hj} \mathbf{1}$.

If there is at least one $I_{hj} > E(I)$, then we conclude that there is a spatial influence for the equation models.

Table 5.1. Data for dependent and exogenous explanatory variables

Time	Location	Variables								
		Dependent			Exogenous explanatory					
		y_1	y_2	y_3	x_{11}	x_{12}	x_{21}	x_{22}	x_{31}	x_{32}
1	1	15	25	20	45	51	46	49	47	48
	2	17	28	23	40	55	42	56	45	53
	3	14	27	21	41	56	40	58	42	51
	4	12	26	24	42	58	43	57	40	55
	5	18	29	22	47	57	45	58	42	51
	6	19	28	26	46	54	44	55	43	54
	7	20	31	29	45	56	47	54	49	57
	8	13	33	31	43	57	46	59	45	52
	9	14	29	28	47	59	48	60	46	59
	10	16	27	25	44	52	43	53	44	52
2	1	16	24	21	50	65	51	64	53	65
	2	17	29	27	51	66	52	67	54	63
	3	15	27	23	59	66	58	68	57	71
	4	14	26	22	58	64	59	66	54	73
	5	17	30	28	57	63	60	62	56	61
	6	20	29	27	61	67	61	68	60	67
	7	18	32	31	63	68	62	65	61	64
	8	14	32	30	62	68	64	66	59	71
	9	15	29	27	64	69	65	68	53	59
	10	18	26	23	58	65	57	69	58	67

Note: data illustration

$\bar{y}_{11} = 15.80$; $\bar{y}_{21} = 28.30$; $\bar{y}_{31} = 24.90$; $\bar{y}_{12} = 16.40$; $\bar{y}_{22} = 28.40$; $\bar{y}_{32} = 25.90$; $I_{11} = -0.2442$; $I_{21} = 0.0539$; $I_{31} = 0.4586$; $I_{12} = -0.2317$; $I_{22} = -0.0878$; $I_{32} = -0.1078$; and $E(I_{hj}) = E(I) = \frac{-1}{n-1} = \frac{-1}{10-1} = -0.1111$.

Based on the above result, by means of R Program version 3.0.3, we obtain that there is a spatial influence for the equation models.

We then continue to estimate parameters by means of FGLS-3SLS. For the first-stage, we estimate all the endogenous explanatory variables in the system in every time period and the results are as in Table 5.2.

Table 5.2. Estimated values for endogenous explanatory variables

Time	Location	Endogenous explanatory variables		
		y_1 -estimate	y_2 -estimate	y_3 -estimate
1	1	16.5625	26.5828	21.7290
	2	15.0373	28.5890	25.1588
	3	16.1904	27.6672	20.7955
	4	12.3775	26.0621	23.9145
	5	16.1804	28.3403	22.0819
	6	17.2918	26.9959	24.8593
	7	18.7007	29.1060	26.5129
	8	12.3543	31.5231	28.7592
	9	16.4805	31.4345	30.8012
	10	16.8246	26.6991	24.3877
2	1	15.5100	25.9073	23.1069
	2	17.3247	27.0314	24.7638
	3	15.8433	26.3597	22.8089
	4	13.3019	25.4562	21.0773
	5	17.3259	29.7492	27.8872
	6	18.1930	30.2621	28.3106
	7	18.0785	31.1379	29.8797
	8	14.7653	32.0924	30.1569
	9	14.9671	29.3371	27.3870
	10	18.6902	26.6667	23.6217

For the second-stage, we estimate $\mathbf{\Sigma}_{\#}$. From (3.5)-(3.8), we obtain

$$\hat{\boldsymbol{\theta}}_1 = \begin{bmatrix} \hat{\boldsymbol{\alpha}}_1 \\ \cdots \\ \hat{\boldsymbol{\beta}}_{-1} \end{bmatrix} = \begin{bmatrix} \hat{\alpha}_{11} \\ \hat{\alpha}_{12} \\ \cdots \\ \hat{\beta}_{12} \\ \hat{\beta}_{13} \end{bmatrix} = \begin{bmatrix} 0.1004 \\ -0.3539 \\ \cdots \\ 0.0145 \\ 0.1207 \end{bmatrix}; \quad \hat{\boldsymbol{\theta}}_2 = \begin{bmatrix} \hat{\boldsymbol{\alpha}}_2 \\ \cdots \\ \hat{\boldsymbol{\beta}}_{-2} \end{bmatrix} = \begin{bmatrix} \hat{\alpha}_{21} \\ \hat{\alpha}_{22} \\ \cdots \\ \hat{\beta}_{21} \\ \hat{\beta}_{23} \end{bmatrix} = \begin{bmatrix} 0.1104 \\ 0.0870 \\ \cdots \\ 0.0280 \\ 0.5193 \end{bmatrix};$$

