



## **A NEW INTEGRAL TRANSFORM METHOD FOR SOLVING FRACTIONAL DIFFERENTIAL EQUATIONS**

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### **Abstract**

In this paper, we construct the polynomial integral transform of fractional derivative and define the convolution of functions, Also, we prove the convolution theorem for the polynomial integral transform and using this we solve the fractional differential equations which give the convergence of the solution faster as compared to other integral transforms defined on  $[1, \infty)$ .

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## 1. Introduction

Fractional calculus began to attract much more attention of physicists and mathematicians, because of the various interdisciplinary applications can be elegantly modeled with the help of fractional derivatives [13]. They were used in modeling of many physical and chemical processes and in engineering [9, 10, 12, 13]. The mathematical aspects of fractional differential equations and methods of their solution were discussed by many authors: the nonlinear oscillation of earthquake can be modeled with fractional derivative [13], the fluid-dynamic traffic model with fractional derivative [12], and the differential equation with fractional order have recently proved to be valuable tool for modeling of many physical phenomena [10]. The analytic results on the existence and uniqueness of solution to the fractional differential equations have been investigated by many authors [6, 9].

During the last decades, several methods have been used to solve fractional differential equations, fractional partial differential equations, fractional integro-differential equations and dynamic systems containing derivatives such as iteration method [18], the series method [7], the Fourier transform method technique [17], special method for fractional differential equations of rational order or for equations of special type, the Laplace transform method [15], and the operational calculus method [11, 20, 21]. Recently, several mathematical methods such as Adomian decomposition method [11, 20, 21], variational iteration method [20], homotopy perturbation method [25] and homotopy analysis method [21] have been developed to obtain the exact and approximate solutions. Some of these methods use transformation in order to reduce equations into simpler equations or systems of equations and other methods give the solution in a series form which converges to the exact solution.

The polynomial integral transforms are either prototypes or have the same applications as the Laplace transforms. In addition, almost all of these integral transforms use exponential function of parameters as their kernels. Using the exponential function, kernel does not only require complex

mathematical structures but also takes a long before the solution is obtained. Using the Mellin-Barnes integral poses the similar challenges as the Laplace integral transform and its prototypes [5].

In [5], the author introduced a new polynomial integral transform, and discussed the integral transform method for solving differential equations, and also presented the definition and given the proof for the polynomial integral transform. Using this polynomial integral transform, it is shown that the solution of the differential equation converges for  $x \in [1, \infty)$ , also some properties of the polynomial integral transform are established.

In this present paper, we develop the solutions for polynomial integral transform of fractional derivative and using polynomial integral transform of fractional derivative we solve some fractional differential equations.

## 2. Technical Background

In this section, we use some definitions and notations which are given in [5] with details, and present technical preparation needed for further discussion.

**Definition 2.1** (Polynomial integral transform) [5]. Let  $f(x)$  be a function defined for  $x \geq 1$ . Then the integral

$$B(f(x)) = \int_1^\infty f(\ln x)x^{-s-1}dx = F(s) \quad (2.1)$$

is the *polynomial integral transform* of  $f(x)$  for  $x \in [1, \infty)$ , provided the integral converges.

### 2.1. The convergence of the polynomial integral transform [5]

It is shown that the polynomial integral transform converges for variable defined  $[1, \infty)$ , by Taylor's series expansion, then

$$B(f(x)) \leq M \int_1^\infty |f(\ln x)| dx, \text{ where } M > 0.$$

It implies that the polynomial integral transform converges uniformly for a given  $s$ . The function  $f(x)$  must be piecewise continuous. Thus,  $f(x)$  has at most a finite number of discontinuities on any interval  $1 \leq x \leq A$ , and the limit of  $f(x)$  exists at every point of discontinuity.

## 2.2. Existence of the polynomial integral transform

It is shown that the polynomial integral transform exists for  $x \in [1, \infty)$ .

**Theorem 2.1** (Existence theorem for the polynomial integral transform) [5]. *Let  $f(x)$  be a piecewise continuous function on  $[1, \infty)$  and of exponential order. Then the polynomial integral transform exists.*

## 2.3. Properties of the polynomial integral transform

**Theorem 2.2.** *The polynomial integral transform is a linear operator:*

$$B[\alpha_1 f(x) + \alpha_2 g(x)] = \alpha_1 B(f(x)) + \alpha_2 B(g(x)).$$

**Theorem 2.3.** *The inverse polynomial integral transform is also a linear operator:*

$$\alpha_1 f(x) + \alpha_2 g(x) = B^{-1}[\alpha_1 F(s) + \alpha_2 G(s)],$$

where  $B(f(x)) = F(s)$  and  $B(g(x)) = G(s)$ , respectively.

**Theorem 2.4** (First shifting theorem) [5]. *If  $B(f(x)) = F(s)$ , then*

$$B(e^{ax} f(x)) = F(s - a) \text{ for } s > 1.$$

**Theorem 2.5** (Second shifting theorem) [5]. *Let*

$$H_c(x) = 0, \quad 0 \leq x \leq c$$

$$= 1, \quad x \geq c$$

be a unit step function. Then

$$B(H_c f(x - c)) = F(s - c).$$

**Theorem 2.6** [5]. *If  $f(x)$  is a piecewise continuous function on  $[1, \infty)$ , but not of exponential order, then a polynomial integral transform*

$$B(f(x)) \rightarrow 0 \text{ as } s \rightarrow \infty.$$

#### 2.4. The polynomial integral transform of derivatives [5]

**Theorem 2.7** [5]. *If  $f, f', f'', \dots, f^{(n-1)}$  are continuous on  $[1, \infty)$  and if  $f^{(n)}(x)$  is piecewise continuous on  $[1, \infty)$ , then*

$$B(f^n(x)) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0),$$

where  $F(s) = B(f(x))$ .

