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NECESSARY AND SUFFICIENT CONDITION FOR FULLY WEAKLY PRIME RINGS FROM ITS NILPOTENT RADICAL

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Abstract

Anderson and Smith [1] defined weakly prime ideal of a commutative ring with identity and considered the necessary and sufficient condition for a ring in which every proper ideal is weakly prime. Hirano et al. [2] generalized the definition of weakly prime ideal for any ring (not necessary commutative nor with identity) and proved some properties of ring in which every proper ideal is weakly prime (then it is called fully weakly prime ring). Necessary and sufficient conditions for fully weakly prime ring are also given. Hirano et al. [2] also studied some possibilities of nilpotent radical of fully weakly prime ring. In this paper, we establish a necessary and sufficient condition for a fully weakly prime ring using the characteristics of its nilpotent radicals.

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1. Introduction

A proper ideal P of commutative ring R is called *prime* if $ab \in P$ implies $a \in P$ or $b \in P$, for any $a, b \in R$. This definition was generalized for any ring. A proper ideal P of any ring R is called *prime* if $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$, for ideals I and J of R. In [1], Anderson and Smith considered the definition of weakly prime ideal of commutative ring with identity. A proper ideal P of ring R is called *weakly prime* if $0 \neq ab \in P$ implies $a \in P$ or $b \in P$, for any $a, b \in R$. Furthermore, Hirano et al. [2] generalized the weakly prime ideal for any ring (not necessary commutative nor with identity). Hirano et al. [2] also proved some possibilities of nilpotent radical of ring in which every proper ideal is weakly prime.

2. Some Results

We begin with the following:

Definition 1 (Hirano et al. [2])**.** Let R be any ring and P be a proper ideal of R. Then P is called a *weakly prime ideal* if and only if for any proper ideals I and J of R with $0 \neq IJ \subseteq P$, $I \subseteq P$ or $J \subseteq P$.

Definition 2 (Hirano et al. [2])**.** A ring whose every proper ideal is weakly prime is called a *fully weakly prime ring*.

Hirano et al. [2] considered necessary and sufficient condition for a fully weakly prime ring by multiplication of its ideals. In the following proposition, it is proved that a ring is fully weakly prime if and only if the multiplication of its ideals satisfies some conditions.

Proposition 3 (Hirano et al. [2])**.** *A ring R is a fully weakly prime ring if and only if for any ideals I and J of R*, $IJ = 0$, $IJ = I$ *or* $IJ = J$.

Hirano et al. [2] also studied nilpotent radical (sum of all nilpotent ideals) of fully weakly prime ring. Moreover, by Proposition 3, we can conclude that every ideal of fully weakly prime ring is idempotent ideal or square zero ideal. Therefore, when we discuss nilpotent radical of fully weakly prime ring, it means we only talk about the sum of all the square zero ideals of the ring. The nilpotent radical of the ring R is denoted by $N(R)$.

Corollary 4 (Hirano et al. [2]). *Let R be a fully weakly prime ring. Then* $(N(R))^2 = 0$ *and every prime ideal of R contains* $N(R) \ldots (*)$, *and one of the following holds*:

(1) $N(R) = R$.

(2) $N(R) = P(R)$ *is a minimal prime ideal. All other ideals are idempotent and linearly ordered. If* $N(R) \neq 0$, *then* $N(R)$ *is the only non-idempotent prime ideal.*

(3) $N(R) = P(R)$ *is not a prime ideal. In this case, there exist two minimal ideals* J_1 *and* J_2, *with* $N(R) = J_1 \cap J_2$ *and* $J_1 J_2 = J_2 J_1 = 0$. *All other ideals containing* $N(R)$ *also contains* $J_1 + J_2$ *and they are linearly ordered.*

By adding some conditions to the possibilities of nilpotent radical in Corollary 4, we can give a sufficient condition for a ring to be fully weakly prime. The following theorem provides a necessary and sufficient condition for a ring to be fully weakly prime.

Theorem 5. *Let R be any ring for an ideal I of which either* $I^2 = 0$ *or* $I^2 = I \neq 0$ *and either* $I \subseteq N(R)$ *or* $N(R) \subset I$. *Then R is fully weakly prime if and only if one of the following holds*:

(1) $N(R) = R$.

(2) *All ideals containing* $N(R)$ *are linearly ordered and for any ideals I and J with* $I \subseteq N(R)$ *and* $N(R) \subset J$, $IJ = 0$ *or* $IJ = I$.

