Far East Journal of Mathematical Sciences (FJMS)
© 2017 Pushpa Publishing House, Allahabad, India
http://www.pphmj.com
http://dx.doi.org/10.17654/MS102112903
Volume 102, Number 11, 2017, Pages 2903-2912 **ISSN: 0972-0871**

AN INTEGRAL FORMULA OF THE MELLIN TRANSFORM TYPE INVOLVING THE EXTENDED WRIGHT-BESSEL FUNCTION

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Received: August 30, 2017; Accepted: October 14, 2017

2010 Mathematics Subject Classification: 33C45, 33C60, 33E12.

Keywords and phrases: Wright-Bessel function, extended Wright-Bessel functions, Mittag-Leffler function and its extensions, generalized hypergeometric function, generalized (Wright) hypergeometric function, Gamma function, Pochhamer symbol, Mellin transform type integral formula.

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Abstract

We present a new Mellin transform type integral formula involving the generalized Wright-Bessel (or Bessel-Maitland) function defined by Ghayasuddin et al. [7, 8] expressed in terms of the generalized (Wright) hypergeometric function. The integral formula presented here, being very general, can be specialized to yield numerous integral formulas involving extended Wright-Bessel function and extended Mittag-Leffler functions, only three of which are demonstrated.

1. Introduction and Preliminaries

Recently, numerous integral formulas involving a variety of special functions and their generalizations have been presented (see, e.g., [2, 7-9, 11]). Also, many integral formulas associated with the Bessel functions of several kinds and their extensions have been established (see, e.g., [3-5]). Those integrals involving Wright-Bessel functions play important roles in many branches of theoretical and applied physics and engineering. Very recently, Abouzaid et al. [1] and Khan and Kashmin [10] have presented certain interesting and new classes of integral formulas involving the generalized Wright-Bessel function, which are expressed in terms of the generalized (Wright) hypergeometric function. In the present sequel to the aforementioned investigations, we present a Mellin transform type integral formula involving generalized Wright-Bessel functions and its variant, which are expressed in terms of the generalized (Wright) hypergeometric function ${}_p\Psi_q$ and the generalized hypergeometric function ${}_pF_q$, respectively. Some particular cases of our main results are also considered.

The Wright-Bessel function $J_\nu^\mu(z)$ is defined by the following series (see [12, equation (8.3)]):

$$J_\nu^\mu(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\,\Gamma(\mu n + \nu + 1)}. \tag{1.1}$$

Singh et al. [19] introduced the following generalization of Wright-Bessel function:

$$J_{\nu, q}^{\mu, \gamma} = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(-z)^n}{\Gamma(\mu n + \nu + 1) n!}$$

$$(\Re(\mu) \geq 0,\ \Re(\nu) \geq -1,\ \Re(\gamma) \geq 0,\ q \in (0, 1) \cup \mathbb{N}), \tag{1.2}$$

where $(\lambda)_\nu$ is the Pochhammer symbol defined $(\text{for } \lambda, \nu \in \mathbb{C})$ by (see [21, p. 2 and p. 5]):

$$(\lambda)_\nu := \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} \quad (\lambda + \nu \in \mathbb{C} \setminus \mathbb{Z}_0^-)$$

$$= \begin{cases} 1 & (\nu = 0) \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (\nu = n \in \mathbb{N}) \end{cases} \tag{1.3}$$

and $\Gamma(\lambda)$ is the familiar Gamma function. Here and in the following, let $\mathbb{C}$, $\mathbb{R}^+$, $\mathbb{N}$ and $\mathbb{Z}_0^-$ be the sets of complex numbers, positive real numbers, positive integers and non-positive integers, respectively, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

In the sequel of the above-cited works, Ghayasuddin et al. [7] introduced and investigated a new extension of Wright-Bessel function as follows:

$$J_{\nu, \gamma, \delta}^{\mu, q, p}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(-z)^n}{\Gamma(\mu n + \nu + 1)(\delta)_{pn}}$$

$$(\Re(\mu) \geq 0,\ \Re(\nu) \geq -1,\ \Re(\gamma) \geq 0,\ \Re(\delta) \geq 0,$$

$$p, q \in \mathbb{R}^+,\ q < \Re(\mu) + p). \tag{1.4}$$

We consider some special cases of the extended Wright-Bessel function $J_{\nu, \gamma, \delta}^{\mu, q, p}(z)$ in (1.4):

(i) We have

$$J^{\mu,\gamma,\delta}_{\nu-1,q,p}(-z) = E^{\mu,\gamma,\delta}_{\nu,q,p}(z), \tag{1.5}$$

where $E^{\mu,\gamma,\delta}_{\nu,q,p}(z)$ is the extended Mittag-Leffler function in [18].