**Corollary** [5]. *Suppose  $f$  is a continuous function and let  $F(s)$  be the polynomial integral transform. Then we have*

$$B(x^n f(x))(s) = (-1)^n F^{(n)}(s).$$

**Definition 2.2.** The *Caputo fractional derivative* of order  $\alpha > 0$  for a continuous function  $f(t) : (0, \infty) \rightarrow \mathbb{R}$  is defined by

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad n-1 < \alpha < n \quad (2.2)$$

provided that the right-hand side is point-wise defined on  $(0, \infty)$  where the Gamma function is defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

### 3. Main Result

**Lemma 3.1.** *Let  $f(x)$  be a function defined for  $x \geq 1$ . Then the polynomial integral transform of fractional derivative is*

$$B[D^\alpha f(t)] = s^\alpha B(f(t)) - \sum_{k=1}^n s^{\alpha-k} f^{(k-1)}(0), \quad \alpha > 0, \quad n-1 < \alpha \leq n,$$

$$f(t) \in C^n(0, \infty), \quad f^{(n)}(t) \in L_1(0, b), \quad b > 0. \quad (3.1)$$

**Proof.** By using (2.1) and (2.2), we have

$$B[D^\alpha f(t)] = \int_1^\infty D^\alpha f(\ln t) t^{-s-1} dt,$$

put  $\ln t = u$ ,  $t = e^u$ ,  $dt = e^u du$ .

When  $t = 1$ ,  $u = 0$ ,  $t \rightarrow \infty$ ,  $u \rightarrow \infty$ ,

$$\begin{aligned} B[D^\alpha f(t)] &= \int_0^\infty D^\alpha f(u) e^{u(-s-1)} e^u du \\ &= \int_0^\infty D^\alpha f(u) e^{-su} du \end{aligned}$$

by Caputo fractional derivative

$$= \int_0^\infty e^{-su} \frac{1}{\Gamma(n-\alpha)} \int_0^u \frac{f^{(n)}(\xi)}{(u-\xi)^{\alpha-n+1}} d\xi du$$

by changing the order of integration we get

$$= \frac{1}{\Gamma(n-\alpha)} \int_0^\infty \int_\xi^\infty e^{-su} \frac{f^{(n)}(\xi)}{(u-\xi)^{\alpha-n+1}} d\xi du,$$

put  $u - \xi = z$ ,  $u = \xi + z$ ,  $du = dz$ ,

$$\begin{aligned} &= \frac{1}{\Gamma(n-\alpha)} \int_0^\infty f^{(n)}(\xi) \int_0^\infty e^{-s(z+\xi)} z^{n-\alpha-1} dz d\xi \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^\infty e^{-s\xi} f^{(n)}(\xi) d\xi \int_0^\infty e^{-sz} z^{n-\alpha-1} dz \end{aligned}$$

again by substituting  $sz = u$ ,  $z = \frac{u}{s}$ ,  $dz = \frac{1}{s} du$ ,

$$\begin{aligned}
&= \frac{1}{\Gamma(n-\alpha)} \int_0^\infty e^{-s\xi} f^{(n)}(\xi) d\xi \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^{n-\alpha-1} \frac{1}{s} du \\
&= \frac{1}{\Gamma(n-\alpha)} \int_0^\infty e^{-s\xi} f^{(n)}(\xi) d\xi \int_0^\infty e^{-u} \frac{u^{n-\alpha-1}}{s^{n-\alpha}} du \\
&= \frac{1}{\Gamma(n-\alpha)} \int_0^\infty e^{-s\xi} f^{(n)}(\xi) d\xi \frac{\Gamma(n-\alpha)}{s^{n-\alpha}} \\
&= s^{\alpha-n} \int_0^\infty e^{-s\xi} f^{(n)}(\xi) d\xi,
\end{aligned}$$

put  $\xi = \ln \eta$ ,  $\eta = e^\xi$ ,  $d\xi = \frac{1}{\eta} d\eta$ ,

$$\begin{aligned}
&= s^{\alpha-n} \int_1^\infty \eta^{-s-1} f^{(n)}(\ln \eta) d\eta \\
&= s^{\alpha-n} B(f^{(n)}(t)) \\
&= s^{\alpha-n} \{s^{(n)} B(f(t)) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0)\} \\
&= s^\alpha B(f(t)) - s^{\alpha-1} f(0) - s^{\alpha-2} f'(0) - \dots - f^{(\alpha-n)} f^{(n-1)}(0) \\
&= s^\alpha B(f(t)) - \sum_{k=1}^n s^{\alpha-k} f^{(k-1)}(0).
\end{aligned}$$

**Theorem 3.2.** Let  $1 < \alpha < 2$  and  $a, b \in \mathbb{R}$ . Then the fractional differential equation

$$D^\alpha y(t) + aDy(t) + by(t) = 0$$

with the initial conditions  $y(0) = k_0$ ,  $y'(0) = k_1$  has its solution

$$\begin{aligned} y(t) &= k_0 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{\Gamma(m+l+1)(-a)^l t^{(\alpha-1)l+\alpha m}}{\Gamma((\alpha-1)l+\alpha m+1)l!} \\ &\quad + k_1 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{\Gamma(m+l+1)(-a)^l t^{(\alpha-1)l+\alpha m+1}}{\Gamma((\alpha-1)l+\alpha m+2)l!} \\ &\quad + ak_0 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{\Gamma(m+l+1)(-a)^l t^{(\alpha-1)l+\alpha m+\alpha-1}}{\Gamma((\alpha-1)l+\alpha m+\alpha)l!}. \end{aligned}$$

**Proof.** The fractional differential equation is

$$D^\alpha y(t) + aDy(t) + by(t) = 0.$$

Applying the polynomial integral transform of fractional derivative, using (3.1), we have

$$\begin{aligned} B[D^\alpha y(t) + aDy(t) + by(t)] &= 0, \\ s^\alpha y(s) - s^{\alpha-1}y(0) - s^{(\alpha-2)}y'(0) + asy(s) - ay(0) + by(s) &= 0, \\ (s^\alpha + as + b)y(s) &= k_0 s^{\alpha-1} + k_1 s^{\alpha-2} + ak_0, \\ y(s) &= \frac{k_0 s^{\alpha-1} + k_1 s^{\alpha-2} + ak_0}{s^\alpha + as + b}, \end{aligned} \tag{3.2}$$

since

$$\begin{aligned} \frac{1}{s^\alpha + as + b} &= \frac{s^{-1}}{s^{\alpha-1} + a + bs^{-1}} \\ &= \frac{s^{-1}}{(s^{\alpha-1} + a)\left(1 + \frac{bs^{-1}}{s^{\alpha-1} + a}\right)} \end{aligned}$$