$$\hat{\boldsymbol{\theta}}_3 = \begin{bmatrix} \hat{\boldsymbol{\alpha}}_3 \\ \cdots \\ \hat{\boldsymbol{\beta}}_{-3} \end{bmatrix} = \begin{bmatrix} \hat{\alpha}_{31} \\ \hat{\alpha}_{32} \\ \cdots \\ \hat{\beta}_{31} \\ \hat{\beta}_{32} \end{bmatrix} = \begin{bmatrix} -0.0477 \\ 0.0741 \\ \cdots \\ 0.1091 \\ 1.3424 \end{bmatrix};$$

$\hat{\mu}_1 = 29.0005$; $\hat{\mu}_2 = 3.6897$; $\hat{\mu}_3 = -16.4294$;

$\hat{\gamma}_{11} = -1.3963$; $\hat{\gamma}_{12} = 1.3963$; $\hat{\gamma}_{21} = 1.4711$; $\hat{\gamma}_{22} = -1.4711$;

$\hat{\gamma}_{31} = -0.2131$; $\hat{\gamma}_{32} = 0.2131$.

Table 5.3. Estimated values for residual errors

Time	Location	Residual errors		
		u_1 -estimate	u_2 -estimate	u_3 -estimate
1	1	-2.0852	-1.2510	-2.1640
	2	1.3894	-0.1567	-2.1566
	3	-0.8172	1.1237	-3.0398
	4	-2.5630	-1.6336	2.1391
	5	2.7690	1.9038	-2.9422
	6	2.4922	-0.1981	2.5667
	7	4.0701	1.6595	2.6443
	8	-2.6813	2.3459	2.2713
	9	-1.6205	-3.1377	-1.5306
	10	-0.9535	-0.6557	2.2118
2	1	0.4177	-1.8521	-1.5421
	2	1.4548	1.8653	2.9468
	3	-1.1030	0.1722	-0.4396
	4	-2.4882	0.2060	-0.2406
	5	-0.6258	0.7948	0.5418
	6	3.3293	-1.0818	-1.4951
	7	1.2802	1.2572	1.6116
	8	-2.6666	0.8980	-0.9222
	9	-1.1394	-0.9537	0.3575
	10	1.5410	-1.3061	-0.8181

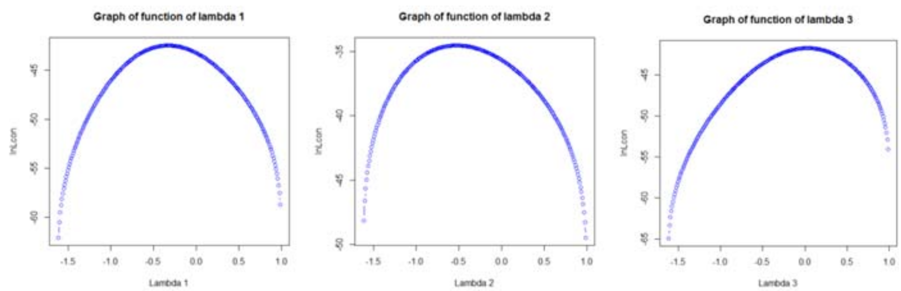

Figure 5.2. Graphs of function of lambda.

By $\mathbf{W}$, we have the acceptable spatial autocorrelation parameter to be $-1.6242 < \lambda_h < 1$. By the method of forming sequence of λ_h with increasing 0.01, we obtain

(1) $\lambda_h = \text{seq}(-1.6142, 0.99, 0.01)$.