(3) *There exist exactly two minimal ideals* P_1 *and* P_2 *of R, such that* $N(R) \subset P_i$ *for* $i \in \{1, 2\}$. *Moreover,* $P_1 P_2 = P_2 P_1 = 0$ *and all other ideals containing* $N(R)$ *also contains* $P_1 + P_2$ *and they are linearly ordered. For any ideals I and J with* $I \subseteq N(R)$ *and* $N(R) \subset J$, $IJ = 0$ *or* $IJ = I$.

Proof. Let R be a fully weakly prime ring. By Corollary 4, then either

(1) $N(R) = R,$ and thus (1) holds.

(2) $N(R)$ is a prime ideal. In this case, all ideals containing $N(R)$ are linearly ordered. Let I and J be any ideals with $I \subseteq N(R)$ and $N(R) \subset J.$ Let $IJ \neq 0.$ By Proposition 3, $IJ = J$ or $IJ = I.$ Since $IJ \subseteq I \bigcap J \subseteq I \subseteq N(R) \subset J,$ $IJ = I.$ Hence, $IJ = 0$ or $IJ = I.$ Thus, (2) holds.

(3) $N(R)$ is not a prime ideal. In this case, there exist exactly two minimal ideals P_1 and P_2 such that $N(R) = P_1 \bigcap P_2 \subseteq J_1$ and $N(R) = P_1 \bigcap P_2 \subseteq J_2.$ Moreover, $P_1P_2 = P_2P_1 = 0$ and all other ideals containing $N(R)$ also contains $P_1 + P_2$ and they are linearly ordered. Let I and J be any ideals with $I \subseteq N(R)$ and $N(R) \subset J.$ Let $IJ \neq 0.$ By Proposition 3, we have $IJ = J$ or $IJ = I.$ Since $IJ \subseteq I \bigcap J \subseteq I \subseteq N(R) \subset J,$ $IJ = I.$ Hence, $IJ = 0$ or $IJ = I.$ Thus, (3) holds.

Conversely, let (1), (2) or (3) hold. First, it will be proved that $(N(R))^2 = 0.$ Since $N(R) = \sum_i N_i,$ with $N_i^2 = 0,$ for every $i \in \Lambda,$ where Λ is an index set. Therefore, there exists a natural number k such that $(N(R))^k = 0.$ But since $(N(R))^2 = 0$ or $(N(R))^2 = N(R) \neq 0,$ $(N(R))^2 = 0.$ Now, we prove the converse as follows:

(1) Let (1) hold. Since $N(R) = R,$ $R^2 = 0.$ Therefore, for any ideals I and J of $R,$ $IJ \subseteq R^2 = 0.$ Thus, R is a fully weakly prime ring.

(2) Let (2) hold. Let K be any nonzero proper ideal of R, and I and J be any ideals with $0 \neq IJ \subseteq K.$

- For $N(R) \subset K.$
  - If $I \subseteq N(R)$ or $J \subseteq N(R),$ then $I \subseteq N(R) \subset K$ or $J \subseteq N(R) \subset K.$

– If $N(R) \subset I$ and $N(R) \subset J$, then $I \subseteq J$ or $J \subseteq I$, since all ideals containing $N(R)$ are linearly ordered. Hence, $I = I^2 \subseteq IJ \subseteq K$ or $J = J^2 \subseteq IJ \subseteq K$.

Thus, K is weakly prime ideal.

- For $K \subseteq N(R)$. Claim $I \subseteq N(R)$ and $N(R) \subset J$.

  – Suppose $I \subseteq N(R)$ and $J \subseteq N(R)$. Then $IJ \subseteq (N(R))^2 = 0$.

  It is a contradiction.

  – Suppose $N(R) \subset I$ and $N(R) \subset J$. Then $I \subseteq J$ or $J \subseteq I$, since all ideals containing $N(R)$ are linearly ordered. Hence, $I = I^2 \subseteq IJ \subseteq K$ or $J = J^2 \subseteq IJ \subseteq K$.

  It is a contradiction.

Therefore, $I \subseteq N(R)$ and $N(R) \subset J$. Consequently, $IJ = 0$ or $IJ = I$. Since $IJ \neq 0$, $IJ = I$ and $I = IJ \subseteq K$. Thus, K is a weakly prime ideal, and R is a fully weakly prime ring.