$$\begin{aligned}
&= \frac{s^{-1}}{(s^{\alpha-1} + a)} \left( 1 + \frac{bs^{-1}}{s^{\alpha-1} + a} \right)^{-1} \\
&= \frac{s^{-1}}{(s^{\alpha-1} + a)} \sum_{m=0}^{\infty} (-1)^m \left( \frac{bs^{-1}}{s^{\alpha-1} + a} \right)^m \\
&= \sum_{m=0}^{\infty} (-b)^m \frac{s^{-m-1}}{(s^{\alpha-1} + a)^{m+1}} \\
&= \sum_{m=0}^{\infty} (-b)^m \frac{s^{-m-1}}{(s^{\alpha-1} (1 + as^{1-\alpha}))^{m+1}} \\
&= \sum_{m=0}^{\infty} (-b)^m s^{-\alpha m - \alpha} \{(1 + as^{1-\alpha})^{m+1}\}^{-1} \\
&= \sum_{m=0}^{\infty} (-b)^m s^{-\alpha m - \alpha} \sum_{l=0}^{\infty} \binom{m+l}{l} (-as^{1-\alpha})^l \\
&= \sum_{m=0}^{\infty} (-b)^m \sum_{l=0}^{\infty} \binom{m+l}{l} (-a)^l s^{l-\alpha l - \alpha m - \alpha}.
\end{aligned}$$

Therefore, by using equation (3.2), we have

$$\begin{aligned}
y(s) &= \{k_0 s^{\alpha-1} + k_1 s^{\alpha-2} + a k_0\} \left\{ \sum_{m=0}^{\infty} (-b)^m \sum_{l=0}^{\infty} \binom{m+l}{l} (-a)^l s^{l-\alpha l - \alpha m - \alpha} \right\} \\
y(s) &= k_0 \sum_{m=0}^{\infty} (-b)^m \sum_{l=0}^{\infty} \binom{m+l}{l} (-a)^l s^{l-\alpha l - \alpha m - 1} \\
&\quad + k_1 \sum_{m=0}^{\infty} (-b)^m \sum_{l=0}^{\infty} \binom{m+l}{l} (-a)^l s^{l-\alpha l - \alpha m - 2} \\
&\quad + a k_0 \sum_{m=0}^{\infty} (-b)^m \sum_{l=0}^{\infty} \binom{m+l}{l} (-a)^l s^{l-\alpha l - \alpha m - \alpha}.
\end{aligned}$$

Now, taking inverse polynomial transform, we get

$$\begin{aligned}
 y(t) &= k_0 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{(m+l)!(-a)^l}{\Gamma((\alpha-1)l+\alpha m+1)} \frac{t^{(\alpha-1)l+\alpha m}}{l!} \\
 &\quad + k_1 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{(m+l)!(-a)^l}{\Gamma((\alpha-1)l+\alpha m+2)} \frac{t^{(\alpha-1)l+\alpha m+1}}{l!} \\
 &\quad + ak_0 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{(m+l)!(-a)^l}{\Gamma((\alpha-1)l+\alpha m+\alpha)} \frac{t^{(\alpha-1)l+\alpha m+\alpha-1}}{l!}.
 \end{aligned}$$

Hence

$$\begin{aligned}
 y(t) &= k_0 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{\Gamma(m+l+1)(-a)^l}{\Gamma((\alpha-1)l+\alpha m+1)} \frac{t^{(\alpha-1)l+\alpha m}}{l!} \\
 &\quad + k_1 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{\Gamma(m+l+1)(-a)^l}{\Gamma((\alpha-1)l+\alpha m+2)} \frac{t^{(\alpha-1)l+\alpha m+1}}{l!} \\
 &\quad + ak_0 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{\Gamma(m+l+1)(-a)^l}{\Gamma((\alpha-1)l+\alpha m+\alpha)} \frac{t^{(\alpha-1)l+\alpha m+\alpha-1}}{l!}.
 \end{aligned}$$

**Lemma 3.3.** If  $a = 0$  in above equation, then

$$D^\alpha y(t) + by(t) = 0, \quad 1 < \alpha \leq 2$$

with initial conditions  $y(0) = k_0$  and  $y'(0) = k_1$  has a solution

$$y(t) = k_0 \sum_{m=0}^{\infty} \frac{(-bt^\alpha)^m}{\Gamma(\alpha m+1)} + k_1 t \sum_{m=0}^{\infty} \frac{(-bt^\alpha)^m}{\Gamma(\alpha m+2)}. \quad (3.3)$$

**Theorem 3.4.** Let  $0 < \alpha < 1$  and  $b \in \mathbb{R}$ . Then the equation

$$D^\alpha y(t) - by(t) = 0$$

with initial condition  $y(0) = k_0$  has a solution

$$y(t) = k_0 \sum_{m=0}^{\infty} \frac{(bt^\alpha)^m}{\Gamma(\alpha m + 1)}. \quad (3.4)$$

**Proof.** The fractional differential equation is

$$D^\alpha y(t) - by(t) = 0.$$

Applying the polynomial integral transform of fractional derivative, using (3.1), we have

$$\begin{aligned} B(D^\alpha y(t) - by(t)) &= 0, \\ s^\alpha y(s) - s^{\alpha-1}y(0) - s^{\alpha-2}y'(0) - \dots - by(s) &= 0, \\ s^\alpha y(s) - s^{\alpha-1}k_0 - by(s) &= 0, \\ (s^\alpha - b)y(s) &= k_0 s^{\alpha-1}, \\ y(s) &= \frac{k_0 s^{\alpha-1}}{s^\alpha - b} = \frac{k_0 s^{-1}}{1 - bs^{-\alpha}} \\ &= k_0 s^{-1} (1 - bs^{-\alpha})^{-1} \\ &= k_0 s^{-1} \sum_{m=0}^{\infty} (bs^{-\alpha})^m \\ &= k_0 \sum_{m=0}^{\infty} b^m s^{-\alpha m - 1}. \end{aligned} \quad (3.5)$$

Using inverse polynomial integral transform, we have

$$\begin{aligned} y(t) &= k_0 \sum_{m=0}^{\infty} b^m B^{-1}(s^{-\alpha m - 1}) \\ &= k_0 \sum_{m=0}^{\infty} \frac{b^m t^{\alpha m}}{\Gamma(\alpha m + 1)}. \end{aligned}$$