(2) For every $\mathbf{u}_{hj}$, $h = 1, 2, 3,$ we insert values of λ_h to (3.10) and find the largest $\ln L_h^{con}$. We use residual error from Table 5.3.

(3) We obtain $\hat{\lambda}_1 = -0.3342$; $\hat{\lambda}_2 = -0.5242$; $\hat{\lambda}_3 = 0.0258$. From (3.12) to (3.15), it follows

$$\hat{\boldsymbol{\theta}}_1 = \begin{bmatrix} \hat{\boldsymbol{\alpha}}_1 \\ \cdots \\ \hat{\boldsymbol{\beta}}_{-1} \end{bmatrix} = \begin{bmatrix} \hat{\alpha}_{11} \\ \hat{\alpha}_{12} \\ \cdots \\ \hat{\beta}_{12} \\ \hat{\beta}_{13} \end{bmatrix} = \begin{bmatrix} 0.1166 \\ -0.3295 \\ \cdots \\ -0.1156 \\ 0.1078 \end{bmatrix}; \quad \hat{\boldsymbol{\theta}}_2 = \begin{bmatrix} \hat{\boldsymbol{\alpha}}_2 \\ \cdots \\ \hat{\boldsymbol{\beta}}_{-2} \end{bmatrix} = \begin{bmatrix} \hat{\alpha}_{21} \\ \hat{\alpha}_{22} \\ \cdots \\ \hat{\beta}_{21} \\ \hat{\beta}_{23} \end{bmatrix} = \begin{bmatrix} 0.1255 \\ 0.0083 \\ \cdots \\ -0.0442 \\ 0.5048 \end{bmatrix};$$

$$\hat{\boldsymbol{\theta}}_3 = \begin{bmatrix} \hat{\boldsymbol{\alpha}}_3 \\ \cdots \\ \hat{\boldsymbol{\beta}}_{-3} \end{bmatrix} = \begin{bmatrix} \hat{\alpha}_{31} \\ \hat{\alpha}_{32} \\ \cdots \\ \hat{\beta}_{31} \\ \hat{\beta}_{32} \end{bmatrix} = \begin{bmatrix} -0.0379 \\ 0.0642 \\ \cdots \\ 0.1036 \\ 1.3270 \end{bmatrix};$$

$\hat{\mu}_1 = 30.4847$; $\quad \hat{\mu}_2 = 9.2146$; $\quad \hat{\mu}_3 = -15.8118$;

$\hat{\gamma}_{11} = -1.1697;$ $\hat{\gamma}_{12} = 1.1697;$ $\hat{\gamma}_{21} = 1.0892;$ $\hat{\gamma}_{22} = -1.0892;$

$\hat{\gamma}_{31} = -0.2252;$ $\hat{\gamma}_{32} = 0.2252.$

Table 5.4. Estimated values for residual errors

Time	Location	Residual errors		
		u_1 -estimate	u_2 -estimate	u_3 -estimate
1	1	-2.0276	-1.7228	-2.2548
	2	1.7354	-0.0777	-2.1559
	3	-0.6877	1.4103	-3.0375
	4	-2.6673	-1.7008	2.1551
	5	2.8813	2.1329	-2.9296
	6	2.5544	-0.0697	2.5846
	7	4.3957	1.7895	2.6734
	8	-2.0044	2.4592	2.2926
	9	-1.0423	-2.6488	-1.4288
	10	-0.8548	-0.7101	2.1931
2	1	0.4360	-2.0388	-1.5636
	2	1.6002	2.0544	2.9229
	3	-1.1994	0.2145	-0.4320
	4	-2.6595	-0.1325	-0.2117
	5	-0.1104	0.5151	0.5204
	6	3.7548	-0.8357	-1.4836
	7	1.7832	1.2665	1.5965
	8	-2.0198	0.7209	-0.8520
	9	-0.9434	-1.0140	0.3263
	10	1.5356	-0.9529	-0.8397

We obtain

$$\hat{\mathbf{\Sigma}}_{\#} = \begin{bmatrix} 60{,}331.13 & 71{,}066.58 & 60{,}705.21 & \cdots & 3{,}943.87 \\ 71{,}066.58 & 83{,}961.02 & 71{,}569.47 & \cdots & 4{,}690.12 \\ 60{,}705.21 & 71{,}569.47 & 61{,}141.64 & \cdots & 3{,}960{,}70 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 3{,}943.87 & 4{,}690.12 & 3{,}983.00 & \cdots & 142{,}414.61 \end{bmatrix}.$$

