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NOTE ON THE EXTENDED WATSON'S SUMMATION THEOREM FOR THE SERIES ${}_4F_3(1)$

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Abstract

We aim to provide a new proof of the extended Watson's summation theorem for the series ${}_4F_3(1)$ due recently to Kim et al. [4]. Several interesting special cases of the extended Watson's summation theorem

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are also considered. We note that the method of proof used here is potentially useful in getting some other summation formulas for the series ${}_pF_q$.

1. Introduction

Throughout this paper, let ${}_pF_q$ denote the generalized hypergeometric series (see, for details, e.g., [1, 6], [7, Section 1.5]). Kim et al. [4] established extensions of various classical summation theorems for the series ${}_2F_1$, ${}_3F_2$ and ${}_4F_3$, two of which, for our present investigation, are recalled:

$$
{}_3F_2\left[\begin{matrix} a,\ b,\ d+1; \\ \frac{1}{2}(a+b+3),\ d; \end{matrix}\ \frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a-\frac{1}{2}b-\frac{3}{2}\right)}
$$
$$
\times\left\{\frac{\frac{1}{2}(a+b-1)-\frac{ab}{d}}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}+\frac{\frac{a+b+1}{d}-2}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)}\right\} \tag{1.1}
$$

and

$$
{}_4F_3\left[\begin{matrix} a,\ b,\ c,\ d+1; \\ \frac{1}{2}(a+b+3),\ 2c,\ d; \end{matrix}\ 1\right]
$$
$$
= \frac{2^{a+b-2}\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{3}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}\right)}{(a-b-1)(a-b+1)\Gamma\left(\frac{1}{2}\right)\Gamma(a)\Gamma(b)}
$$
$$
\times\left\{c_1\frac{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)}{\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)}+c_2\frac{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(c-\frac{1}{2}a\right)\Gamma\left(c-\frac{1}{2}b\right)}\right\}, \tag{1.2}
$$

provided $d \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $\Re(2c - a - b) > 1$ and where

$$c_1 := a(2c - a) + b(2c - b) - 2c + 1 - \frac{ab}{d}(4c - a - b - 1)$$

and $c_2 := \frac{4}{d}(a + b + 1) - 8.$

The result (1.1) is an extension of the following classical Gauss' second summation theorem (see, e.g., [1, p. 11], [6, p. 69] and [7, p. 350]):

$$ {}_2F_1\left[\begin{matrix} a,\ b; \\ \frac{1}{2}(a+b+1); \end{matrix}\ \frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}b + \frac{1}{2}\right)} \tag{1.3}$$

and the result (1.2) is an extension of the following classical Watson's summation theorem (see, e.g., [1, p. 16] and [7, p. 351]):

$$ {}_3F_2\left[\begin{matrix} a,\ b,\ c; \\ \frac{1}{2}(a+b+1),\ 2c; \end{matrix}\ 1\right]$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}\right)\Gamma\left(c - \frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}b + \frac{1}{2}\right)\Gamma\left(c - \frac{1}{2}a + \frac{1}{2}\right)\Gamma\left(c - \frac{1}{2}b + \frac{1}{2}\right)}$$

$$(\Re(2c - a - b) > -1). \tag{1.4}$$

Here and in the following, each of denominator parameters in the ${}_pF_q$ is assumed to be nonzero.

Kim et al. [4] established the summation formula (1.2) by using some results contiguous to Gauss' second summation theorem obtained by Lavoie et al. [5]. Here, we provide a new proof of the extended Watson's summation theorem (1.2). We also consider some interesting special cases of (1.2). For an interesting proof of the classical Watson's summation theorem (1.4), the reader may be referred to [3].