**Theorem 3.5.** Let  $1 < \alpha < 2$  and  $a, b \in \mathbb{R}$ . Then the fractional differential equation

$$D^2 y(t) + aD^\alpha y(t) + by(t) = 0$$

with initial conditions  $y(0) = k_0$ ,  $y'(0) = k_1$  has a solution

$$\begin{aligned} y(t) = & k_0 \sum_{m=0}^{\infty} \frac{(-b)^m t^{2m}}{m!} \sum_{l=0}^{\infty} \frac{\Gamma(m+l+1)(-at^{2-\alpha})^l}{\Gamma((2-\alpha)l+2m+1)l!} \\ & + k_1 \sum_{m=0}^{\infty} \frac{(-b)^m t^{2m+1}}{m!} \sum_{l=0}^{\infty} \frac{\Gamma(m+l+1)(-at^{2-\alpha})^l}{\Gamma((2-\alpha)l+2m+2)l!} \\ & + ak_0 \sum_{m=0}^{\infty} \frac{(-b)^m t^{2m-\alpha+2}}{m!} \sum_{l=0}^{\infty} \frac{\Gamma(m+l+1)(-at^{2-\alpha})^l}{\Gamma((2-\alpha)l+2m-\alpha+3)l!} \\ & + ak_1 \sum_{m=0}^{\infty} \frac{(-b)^m t^{2m-\alpha+3}}{m!} \sum_{l=0}^{\infty} \frac{\Gamma(m+l+1)(-at^{2-\alpha})^l}{\Gamma((2-\alpha)l+2m-\alpha+4)l!}. \end{aligned}$$

**Proof.** The fractional differential equation is

$$D^2 y(t) + aD^\alpha y(t) + by(t) = 0.$$

Applying the polynomial integral transform of fractional derivative, using (3.1), we have

$$\begin{aligned} B(D^2 y(t)) + aB(D^\alpha y(t)) + bB(y(t)) &= 0, \\ s^2 y(s) - sy(0) - y'(0) + a\{s^\alpha y(s) - s^{\alpha-1}y(0) - s^{\alpha-2}y'(0) - \dots\} + by(s) &= 0, \\ s^2 y(s) - sk_0 - k_1 + as^\alpha y(s) - as^{\alpha-1}k_0 - as^{\alpha-2}k_1 + by(s) &= 0, \\ (s^2 + as^\alpha + b)y(s) &= sk_0 + k_1 + as^{\alpha-1}k_0 + as^{\alpha-2}k_1, \\ y(s) &= \frac{sk_0 + k_1 + as^{\alpha-1}k_0 + as^{\alpha-2}k_1}{s^2 + as^\alpha + b}, \end{aligned} \tag{3.6}$$

since

$$\begin{aligned}
\frac{1}{s^2 + as^\alpha + b} &= \frac{s^{-\alpha}}{s^{2-\alpha} + a + bs^{-\alpha}} \\
&= \frac{s^{-\alpha}}{(s^{2-\alpha} + a) \left\{ 1 + \frac{bs^{-\alpha}}{s^{2-\alpha} + a} \right\}} \\
&= \frac{s^{-\alpha}}{(s^{2-\alpha} + a) \left\{ 1 + \frac{bs^{-\alpha}}{s^{2-\alpha} + a} \right\}^{-1}} \\
&= \frac{s^{-\alpha}}{(s^{2-\alpha} + a)} \sum_{m=0}^{\infty} \left\{ \frac{-bs^{-\alpha}}{s^{2-\alpha} + a} \right\}^m \\
&= \sum_{m=0}^{\infty} \frac{(-b)^m s^{-\alpha m - \alpha}}{(s^{2-\alpha} + a)^{m+1}} \\
&= \sum_{m=0}^{\infty} \frac{(-b)^m s^{-\alpha m - \alpha}}{[s^{2-\alpha} (1 + as^{\alpha-2})]^{m+1}} \\
&= \sum_{m=0}^{\infty} \frac{(-b)^m s^{-2m-2}}{(1 + as^{\alpha-2})^{m+1}} \\
&= \sum_{m=0}^{\infty} (-b)^m s^{-2m-2} \{(1 + as^{\alpha-2})^{m+1}\}^{-1} \\
&= \sum_{m=0}^{\infty} (-b)^m s^{-2m-2} \sum_{l=0}^{\infty} \binom{m+1}{l} (-as^{\alpha-2})^l \\
&= \sum_{m=0}^{\infty} (-b)^m \sum_{l=0}^{\infty} \binom{m+l}{l} (-a)^l s^{(\alpha-2)l-2m-2}.
\end{aligned}$$

Therefore, by using equation (3.6), we have

$$\begin{aligned}
 y(s) &= \{k_0 s + k_1 + ak_0 s^{\alpha-1} + ak_1 s^{\alpha-2}\} \left\{ \sum_{m=0}^{\infty} (-b)^m \sum_{l=0}^{\infty} \binom{m+l}{l} (-a)^l s^{(\alpha-2)l-2m-2} \right\} \\
 y(s) &= k_0 \sum_{m=0}^{\infty} (-b)^m \sum_{l=0}^{\infty} \binom{m+l}{l} (-a)^l s^{(\alpha-2)l-2m-1} \\
 &\quad + k_1 \sum_{m=0}^{\infty} (-b)^m \sum_{l=0}^{\infty} \binom{m+l}{l} (-a)^l s^{(\alpha-2)l-2m-2} \\
 &\quad + ak_0 \sum_{m=0}^{\infty} (-b)^m \sum_{l=0}^{\infty} \binom{m+l}{l} (-a)^l s^{(\alpha-2)l-2m+\alpha-3} \\
 &\quad + ak_1 \sum_{m=0}^{\infty} (-b)^m \sum_{l=0}^{\infty} \binom{m+l}{l} (-a)^l s^{(\alpha-2)l-2m+\alpha-4}.
 \end{aligned}$$