(ii) We get

$$J^{\mu,\gamma,1}_{\nu-1,q,1}(-z) = E^{\mu,\gamma}_{\nu,q}(z), \tag{1.6}$$

where $E^{\mu,\gamma}_{\nu,q}(z)$ is the extended Mittag-Leffler function in [20].

(iii) We find

$$J^{\mu,\gamma,1}_{\nu-1,1,1}(-z) = E^{\gamma}_{\mu,\nu}(z), \tag{1.7}$$

where $E^{\gamma}_{\mu,\nu}(z)$ is the extended Mittag-Leffler function in [15].

(iv) We obtain

$$J^{\mu,1,1}_{\nu-1,1,1}(-z) = E_{\mu,\nu}(z), \tag{1.8}$$

where $E_{\mu,\nu}(z)$ is the Mittag-Leffler function in [23].

(v) We see

$$J^{\mu,1,1}_{0,1,1}(-z) = E_{\mu}(z), \tag{1.9}$$

where $E_{\mu}(z)$ is the Mittag-Leffler function in [13].

An interesting generalization of the generalized hypergeometric series ${}_pF_q$ (see, e.g., [21, Section 1.5]) is due to Fox [6] and Wright [24-26] who studied the asymptotic expansion of the generalized (Wright) hypergeometric function defined by (see [22, p. 21]; see also [17])

$$ {}_p\Psi_q\left[\begin{matrix}(\alpha_1, A_1), \dots, (\alpha_p, A_p); \\ (\beta_1, B_1), \dots, (\beta_q, B_q);\end{matrix}\; z\right] = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p}\Gamma(\alpha_j + A_j k)}{\prod_{j=1}^{q}\Gamma(\beta_j + B_j k)} \frac{z^k}{k!}, \tag{1.10} $$

where the coefficients $A_1, \dots, A_p \in \mathbb{R}^+$ and $B_1, \dots, B_q \in \mathbb{R}^+$ such that

$$ 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geq 0. \tag{1.11} $$

A special case of (1.10) is

$$ {}_p\Psi_q\left[\begin{matrix}(\alpha_1, 1), \dots, (\alpha_p, 1); \\ (\beta_1, 1), \dots, (\beta_q, 1);\end{matrix}\; z\right] = \frac{\prod_{j=1}^{p}\Gamma(\alpha_j)}{\prod_{j=1}^{q}\Gamma(\beta_j)}\, {}_pF_q\left[\begin{matrix}\alpha_1, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q;\end{matrix}\; z\right]. \tag{1.12} $$

We also need to recall the following integral formula of the Mellin transform type (see [14, p. 22, Entry 2.47]):

$$ \int_0^{\infty} x^{\mu-1}\left(x + a + \sqrt{x^2 + 2ax}\right)^{-\lambda} dx = 2\lambda a^{-\lambda}\left(\frac{a}{2}\right)^{\mu} \frac{\Gamma(2\mu)\Gamma(\lambda - \mu)}{\Gamma(1 + \lambda + \mu)} $$

$$ (a \in \mathbb{R}^+;\ 0 < \Re(\mu) < \lambda). \tag{1.13} $$

2. Integral Formulas

Here, we establish a Mellin transform tpye integral formula involving the extended Wright-Bessel function (1.4) and its variant.