3. Let (3) hold. Let K be any nonzero proper ideal of R, and I and J be any ideals with $0 \neq IJ \subseteq K$.

- For $P_1 + P_2 \subseteq K$.

  – If $I \subset P_1 + P_2$ or $J \subset P_1 + P_2$, then we have $I \subset P_1 + P_2 \subseteq K$ or $J \subset P_1 + P_2 \subseteq K$.

  – If $P_1 + P_2 \subseteq I$ and $P_1 + P_2 \subseteq J$, then $I \subseteq J$ or $J \subseteq I$, since all ideals containing $N(R)$ other than P_1 and P_2 are linearly ordered. Therefore, $I = I^2 \subseteq IJ \subseteq K$ or $J = J^2 \subseteq IJ \subseteq K$.

Thus, K is a weakly prime ideal.

- For $K = P_1$.
  - If $I \subseteq N(R)$ or $J \subseteq N(R)$, then $I \subseteq N(R) \subset P_1$ or $J \subseteq N(R) \subset P_1$.
  - If $N(R) \subset I$ and $N(R) \subset J$.
    * In case, $P_1 + P_2 \subseteq I$ or $P_1 + P_2 \subseteq J$. Without loss of generality, assume $P_1 + P_2 \subseteq I$. We claim $J = P_1$.

      Suppose $J = P_2$. Then

      $$P_2 = (P_2)^2 = P_1P_2 + (P_2)^2 = (P_1 + P_2)P_2 \subseteq IJ \subseteq K = P_1.$$

      It is a contradiction.

      Suppose $P_1 + P_2 \subseteq J$. Then $I \subseteq J$ or $J \subseteq I$, since all ideals containing $P_1 + P_2$ are linearly ordered. Therefore, $I = I^2 \subseteq IJ \subseteq K$ or $J = J^2 \subseteq IJ \subseteq K$.

      It is a contradiction. Hence, $J = P_1 \subseteq K$.
    * If $I \subset P_1 + P_2$ and $J \subset P_1 + P_2$, then $I = P_i$ and $J = P_j$, for $i, j \in \{1, 2\}$. Therefore, $I = P_1$ or $J = P_1$. Hence, $I = P_1 \subseteq K$ or $J = P_1 \subseteq K$.

Thus, K is a weakly prime ideal. With the same argument as above, it can be proved that $K = P_2$ is also a weakly prime ideal.

- For $K \subseteq N(R)$. We claim $I \subseteq N(R)$ and $N(R) \subset J$.
  - Suppose $I \subseteq N(R)$ and $J \subseteq N(R)$. Then $IJ \subseteq (N(R))^2 = 0$.

    It is a contradiction.

– Suppose $N(R) \subset I$ and $N(R) \subset J$.

* If $P_1 + P_2 \subseteq I$ or $P_1 + P_2 \subseteq J$. Without loss of generality, assume $P_1 + P_2 \subseteq I$.

  Suppose $J = P_i$, for $i \in \{1, 2\}$. Then

  $$P_i = (P_i)^2 = P_1 P_2 + (P_i)^2 = (P_1 + P_2) P_i \subseteq IJ \subseteq K.$$

  It is a contradiction.

  Suppose $P_1 + P_2 \subseteq J$. Then $I \subseteq J$ or $J \subseteq I$, since all ideals containing $P_1 + P_2$ are linearly ordered. Therefore, $I = I^2 \subseteq IJ \subseteq K$ or $J = J^2 \subseteq IJ \subseteq K$.

  It is a contradiction.

* If $P_1 + P_2 \subset I$ and $J \subset P_1 + P_2$, then $I = P_i$ and $J = P_j$, with $i, j \in \{1, 2\}$.

  Suppose $I = P_1$ and $J = P_2$. Then $IJ = P_1 P_2 = 0$.

  It is a contradiction.

  Suppose $I = P_i = J$, for some $i \in \{1, 2\}$. Then $P_i = (P_i)^2 = IJ \subseteq K$.

  It is a contradiction.

Therefore, $I \subseteq N(R)$ and $N(R) \subset J$. Consequently, $IJ = 0$ or $IJ = I$. Since $IJ \neq 0$, $IJ = I$ and $I = IJ \subseteq K$.

Thus, K is a weakly prime ideal, and R is a fully weakly prime ring.

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