Now, by taking inverse polynomial integral transform, we have

$$\begin{aligned}
 y(t) &= k_0 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{(m+l)!(-a)^l t^{(2-\alpha)l+2m}}{\Gamma((2-\alpha)l+2m+1)l!} \\
 &\quad + k_1 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{(m+l)!(-a)^l t^{(2-\alpha)l+2m+1}}{\Gamma((2-\alpha)l+2m+2)l!} \\
 &\quad + ak_0 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{(m+l)!(-a)^l t^{(2-\alpha)l+2m-\alpha+2}}{\Gamma((2-\alpha)l+2m-\alpha+3)l!} \\
 &\quad + ak_1 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{(m+l)!(-a)^l t^{(2-\alpha)l+2m-\alpha+3}}{\Gamma((2-\alpha)l+2m-\alpha+4)l!} \\
 y(t) &= k_0 \sum_{m=0}^{\infty} \frac{(-b)^m t^{2m}}{m!} \sum_{l=0}^{\infty} \frac{\Gamma(m+l+1)(-at^{2-\alpha})^l}{\Gamma((2-\alpha)l+2m+1)l!}
 \end{aligned}$$

$$\begin{aligned}
& + k_1 \sum_{m=0}^{\infty} \frac{(-b)^m t^{2m+1}}{m!} \sum_{l=0}^{\infty} \frac{\Gamma(m+l+1)(-at^{2-\alpha})^l}{\Gamma((2-\alpha)l+2m+2)l!} \\
& + ak_0 \sum_{m=0}^{\infty} \frac{(-b)^m t^{2m-\alpha+2}}{m!} \sum_{l=0}^{\infty} \frac{\Gamma(m+l+1)(-at^{2-\alpha})^l}{\Gamma((2-\alpha)l+2m-\alpha+3)l!} \\
& + ak_1 \sum_{m=0}^{\infty} \frac{(-b)^m t^{2m-\alpha+3}}{m!} \sum_{l=0}^{\infty} \frac{\Gamma(m+l+1)(-at^{2-\alpha})^l}{\Gamma((2-\alpha)l+2m-\alpha+4)l!}.
\end{aligned}$$

**Definition 3.1.** The convolution of functions  $f(t)$  and  $g(t)$  is denoted by  $f(t) * g(t)$  and is defined by

$$f(t) * g(t) = \int_1^t f(\ln u) g(\ln t - \ln u) \frac{1}{u} du$$

put

$$\ln t - \ln u = \ln v,$$

$$\ln u = \ln \frac{t}{v},$$

$$u = \frac{t}{v}, \quad du = -\frac{1}{v^2} dv$$

when

$$u = 1, v = t \quad \text{and} \quad u = t, v = 1,$$

then

$$\begin{aligned}
f(t) * g(t) &= \int_1^t g(\ln v) f(\ln t - \ln u) \frac{1}{v} du \\
&= g(t) * f(t).
\end{aligned}$$

This shows that the convolution of  $f(t)$  and  $g(t)$  obeys the commutative law of algebra.

**Theorem 3.6** (Convolution theorem). *If  $B[f(t)] = F(s)$  and  $B[g(t)] = G(s)$ , then*

$$B[f(t) * g(t)] = B\left[\int_1^t f(\ln u)g(\ln t - \ln u)\frac{1}{u}du\right] = F(s)G(s).$$

**Proof.** By (3.1), we have

$$\begin{aligned} B[f(t) * g(t)] &= B\left[\int_1^t f(\ln u)g(\ln t - \ln u)\frac{1}{u}du\right] \\ &= \int_{t=1}^{\infty} t^{-s-1} \left[ \int_1^t f(\ln u)g(\ln t - \ln u)\frac{1}{u}du \right] dt \end{aligned}$$

by changing the order of integration, we get

$$\begin{aligned} &= \int_{u=1}^{\infty} \left[ \int_{t=u}^{t=\infty} t^{-s-1} f(\ln u)g(\ln t - \ln u)\frac{1}{u}dt \right] du \\ &= \int_{u=1}^{\infty} \frac{f(\ln u)}{u} \left[ \int_{t=u}^{t=\infty} t^{-s-1} g(\ln t - \ln u)dt \right] du \end{aligned}$$

put

$$\ln t - \ln u = \ln v,$$

$$\ln t = \ln(uv),$$

$$t = uv, \quad dt = udv$$

when

$$t = u, v = 1 \quad \text{and} \quad t \rightarrow \infty, v \rightarrow \infty,$$

then

$$\begin{aligned} B[f(t) * g(t)] &= \int_{u=1}^{\infty} \frac{f(\ln u)}{u} \left[ \int_{v=1}^{\infty} (uv)^{-s-1} g(\ln v) u dv \right] du \\ &= \int_{u=1}^{\infty} u^{-s-1} f(\ln u) du \int_{v=1}^{\infty} v^{-s-1} g(\ln v) dv \\ &= F(s)G(s). \end{aligned}$$

**Theorem 3.7.** Let  $0 < \alpha < 1$  and  $a, b \in \mathbb{R}$ . Then the fractional differential equation

$$D^\alpha y(t) + aDy(t) + by(t) = f(t)$$

with initial conditions  $y(0) = k_0$ ,  $y'(0) = k_1$ ,  $y''(0) = k_2$  has its solution

$$\begin{aligned} y(t) &= \sum_{m=0}^{\infty} (-b)^m \sum_{l=0}^{\infty} \binom{m+l}{l} (-a)^l G(t) \\ &+ ak_0 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{(m+l)!(-a)^l t^{(\alpha-1)l+\alpha m-1}}{\Gamma((\alpha-1)l+\alpha m+\alpha)l!} \\ &+ k_0 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{(m+l)!(-a)^l t^{(\alpha-1)l+\alpha m}}{\Gamma((\alpha-1)l+\alpha m+1)l!} \\ &+ k_1 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{(m+l)!(-a)^l t^{(\alpha-1)l+\alpha m+1}}{\Gamma((\alpha-1)l+\alpha m+2)l!} \\ &+ k_2 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{(m+l)!(-a)^l t^{(\alpha-1)l+\alpha m+2}}{\Gamma((\alpha-1)l+\alpha m+3)l!}. \end{aligned}$$

**Proof.** The fractional differential equation is

$$D^\alpha y(t) + aDy(t) + by(t) = f(t)$$

with initial conditions  $y(0) = k_0$ ,  $y'(0) = k_1$ ,  $y''(0) = k_2$ .