2. Derivation of (1.2)

Let $\mathcal{L}$ be the left side of (1.2). Expressing ${}_4F_3$ as the series, we have

$$\mathcal{L} = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k (d+1)_k}{\left(\frac{1}{2}(a+b+3)\right)_k (d)_k 2^k k!} \left\{ \frac{2^k (c)_k}{(2c)_k} \right\}, \tag{2.1}$$

where $(\lambda)_n$ is the Pochhammer symbol defined $(\text{for } \lambda \in \mathbb{C})$ by (see [7, p. 2 and pp. 4-6]):

$$(\lambda)_n := \begin{cases} 1 & (n = 0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N}) \end{cases}$$

$$= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-), \tag{2.2}$$

where Γ is the familiar Gamma function. Here and in what follows, let $\mathbb{C}$, $\mathbb{N}$ and $\mathbb{Z}_0^-$ be the sets of complex numbers, positive integers and non-positive integers, respectively, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Applying a known identity (see, e.g., [6, p. 49])

$${}_2F_1\left[\begin{matrix} -\frac{1}{2}k, -\frac{1}{2}k + \frac{1}{2}; \\ c + \frac{1}{2}; \end{matrix} \; 1\right] = \frac{2^k (c)_k}{(2c)_k} \quad (\Re(c) > 0;\ k \in \mathbb{N}_0)$$

to (2.1), we obtain

$$\mathcal{L} = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k (d+1)_k}{\left(\frac{1}{2}(a+b+3)\right)_k (d)_k 2^k k!} \, {}_2F_1\left[\begin{matrix} -\frac{1}{2}k, -\frac{1}{2}k + \frac{1}{2}; \\ c + \frac{1}{2}; \end{matrix} \; 1\right]. \tag{2.3}$$

It easily follows from (2.2) that

$$(\lambda)_{2n} = 2^{2n} \left(\frac{1}{2}\lambda\right)_n \left(\frac{1}{2}\lambda + \frac{1}{2}\right)_n \quad (n \in \mathbb{N}_0). \tag{2.4}$$

Using (2.4), we get

$$\left(-\frac{1}{2}k\right)_m\left(-\frac{1}{2}k+\frac{1}{2}\right)_m = 2^{-2m}(-k)_{2m} = \frac{2^{-2m}k!}{(k-2m)!} \quad (k,\ m \in \mathbb{N}_0). \quad (2.5)$$

Expressing ${}_2F_1$ in (2.3) as the series with the aid of (2.5), we obtain

$$\mathcal{L} = \sum_{k=0}^{\infty}\sum_{m=0}^{\left[\frac{k}{2}\right]}\frac{(a)_k(b)_k(d+1)_k\,2^{-2m}}{\left(\frac{1}{2}(a+b+3)\right)_k 2^k(d)_k\left(c+\frac{1}{2}\right)_m m!(k-2m)!}. \qquad (2.6)$$

Applying the following formal manipulation of double series (see, e.g., [2] and [6, p. 57, Lemma 11]):

$$\sum_{k=0}^{\infty}\sum_{m=0}^{\left[\frac{k}{2}\right]}A(m,\ k) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}A(m,\ k+2m)$$

to (2.6) and using the identity

$$(a)_{k+2m} = (a)_{2m}(a+2m)_k$$

in the resulting double infinite series, we have

$$\mathcal{L} = \sum_{m=0}^{\infty}\frac{(a)_{2m}(b)_{2m}(1+d)_{2m}}{\left(\frac{k}{2}(a+b+3)\right)_{2m}2^{4m}(d)_{2m}\left(c+\frac{1}{2}\right)_m m!}$$

$$\times\sum_{k=0}^{\infty}\frac{(a+2m)_k(b+2m)_k(d+1+2m)_k}{\left(\frac{1}{2}(a+b+3)+2m\right)_k(d+2m)_k 2^k k!}. \qquad (2.7)$$

Expressing the inner series in (2.7) as a ${}_3F_2$, we get

$$\mathcal{L} = \sum_{m=0}^{\infty}\frac{(a)_{2m}(b)_{2m}(1+d)_{2m}}{\left(\frac{k}{2}(a+b+3)\right)_{2m}2^{4m}(d)_{2m}\left(c+\frac{1}{2}\right)_m m!}$$

$$\times\ {}_3F_2\left[\begin{matrix} a+2m, b+2m, 1+d+2m; \\ \frac{1}{2}(a+b+3)+2m,\ d+2m; \end{matrix}\ \frac{1}{2}\right]. \tag{2.8}$$

Now, we can evaluate the ${}_3F_2$ series in (2.8) with the help of (1.1), after simplification, using Gauss' second summation theorem (1.3), we arrive at the right side of (1.2).

3. Special Cases of (1.2)

Here, we consider some interesting and potentially useful special cases of (1.2).