Theorem 2.1. *Let* $\delta, \eta, \nu \in \mathbb{C}$ *with* $\Re(\delta) > 0$, $\Re(\eta) > 0$. *Also, let* $a, p, q, \gamma, \lambda, \mu \in \mathbb{R}^+$ *with* $\mu + p - q \geq 0$, $0 < \Re(\delta) < \lambda$, *and* $\Re(\nu) > -1 - \mu$. *Then*

$$\int_0^\infty x^{\delta-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} J_{\nu,\eta,p}^{\mu,\gamma,q}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= 2^{1-\delta} a^{\delta-\lambda} \frac{\Gamma(2\delta)\Gamma(\eta)}{\Gamma(\gamma)}$$

$$\times {}_4\Psi_4\left[\begin{matrix}(\gamma,\ q),\ (\lambda-\delta,\ 1),\ (\lambda+1,\ 1),\ (1,\ 1); \\ (\nu+1,\ \mu),\ (\lambda,\ 1),\ (1+\lambda+\delta,\ 1),\ (\eta,\ p);\end{matrix}\ -\frac{y}{a}\right]. \tag{2.1}$$

Proof. Let $\mathcal{L}$ be the left side of (2.1). Using (1.4) and changing the order of integral and summation, which is verified under the given conditions, we obtain

$$\mathcal{L} = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(-y)^n}{\Gamma(\mu n+\nu+1)(\eta)_{pn}} \int_0^\infty x^{\delta-1}(x+a+\sqrt{x^2+2ax})^{-\lambda-n}\,dx. \tag{2.2}$$

Applying the formula (1.13) to the integral in (2.2), we obtain

$$\mathcal{L} = 2^{1-\delta} a^{\delta-\lambda} \frac{\Gamma(2\delta)\Gamma(\eta)}{\Gamma(\gamma)}$$

$$\times \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+qn)\Gamma(\lambda+n-\delta)\Gamma(\lambda+n+1)\Gamma(n+1)}{\Gamma(\mu n+\nu+1)\Gamma(\lambda+n)\Gamma(1+\lambda+n+\delta)\Gamma(\eta+pn)n!}\left(-\frac{y}{a}\right)^n,$$

which, upon expressing in terms of the generalized (Wright) hypergeometric function (1.10), leads to the right side of (2.1). $\square$

Under a little stronger condition, by using the following multiplication formula (see, e.g., [21, p. 6, equation (30)]):

$$(\lambda)_{mn} = m^{mn}\prod_{j=1}^{m}\left(\frac{\lambda+j-1}{m}\right)_n \quad (m\in\mathbb{N};\ n\in\mathbb{N}_0),$$

the integral formula (2.1) can be expressed in terms of the generalized hypergeometric function ${}_pF_q$ which is asserted in the following theorem.

Theorem 2.2. *Let* $\delta, \eta, \nu \in \mathbb{C}$ *with* $\Re(\delta) > 0$, $\Re(\eta) > 0$. *Also, let* $a, \gamma, \lambda \in \mathbb{R}^+$ *and* $p, q, \mu \in \mathbb{N}$ *with* $\mu + p - q \geq 0$, $0 < \Re(\delta) < \lambda$, *and* $\Re(\nu) > -1-\mu$. *Then*

$$\int_0^\infty x^{\delta-1}(x + a + \sqrt{x^2 + 2ax})^{-\lambda} J_{\nu, \eta, p}^{\mu, \gamma, q}\left(\frac{y}{x + a + \sqrt{x^2 + 2ax}}\right)dx$$

$$= 2^{1-\delta} a^{\delta-\lambda} \frac{\Gamma(\lambda+1)\Gamma(2\delta)\Gamma(\lambda-\delta)}{\Gamma(\lambda)\Gamma(\nu+1)\Gamma(\lambda+\delta+1)}$$

$$\times {}_{q+3}F_{\mu+p+2}\left[\begin{matrix}\Delta(q; \gamma), \lambda+1, \lambda-\delta, 1; \\ \Delta(\mu; \nu+1), \Delta(p; \eta), \lambda, 1+\lambda+\delta;\end{matrix} -\frac{q^q y}{\mu^\mu p^p a}\right], \tag{2.3}$$

where $\Delta(m; \ell)$ *abbreviates the array of m parameters*

$$\frac{\ell}{m}, \frac{\ell+m}{m}, \ldots, \frac{\ell+m-1}{m} \quad (m \in \mathbb{N}).$$

3. Special Cases

Here, among numerous special cases of the results in Section 2, only three of which are presented.