Applying the polynomial integral transform of fractional derivative, using (3.1), we have

$$\begin{aligned} B[D^\alpha y(t) + aDy(t) + by(t)] &= B[f(t)], \\ s^\alpha y(s) - s^{\alpha-1}y(0) - s^{\alpha-2}y'(0) - s^{\alpha-3}y''(0) - \cdots + asy(s) - ay(0) + by(s) \\ &= B(f(t)), \\ (s^\alpha + as + b)y(s) &= F(s) + k_0s^{\alpha-1} + k_1s^{\alpha-2} + k_2s^{\alpha-3} + ak_0, \end{aligned}$$

$$y(s) = \frac{F(s) + k_0 s^{\alpha-1} + k_1 s^{\alpha-2} + k_2 s^{\alpha-3} + ak_0}{s^\alpha + as + b}, \quad (3.7)$$

$$\begin{aligned} y(s) &= \frac{F(s)}{s^\alpha + as + b} + \frac{ak_0}{s^\alpha + as + b} + \frac{k_0 s^{\alpha-1}}{s^\alpha + as + b} \\ &\quad + \frac{k_1 s^{\alpha-2}}{s^\alpha + as + b} + \frac{k_2 s^{\alpha-3}}{s^\alpha + as + b}, \end{aligned} \quad (3.8)$$

since

$$\begin{aligned} \frac{1}{s^\alpha + as + b} &= \frac{s^{-1}}{s^{\alpha-1} + a + bs^{-1}} \\ &= \frac{s^{-1}}{(s^{\alpha-1} + a) \left( 1 + \frac{bs^{-1}}{s^{\alpha-1} + a} \right)} \\ &= \frac{s^{-1}}{s^{\alpha-1} + a} \left( 1 + \frac{bs^{-1}}{s^{\alpha-1} + a} \right)^{-1} \\ &= \frac{s^{-1}}{s^{\alpha-1} + a} \sum_{m=0}^{\infty} (-1)^m \left( \frac{bs^{-1}}{s^{\alpha-1} + a} \right)^m \\ &= \sum_{m=0}^{\infty} (-b)^m \frac{s^{-m-1}}{(s^{\alpha-1} + a)^{m+1}} \\ &= \sum_{m=0}^{\infty} (-b)^m \frac{s^{-m-1}}{(s^{\alpha-1} (1 + as^{1-\alpha}))^{m+1}} \\ &= \sum_{m=0}^{\infty} (-b)^m \frac{s^{-\alpha m - \alpha}}{(1 + as^{1-\alpha})^{m+1}} \\ &= \sum_{m=0}^{\infty} (-b)^m s^{-\alpha m - \alpha} \{(1 + as^{1-\alpha})^{m+1}\}^{-1} \end{aligned}$$

$$\begin{aligned}
&= \sum_{m=0}^{\infty} (-b)^m s^{-\alpha m - \alpha} \sum_{l=0}^{\infty} \binom{m+l}{l} (-as^{1-\alpha})^l \\
&= \sum_{m=0}^{\infty} (-b)^m \sum_{l=0}^{\infty} \binom{m+l}{l} (-a)^l s^{l-\alpha l - \alpha m - \alpha}.
\end{aligned}$$

Therefore, using equation (3.7), we have

$$\begin{aligned}
y(s) &= F(s) \sum_{m=0}^{\infty} (-b)^m \sum_{l=0}^{\infty} \binom{m+l}{l} (-a)^l s^{l-\alpha l - \alpha m - \alpha} \\
&\quad + ak_0 \sum_{m=0}^{\infty} (-b)^m \sum_{l=0}^{\infty} \binom{m+l}{l} (-a)^l s^{l-\alpha l - \alpha m - \alpha} \\
&\quad + k_0 \sum_{m=0}^{\infty} (-b)^m \sum_{l=0}^{\infty} \binom{m+l}{l} (-a)^l s^{l-\alpha l - \alpha m - 1} \\
&\quad + k_1 \sum_{m=0}^{\infty} (-b)^m \sum_{l=0}^{\infty} \binom{m+l}{l} (-a)^l s^{l-\alpha l - \alpha m - 2} \\
&\quad + k_2 \sum_{m=0}^{\infty} (-b)^m \sum_{l=0}^{\infty} \binom{m+l}{l} (-a)^l s^{l-\alpha l - \alpha m - 3}.
\end{aligned}$$

Now taking the inverse polynomial integral transform, we have

$$\begin{aligned}
y(t) &= B^{-1} \left\{ \left( F(s) + ak_0 \right) \sum_{m=0}^{\infty} (-b)^m \sum_{l=0}^{\infty} \binom{m+l}{l} (-a)^l s^{l-\alpha l - \alpha m - \alpha} \right. \\
&\quad + k_0 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{(m+l)!(-a)^l t^{(\alpha-1)l+\alpha m}}{\Gamma((\alpha-1)l+\alpha m+1)l!} \\
&\quad + k_1 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{(m+l)!(-a)^l t^{(\alpha-1)l+\alpha m+1}}{\Gamma((\alpha-1)l+\alpha m+2)l!} \\
&\quad \left. + k_2 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{(m+l)!(-a)^l t^{(\alpha-1)l+\alpha m+2}}{\Gamma((\alpha-1)l+\alpha m+3)l!} \right\}.
\end{aligned}$$

The inverse of the first term depends upon the nature of  $F(s)$ . We apply partial fraction or convolution theorem. Let the inverse of first term be  $G(t)$ .

Then

$$\begin{aligned}
 y(t) = & \sum_{m=0}^{\infty} (-b)^m \sum_{l=0}^{\infty} \binom{m+l}{l} (-a)^l G(t) \\
 & + ak_0 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{(m+l)!(-a)^l t^{(\alpha-1)l+\alpha m-1}}{\Gamma((\alpha-1)l+\alpha m+\alpha)l!} \\
 & + k_0 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{(m+l)!(-a)^l t^{(\alpha-1)l+\alpha m}}{\Gamma((\alpha-1)l+\alpha m+1)l!} \\
 & + k_1 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{(m+l)!(-a)^l t^{(\alpha-1)l+\alpha m+1}}{\Gamma((\alpha-1)l+\alpha m+2)l!} \\
 & + k_2 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{(m+l)!(-a)^l t^{(\alpha-1)l+\alpha m+2}}{\Gamma((\alpha-1)l+\alpha m+3)l!}.
 \end{aligned}$$