(i) First, setting $b = -2n$ and replacing a by $a + 2n$ and second, letting $b = -2n - 1$ and substituting $a + 2n + 1$ for a $(n \in \mathbb{N}_0)$ in (1.2), we see that, in each case, one of the two terms on the right side of the resulting identities (1.2) will vanish. We thus have

$$ {}_4F_3\left[\begin{matrix} -2n,\ a+2n,\ c,\ d+1; \\ \frac{1}{2}(a+3),\ 2c,\ d; \end{matrix}\ 1\right]$$

$$= \frac{\alpha(a+1)}{(2c-a-1)(a+4n+1)(a+4n-1)} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}-c+\frac{1}{2}a\right)_n}{\left(\frac{1}{2}a+\frac{1}{2}\right)_n \left(c+\frac{1}{2}\right)_n}, \tag{3.1}$$

where $n \in \mathbb{N}_0$ and

$$\alpha := (a+2n)(2c-a-2n) - 4n(c+n) - 2c + 1 + \frac{2n(a+2n)}{d}(4c-a-1);$$

$$ {}_4F_3\left[\begin{matrix} -2n-1,\ a+2n+1,\ c,\ d+1; \\ \frac{1}{2}(a+3),\ 2c,\ d; \end{matrix}\ 1\right] \quad (n \in \mathbb{N}_0)$$

$$= \left(1 - \frac{a+1}{2d}\right) \frac{2(a+1)}{(a+4n+1)(a+4n+3)} \frac{\left(\frac{3}{2}\right)_n \left(\frac{3}{2}-c+\frac{1}{2}a\right)_n}{\left(\frac{1}{2}a+\frac{1}{2}\right)_n \left(c+\frac{1}{2}\right)_n}. \tag{3.2}$$

(ii) Also, setting $d = \frac{1}{2}(a+1)$ and $d = 2c-1$ in (3.1), respectively, we obtain the following summation formulas:

$$ {}_3F_2\left[\begin{matrix} -2n,\ a+2n,\ c; \\ \frac{1}{2}(a+1),\ 2c; \end{matrix}\ 1\right] = \frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}-c+\frac{1}{2}a\right)_n}{\left(\frac{1}{2}a+\frac{1}{2}\right)_n\left(c+\frac{1}{2}\right)_n} \tag{3.3} $$

and

$$ {}_3F_2\left[\begin{matrix} -2n,\ a+2n,\ c; \\ \frac{1}{2}(a+3),\ 2c-1; \end{matrix}\ 1\right] = \beta\,\frac{\left(\frac{1}{2}\right)_n\left(\frac{3}{2}-c+\frac{1}{2}a\right)_n}{\left(\frac{1}{2}a+\frac{1}{2}\right)_n\left(c-\frac{1}{2}\right)_n}, \tag{3.4} $$

where $n \in \mathbb{N}_0$ and

$$ \beta := \frac{(a+1)(a-1)}{(a+4n+1)(a+4n-1)}. $$

(iii) Further, setting $d = \frac{1}{2}(a+1)$ and $d = 2c-1$ in (3.2), respectively, we get

$$ {}_3F_2\left[\begin{matrix} -2n-1,\ a+2n+1,\ c; \\ \frac{1}{2}(a+1),\ 2c; \end{matrix}\ 1\right] = 0 \tag{3.5} $$

and

$$ {}_3F_2\left[\begin{matrix} -2n-1,\ a+2n+1,\ c; \\ \frac{1}{2}(a+3),\ 2c-1; \end{matrix}\ 1\right] = \gamma\,\frac{\left(\frac{3}{2}\right)_n\left(\frac{1}{2}a-c+\frac{3}{2}\right)_n}{\left(\frac{1}{2}a+\frac{1}{2}\right)_n\left(c+\frac{1}{2}\right)_n}, \tag{3.6} $$

where $n \in \mathbb{N}_0$ and

$$ \gamma := \frac{(a+1)(4c-a-3)}{(a+4n+1)(a+4n+3)(2c-1)}. $$

We note that the identities (3.3)-(3.6) have been presented by Lavoie et al. [5] who employed other methods different from those used here. We also remark that the method of proof used here is potentially useful in getting some other summation formulas for the ${}_pF_q$.

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