(i) Using the relation (1.5) in the formula (2.1), we have

$$\int_0^\infty x^{\delta-1}(x + a + \sqrt{x^2 + 2ax})^{-\lambda} E_{\nu, \eta, p}^{\mu, \gamma, q}\left(\frac{y}{x + a + \sqrt{x^2 + 2ax}}\right)dx$$

$$= 2^{1-\delta} a^{\delta-\lambda} \frac{\Gamma(2\delta)\Gamma(\eta)}{\Gamma(\gamma)}$$

$$\times {}_4\Psi_4\left[\begin{matrix}(\gamma, q), (\lambda-\delta, 1), (\lambda+1, 1), (1, 1); \\ (\nu, \mu), (\lambda, 1), (1+\lambda+\delta, 1), (\eta, p);\end{matrix} \frac{y}{a}\right], \tag{3.1}$$

provided $\delta, \eta, \nu \in \mathbb{C}$ with $\min\{\Re(\delta), \Re(\eta)\} > 0$ and $a, p, q, \gamma, \lambda, \mu \in \mathbb{R}^+$ with $\mu + p - q \geq 0$, $0 < \Re(\delta) < \lambda$, and $\Re(\nu) > -\mu$.

(ii) Using the relation (1.5) in the formula (2.3), we obtain

$$\int_0^\infty x^{\delta-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} E_{\nu,\eta,p}^{\mu,\gamma,q}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= 2^{1-\delta} a^{\delta-\lambda} \frac{\Gamma(\lambda+1)\Gamma(2\delta)\Gamma(\lambda-\delta)}{\Gamma(\lambda)\Gamma(\nu)\Gamma(\lambda+\delta+1)}$$

$$\times {}_{q+3}F_{\mu+p+2}\left[\begin{matrix}\Delta(q;\gamma),\ \lambda+1,\ \lambda-\delta,\ 1; \\ \Delta(\mu;\nu),\ \Delta(p;\eta),\ \lambda,\ 1+\lambda+\delta;\end{matrix}\ \frac{q^q y}{\mu^\mu p^p a}\right], \tag{3.2}$$

provided $\delta, \eta, \nu \in \mathbb{C}$ with $\min\{\Re(\delta), \Re(\eta)\} > 0$ and $a, \gamma, \lambda \in \mathbb{R}^+$ and $p, q, \mu \in \mathbb{N}$ with $\mu + p - q \geq 0$, $0 < \Re(\delta) < \lambda$, and $\Re(\nu) > -\mu$.

(iii) Setting $\nu = 0$, $\eta = \gamma = p = q = 1$ in (2.3) and using (1.9), we get

$$\int_0^\infty x^{\delta-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} E_{\mu}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= 2^{1-\delta} a^{\delta-\lambda} \frac{\Gamma(2\delta)\Gamma(\lambda+1)\Gamma(\lambda-\delta)}{\Gamma(\lambda)\Gamma(1+\lambda+\delta)}$$

$$\times {}_3F_{\mu+2}\left[\begin{matrix}\lambda+1,\ \lambda-\delta,\ 1; \\ \Delta(\mu;1),\ \lambda,\ 1+\lambda+\delta;\end{matrix}\ \frac{y}{\mu^\mu a}\right], \tag{3.3}$$

provided $\Re(\delta) > 0$, $\lambda \in \mathbb{R}^+$, and $\mu \in \mathbb{N}$.

In addition to three formulas (3.1), (3.2) and (3.3), by suitably specializing the integral formulas in Section 2, we can present other numerous integral formulas associated with extended Wright-Bessel functions and extended Mittag-Leffler functions.

References

[1] M. S. Abouzaid, A. H. Abusufian and K. S. Nisar, Some unified integrals associated with generalized Bessel-Maitland function, Int. Bull. Math. Res. 3(1) (2016), 18-23.

[2] S. Ali, On an interesting integral involving Gauss's hypergeometric function, Adv. Comput. Math. Appl. 1(4) (2012), 244-246.