**Theorem 3.8.** Let  $0 < \alpha < 1$  and  $a, b \in \mathbb{R}$ . Then the fractional differential equation

$$D^2 y(t) + a D^\alpha y(t) + b y(t) = f(t)$$

with initial conditions  $y(0) = k_0$ ,  $y'(0) = k_1$ ,  $y''(0) = k_2$  has its solution

$$\begin{aligned}
 y(t) = & \sum_{m=0}^{\infty} (-b)^m \sum_{l=0}^{\infty} \binom{m+l}{l} (-a)^l G(t) \\
 & + k_1 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{(m+l)!(-a)^l t^{(2-\alpha)l+2m+1}}{\Gamma((2-\alpha)l+2m+2)l!} \\
 & + k_0 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{(m+l)!(-a)^l t^{(2-\alpha)l+2m}}{\Gamma((2-\alpha)l+2m+1)l!}
 \end{aligned}$$

$$\begin{aligned}
& + ak_0 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{(m+l)!(-a)^l t^{(2-\alpha)l+2m-\alpha+2}}{\Gamma((2-\alpha)+2m-\alpha+3)l!} \\
& + ak_1 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{(m+l)!(-a)^l t^{(2-\alpha)l+2m-\alpha+3}}{\Gamma((2-\alpha)l+2m-\alpha+4)l!} \\
& + ak_2 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{(m+l)!(-a)^l t^{(2-\alpha)l+2m-\alpha+4}}{\Gamma((2-\alpha)l+2m-\alpha+5)l!}.
\end{aligned}$$

**Proof.** The fractional differential equation is

$$D^2 y(t) + aD^\alpha y(t) + by(t) = f(t)$$

with initial conditions  $y(0) = k_0$ ,  $y'(0) = k_1$ ,  $y''(0) = k_2$ .

Applying the polynomial integral transform of fractional derivative, using (3.1), we have

$$\begin{aligned}
B[D^2 y(t) + aD^\alpha y(t) + by(t)] &= B[f(t)], \\
s^2 y(s) - sy(0) - y'(0) + a\{s^\alpha y(s) - s^{\alpha-1}y(0) - s^{\alpha-2}y'(0) - \dots\} + by(s) \\
&= B(f(t)), \\
s^2 y(s) - sk_0 - k_1 + as^\alpha y(s) - ak_0 s^{\alpha-1} - ak_1 s^{\alpha-2} - ak_2 s^{\alpha-3} + by(s) \\
&= F(s), \\
(s^2 + as^\alpha + b)y(s) &= k_0 s + k_1 + ak_0 s^{\alpha-1} + ak_1 s^{\alpha-2} + ak_2 s^{\alpha-3} + F(s), \\
y(s) &= \frac{F(s) + k_0 s + k_1 + ak_0 s^{\alpha-1} + ak_1 s^{\alpha-2} + ak_2 s^{\alpha-3}}{s^2 + as^\alpha + b}, \tag{3.9}
\end{aligned}$$

since

$$\begin{aligned}
\frac{1}{s^2 + as^\alpha + b} &= \frac{s^{-\alpha}}{s^{2-\alpha} + a + bs^{-\alpha}} \\
&= \frac{s^{-\alpha}}{s^{2-\alpha} + a} \left\{ 1 + \frac{bs^{-\alpha}}{s^{2-\alpha} + a} \right\}^{-1}
\end{aligned}$$

$$\begin{aligned}
&= \frac{s^{-\alpha}}{s^{2-\alpha} + a} \sum_{m=0}^{\infty} \left( \frac{-bs^{-\alpha}}{s^{2-\alpha} + a} \right)^m \\
&= \sum_{m=0}^{\infty} \frac{(-b)^m s^{-\alpha m - \alpha}}{(s^{2-\alpha} + a)^{m+1}} \\
&= \sum_{m=0}^{\infty} \frac{(-b)^m s^{-\alpha m - \alpha}}{(s^{2-\alpha} (1 + as^{\alpha-1}))^{m+1}} \\
&= \sum_{m=0}^{\infty} \frac{(-b)^m s^{-2m-2}}{(1 + as^{\alpha-2})^{m+1}} \\
&= \sum_{m=0}^{\infty} (-b)^m s^{-2m-2} \{ (1 + as^{\alpha-2})^{m+1} \}^{-1} \\
&= \sum_{m=0}^{\infty} (-b)^m s^{-2m-2} \sum_{l=0}^{\infty} \binom{m+l}{l} (-as^{\alpha-2})^l \\
&= \sum_{m=0}^{\infty} (-b)^m \sum_{l=0}^{\infty} \binom{m+l}{l} (-a)^l s^{(\alpha-2)l-2m-2}.
\end{aligned}$$

Therefore, using equation (3.9), we have

$$\begin{aligned}
y(s) &= \{F(s) + k_0 s + k_1 + ak_0 s^{\alpha-1} + ak_1 s^{\alpha-2} + ak_2 s^{\alpha-3}\} \\
&\quad \cdot \left\{ \sum_{m=0}^{\infty} (-b)^m \sum_{l=0}^{\infty} \binom{m+l}{l} (-a)^l s^{(\alpha-2)l-2m-2} \right\} \\
y(s) &= (F(s) + k_1) \sum_{m=0}^{\infty} (-b)^m \sum_{l=0}^{\infty} \binom{m+l}{l} (-a)^l s^{(\alpha-2)l-2m-2} \\
&\quad + k_0 \sum_{m=0}^{\infty} (-b)^m \sum_{l=0}^{\infty} \binom{m+l}{l} (-a)^l s^{(\alpha-2)l-2m-1}
\end{aligned}$$

$$\begin{aligned}
& + ak_0 \sum_{m=0}^{\infty} (-b)^m \sum_{l=0}^{\infty} \binom{m+l}{l} (-a)^l s^{(\alpha-2)l-2m+\alpha-3} \\
& + ak_1 \sum_{m=0}^{\infty} (-b)^m \sum_{l=0}^{\infty} \binom{m+l}{l} (-a)^l s^{(\alpha-2)l-2m+\alpha-4} \\
& + k_2 \sum_{m=0}^{\infty} (-b)^m \sum_{l=0}^{\infty} \binom{m+l}{l} (-a)^l s^{(\alpha-2)l-2m+\alpha-5}.
\end{aligned}$$