For the last-stage, we estimate the parameters of equation models (5.1). By (3.16) to (3.18), we obtain

$$\hat{\boldsymbol{\theta}} = \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_{21} \\ \alpha_{22} \\ \alpha_{31} \\ \alpha_{32} \\ \dots \\ \beta_{12} \\ \beta_{13} \\ \beta_{21} \\ \beta_{23} \\ \beta_{31} \\ \beta_{32} \end{bmatrix} = \begin{bmatrix} 0.1286 \\ -0.6923 \\ 0.0440 \\ -0.1209 \\ -0.0037 \\ -0.0974 \\ \dots \\ -0.0335 \\ 0.1649 \\ -0.2279 \\ 0.5671 \\ -0.1679 \\ 1.0580 \end{bmatrix}; \quad \hat{\boldsymbol{\mu}} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} = \begin{bmatrix} 48.1897 \\ 22.6601 \\ 4.1290 \end{bmatrix};$$

$$\hat{\boldsymbol{\gamma}}_1 = \begin{bmatrix} \gamma_{11} \\ \gamma_{21} \\ \gamma_{31} \end{bmatrix} = \begin{bmatrix} -2.9458 \\ -0.2138 \\ -1.1473 \end{bmatrix}; \quad \hat{\boldsymbol{\gamma}}_2 = \begin{bmatrix} \gamma_{12} \\ \gamma_{22} \\ \gamma_{32} \end{bmatrix} = \begin{bmatrix} 2.9458 \\ 0.2138 \\ 1.1473 \end{bmatrix}$$

and the estimated equation models (5.1) are

$$\begin{aligned} \hat{y}_{1i1} &= 48.1897 + 0.1286x_{11i1} - 0.6923x_{12i1} - 0.0335y_{2i1} \\ &\quad + 0.1649y_{3i1} - 2.9458 + \hat{u}_{1i1}, \\ \hat{y}_{2i1} &= 22.6601 + 0.0440x_{21i1} - 0.1209x_{22i1} - 0.2279y_{1i1} \\ &\quad + 0.5671y_{3i1} - 0.2138 + \hat{u}_{2i1}, \\ \hat{y}_{3i1} &= 4.1290 - 0.0037x_{31i1} - 0.0974x_{32i1} - 0.1679y_{1i1} \\ &\quad + 1.0580y_{2i1} - 1.1473 + \hat{u}_{3i1}, \\ \hat{u}_{1i1} &= -0.3342\mathbf{w}_i^t\hat{\mathbf{u}}_{11}, \\ \hat{u}_{2i1} &= -0.5242\mathbf{w}_i^t\hat{\mathbf{u}}_{21}, \end{aligned}$$

$$\hat{u}_{3i1} = 0.0258\mathbf{w}_i^t\hat{\mathbf{u}}_{31},$$

$$\hat{y}_{1i2} = 48.1897 + 0.1286x_{11i2} - 0.6923x_{12i2} - 0.0335y_{2i2}$$
$$+ 0.1649y_{3i2} + 2.9458 + \hat{u}_{1i2},$$

$$\hat{y}_{2i2} = 22.6601 + 0.0440x_{21i2} - 0.1209x_{22i2} - 0.2279y_{1i2}$$
$$+ 0.5671y_{3i2} + 0.2138 + \hat{u}_{2i2},$$

$$\hat{y}_{3i2} = 4.1290 - 0.0037x_{31i2} - 0.0974x_{32i2} - 0.1679y_{1i2}$$
$$+ 1.0580y_{2i2} + 1.1473 + \hat{u}_{3i2},$$

$$\hat{u}_{1i2} = -0.3342\mathbf{w}_i^t\hat{\mathbf{u}}_{12},$$

$$\hat{u}_{2i2} = -0.5242\mathbf{w}_i^t\hat{\mathbf{u}}_{22},$$

$$\hat{u}_{3i2} = 0.0258\mathbf{w}_i^t\hat{\mathbf{u}}_{32}.$$

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