[3] J. Choi and P. Agarwal, Certain unified integrals involving a product of Bessel functions of the first kind, Honam Math. J. 35(4) (2013), 667-677.

[4] J. Choi and P. Agarwal, Certain unified integrals associated with Bessel functions, Bound. Value Probl. 2013 (2013), Art. ID 95, 9 pp.

[5] J. Choi, P. Agarwal, S. Mathur and S. D. Purohit, Certain new integral formulas involving the generalized Bessel functions, Bull. Korean Math. Soc. 51(4) (2014), 995-1003.

[6] C. Fox, The asymptotic expansion of generalized hypergeometric functions, Proc. London Math. Soc. 27(4) (1928), 389-400.

[7] M. Ghayasuddin, W. A. Khan and S. Araci, A new extension of Bessel-Maitland function and its properties, submitted.

[8] M. Ghayasuddin, W. A. Khan and L. N. Mishra, On Hankel type integral transform associated with Whittaker and hypergeometric functions, Thai J. Math. (2017), accepted.

[9] N. U. Khan, M. Ghayasuddin and M. Talha, On certain integral formulas involving the product of Bessel function and Jacobi polynomial, Tamkang J. Math. 47(3) (2016), 339-349.

[10] N. U. Khan and T. Kashmin, Some integrals for the generalized Bessel-Maitland functions, Elec. J. Math. Anal. Appl. 4(2) (2016), 139-149.

[11] N. U. Khan, S. W. Khan and M. Ghayasuddin, Some new results associated with Bessel-Struve kernel function, Acta Univ. Apulensis Math. Inform. 48 (2016), 89-101.

[12] O. I. Marichev, Handbook of Integral Transform and Higher Transcendental Functions, Ellis Harwood, John Wiley and Sons, Chichester, New York, 1983.

[13] G. M. Mittag-Leffler, Sur la nouvelle function $E_{\alpha}(x)$, C. R. Acad. Sci. Paris 137 (1903), 554-558.

[14] F. Oberhettinger, Tables of Mellin Transforms, Springer-Verlag, New York, 1974.

[15] T. R. Prabhakar, A singular integral equation with a generalized Mittag-Leffler function in the kernel, Yokohama Math. J. 19 (1971), 7-15.

[16] E. D. Rainville, Special Functions, Macmillan Company, New York, 1960; Reprinted by Chelsea Publishing Company, Bronx, New York, 1971.

[17] A. K. Rathie, A new generalization of generalized hypergeometric function, Le Matematiche 52 (1997), 297-310.

[18] T. O. Salim and A. W. Faraj, A generalization of Mittag-Leffler function and integral operator associated with the fractional calculus, J. Fract. Calc. Appl. 3(5) (2012), 1-13.

[19] M. Singh, M. A. Khan and A. H. Khan, On some properties of a generalization of Bessel-Maitland function, Int. J. Math. Trends Tech. 14(1) (2014), 46-54.

[20] A. K. Shukla and J. C. Prajapati, On a generalization of Mittag-Leffler function and its properties, J. Math. Anal. Appl. 336(2) (2007), 797-811.

[21] H. M. Srivastava and J. Choi, Zeta and q-Zeta Functions and Associated Series and Integrals, Elsevier Science Publishers, Amsterdam, London and New York, 2012.

[22] H. M. Srivastava and P. W. Karlsson, Multiple Gaussian hypergeometric series, Ellis Horwood Series: Mathematics and its Applications, Ellis Horwood Ltd., Halsted Press, John Wiley and Sons, Inc., Chichester, New York, 1985.

[23] A. Wiman, Über den fundamentalsatz in der theorie der funktionen $E_{\alpha}(x)$, Acta Math. 29 (1905), 191-201.

[24] E. M. Wright, The asymptotic expansion of the generalized hypergeometric functions, J. London Math. Soc. 10 (1935), 286-293.

[25] E. M. Wright, The asymptotic expansion of integral functions defined by Taylor series, Philos. Trans. Roy. Soc. London, Ser. A 238 (1940), 423-451.

[26] E. M. Wright, The asymptotic expansion of the generalized hypergeometric function II, Proc. London Math. Soc. (2) 46 (1940), 389-408.