Now, taking the inverse polynomial integral transform, we have

$$\begin{aligned}
y(t) = & B^{-1} \left\{ \left( F(s) + k_1 \right) \sum_{m=0}^{\infty} (-b)^m \sum_{l=0}^{\infty} \binom{m+l}{l} (-a)^l s^{(\alpha-2)l-2m-2} \right. \\
& + k_0 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{(m+l)!(-a)^l t^{(2-\alpha)l+2m}}{\Gamma((2-\alpha)l+2m+1)l!} \\
& + ak_0 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{(m+l)!(-a)^l t^{(2-\alpha)l+2m-\alpha+2}}{\Gamma((2-\alpha)+2m-\alpha+3)l!} \\
& + ak_1 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{(m+l)!(-a)^l t^{(2-\alpha)l+2m-\alpha+3}}{\Gamma((2-\alpha)l+2m-\alpha+4)l!} \\
& \left. + ak_2 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{(m+l)!(-a)^l t^{(2-\alpha)l+2m-\alpha+4}}{\Gamma((2-\alpha)l+2m-\alpha+5)l!} \right\}.
\end{aligned}$$

The inverse of the first term depends on the nature of  $F(s)$  and depending on its nature, we will apply inverse by partial fraction or by convolution theorem.

Let the inverse of the first term be  $G(t)$ . Then we get

$$\begin{aligned}
 y(t) &= \sum_{m=0}^{\infty} (-b)^m \sum_{l=0}^{\infty} \binom{m+l}{l} (-a)^l G(t) \\
 &\quad + k_1 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{(m+l)!(-a)^l t^{(2-\alpha)l+2m+1}}{\Gamma((2-\alpha)l+2m+2)l!} \\
 &\quad + k_0 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{(m+l)!(-a)^l t^{(2-\alpha)l+2m}}{\Gamma((2-\alpha)l+2m+1)l!} \\
 &\quad + ak_0 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{(m+l)!(-a)^l t^{(2-\alpha)l+2m-\alpha+2}}{\Gamma((2-\alpha)+2m-\alpha+3)l!} \\
 &\quad + ak_1 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{(m+l)!(-a)^l t^{(2-\alpha)l+2m-\alpha+3}}{\Gamma((2-\alpha)l+2m-\alpha+4)l!} \\
 &\quad + ak_2 \sum_{m=0}^{\infty} \frac{(-b)^m}{m!} \sum_{l=0}^{\infty} \frac{(m+l)!(-a)^l t^{(2-\alpha)l+2m-\alpha+4}}{\Gamma((2-\alpha)l+2m-\alpha+5)l!}.
 \end{aligned}$$

**Example 3.1.** The fractional differential equation

$$D^{\frac{5}{4}}y(t) - 2Dy(t) - 3y(t) = 0$$

with initial conditions  $y(0) = k_0$  and  $y'(0) = k_1$  has a solution

$$\begin{aligned}
 y(t) &= k_0 \sum_{m=0}^{\infty} \frac{3^m}{m!} \sum_{l=0}^{\infty} \frac{\Gamma(m+l+1)t^{\frac{l}{4}+\frac{5}{4}m}}{\Gamma\left(\frac{l}{4}+\frac{5m}{4}+1\right)l!} \\
 &\quad + k_1 \sum_{m=0}^{\infty} \frac{3^m}{m!} \sum_{l=0}^{\infty} \frac{\Gamma(m+l+1)2^l t^{\frac{l}{4}+\frac{5}{4}m+1}}{\Gamma\left(\frac{l}{4}+\frac{5m}{4}+2\right)l!} \\
 &\quad + 2k_0 \sum_{m=0}^{\infty} \frac{3^m}{m!} \sum_{l=0}^{\infty} \frac{\Gamma(m+l+1)2^l t^{\frac{l}{4}+\frac{5}{4}m+\frac{5}{4}}}{\Gamma\left(\frac{l}{4}+\frac{5m}{4}+\frac{5}{4}\right)l!}.
 \end{aligned}$$

**Example 3.2.** The fractional differential equation

$$D^2 y(t) - D^{\frac{6}{5}} y(t) - 5y(t) = 0$$

with initial conditions  $y(0) = k_0$  and  $y'(0) = k_1$  has a solution

$$\begin{aligned} y(t) &= k_0 \sum_{m=0}^{\infty} \frac{5^m t^{2m}}{m!} \sum_{l=0}^{\infty} \frac{\Gamma(m+l+1)(t^{\frac{4}{5}})^l}{\Gamma\left(\frac{4}{5}l + 2m + 1\right)l!} \\ &\quad + k_1 \sum_{m=0}^{\infty} \frac{5^m t^{2m+1}}{m!} \sum_{l=0}^{\infty} \frac{\Gamma(m+l+1)(t^{\frac{4}{5}})^l}{\Gamma\left(\frac{4}{5}l + 2m + 2\right)l!} \\ &\quad - k_0 \sum_{m=0}^{\infty} \frac{5^m t^{2m-\frac{4}{5}}}{m!} \sum_{l=0}^{\infty} \frac{\Gamma(m+l+1)(t^{\frac{4}{5}})^l}{\Gamma\left(\frac{4}{5}l + 2m - \frac{9}{5}\right)l!} \\ &\quad - k_1 \sum_{m=0}^{\infty} \frac{5^m t^{2m-\frac{9}{5}}}{m!} \sum_{l=0}^{\infty} \frac{\Gamma(m+l+1)(t^{\frac{4}{5}})^l}{\Gamma\left(\frac{4}{5}l + 2m - \frac{14}{5}\right)l!}. \end{aligned}$$

**Example 3.3.** The fractional differential equation

$$D^{\frac{4}{5}} y(t) - 2y(t) = 0$$

with initial conditions  $y(0) = k_0$  and  $y'(0) = k_1$  has a solution

$$y(t) = k_0 \sum_{m=0}^{\infty} \frac{(2t^{\frac{4}{5}})^m}{\Gamma(am+1)}.$$

**Example 3.4.** The fractional differential equation

$$D^{\frac{3}{2}} y(t) + 4y(t) = 0$$

with initial condition  $y(0) = k_0$  has a solution

$$y(t) = k_0 \sum_{m=0}^{\infty} \frac{(-4t^{\frac{3}{2}})^m}{\Gamma(\alpha m + 1)} + k_1 t \sum_{m=0}^{\infty} \frac{(-4t^{\frac{3}{2}})^m}{\Gamma(\alpha m + 2)}. \quad (3.10)$$

#### 4. Conclusion

Hence, by the polynomial integral transform, we can easily obtain the solution of the fractional differential equations.

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