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ON THE FORM AND PROPERTIES OF AN INTEGRAL TRANSFORM WITH STRENGTH IN INTEGRAL TRANSFORMS

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Abstract

We propose and study the properties of an integral transform that has strengths in transforms of integrals. This result is from the intrinsic structure and properties of Laplace-typed integral transforms. Also, this form is well adapted to solving Volterra integral equation and semi-infinite string.

1. Introduction

The method of integral transforms has been developed because of the easy accessibility and the high application. The form is defined by

$$\int_0^{\infty} K(s, t) f(t) dt,$$

where the kernel $K(s, t)$ is doing the role which transforms one space to the other space in order to solve the solution. In the previous researches, the

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nature of integral transform is mentioned well in [2-4]. The shifted data problem [14], the Laplace transform of derivative expressed by Heaviside functions [15], and the solution of Volterra integral equation of the second kind by using the Elzaki transform [18] were proposed. As a study of comprehensive forms, we have proposed the intrinsic structure and properties of Laplace-typed integral transforms in [13] as

$$F(u) = G(f) = u^{\alpha}\int_0^{\infty} e^{-\frac{t}{u}} f(t)dt \tag{1}$$

for an integer α and for a generalized integral transform G. In general, Laplace transform has a strong point in the transforms of derivatives, that is, the differentiation of a function $f(t)$ corresponds to multiplication of its transform $£(f)$ by s. While, if we choose $G_{-2}(f)$ as

$$G_{-2}(f) = \frac{1}{u^2}\int_0^{\infty} e^{-\frac{t}{u}} f(t)dt, \tag{2}$$

then this transform is giving a simple tool for transforms of integrals. That is, the integration of a function $f(t)$ corresponds to multiplication of $G_{-2}(f)$ by u, whereas the differentiation of $f(t)$ corresponds to division of $G_{-2}(f)$ by u. This means that the integer α is applicable to –2 in (1).

The form of Laplace transform is a well-known fact, and it is defined by

$$£(f) = \int_0^{\infty} e^{-st} f(t)dt.$$

Since the Laplace transform $£(f)$ can be rewritten as

$$\int_0^{\infty} e^{-\frac{t}{u}} f(t)dt$$

for $s = 1/u$, the value of α is applicable to 0 in (1). Similarly, Sumudu transform [7, 8, 19] has the value of $\alpha = -1$ and Elzaki [9-11, 16] has the value of $\alpha = 1$.

In this article, we investigate the properties of the integral transform which is $\alpha = -2$ in a generalized integral transform G and we apply it to semi-infinite string.

2. The Properties of an Integral Transform that has Strengths in Transforms of Integrals

Theorem 1. *The following properties are valid*:

(A) (*u-shifting*) *If* $f(t)$ *has the transform* $F(u)$, *then* $e^{at}f(t)$ *has the transform*

$$F\left(\frac{u}{1-au}\right).$$

That is,

$$G_{-2}[e^{at}f(t)] = F\left(\frac{u}{1-au}\right).$$

(B) (*t-shifting*) *If* $f(t)$ *has the transform* $F(u)$, *then the shifted function* $f(t-a)h(t-a)$ *has the transform* $e^{-a/u}F(u)$. *In formulas,*

$$G_{-2}[f(t-a)h(t-a)] = e^{-a/u}F(u)$$

where $h(t-a)$ *is a Heaviside function* (*we write h since we need u to denote u-space*.)

(Ca)

$$G_{-2}(f') = \frac{1}{u}Y - \frac{1}{u^2}f(0).$$

(Cb)

$$G_{-2}(f'') = \frac{1}{u^2}Y - \frac{1}{u^3}f(0) - \frac{1}{u^2}f'(0)$$

for $Y = G_{-2}(f)$ *and where f is differentiable n-times.*

(Cc)

$$G_{-2}(f^{(n)}) = \frac{1}{u^n}Y - \frac{1}{u^{n+1}} f(0) - \frac{1}{u^n} f'(0) - \frac{1}{u^{n-1}} f''(0)$$

$$- \cdots - \frac{1}{u^2} f^{(n-1)}(0)$$

for n is an arbitrary natural number.

(D) *Let* $F(u)$ *denote the transform of an integrable function* $f(t)$, *i.e.*, $F(u) = G_{-2}[f(t)]$. *Then*

$$G_{-2}\left[\int_0^t f(\tau)d\tau\right] = uF(u)$$

holds for $t > 0$.

(E)

$$G_{-2}(f * g) = u^2 G_{-2}(f) G_{-2}(g),$$

where $*$ *is the convolution of f and g.*

(Fa)

$$G_{-2}(f)'(u) = \frac{dG}{du} = -\frac{2}{u}Y + \frac{1}{u^2}G_{-2}(tf(t)).$$

(Fb)

$$G_{-2}(f)''(u) = \frac{6}{u^2}Y - \frac{2}{u}G_{-2}(tf(t))\left(1 + \frac{1}{u^2}\right) + \frac{1}{u^2}G_{-2}(t^2 f(t)).$$

(Fc)

$$\int_u^\infty G_{-2}(f)(s)ds = a - u^2 G_{-2}\left(\frac{f(t)}{t}\right)$$

for a constant $a = \int_0^\infty f(t)/tdt$ *and for* $Y = G_{-2}(f)(u)$ *under the condition of the limit of* $f(t)/t$, *as t approaches* 0 *from the right, exists.*

(Fd) $G_{-2}(ty') = Y + u(dY/du)$ *and* $G_{-2}(ty'') = dY/du + (1/u^2)\, y(0)$ *for* $Y = G_{-2}(f)(u)$.

Proof. Since these can be obtained by modifying the results of a generalized integral transform G, we would just like to check a few and skip the rest.

The proof of (D). Using (C) and putting $g(t) = \int_0^t f(\tau)d\tau$,

$$G_{-2}(f(t)) = G_{-2}(g'(t)) = \frac{1}{u} G_{-2}(g) - \frac{1}{u^2} g(0) = \frac{1}{u} G_{-2}(g).$$

The proof of (E). Let us put

$$G_{-2}(f) = \frac{1}{u^2} \int_0^\infty e^{-\frac{\tau}{u}} f(\tau)d\tau$$

and

$$G_{-2}(g) = \frac{1}{u^2} \int_0^\infty e^{-\frac{v}{u}} g(v)dv.$$

Then

$$G_{-2}(f)G_{-2}(g) = \frac{1}{u^4} \int_0^\infty e^{-\frac{\tau}{u}} f(\tau)d\tau \cdot \int_0^\infty e^{-\frac{v}{u}} g(v)dv.$$

As let us put $t = v + \tau$, where τ is at first constant. Then $v = t - \tau$ and so, we get

$$G_{-2}(g) = \frac{1}{u^2} \int_\tau^\infty e^{-\frac{t-\tau}{u}} g(t-\tau)dt = \frac{1}{u^2} e^{\frac{\tau}{u}} \int_\tau^\infty e^{-\frac{t}{u}} g(t-\tau)dt.$$

Since we can change the order of integration by using dominated convergence theorem [5, 12],

$$G_{-2}(f)G_{-2}(g) = \frac{1}{u^4} \int_0^\infty f(\tau) \int_\tau^\infty e^{-\frac{t}{u}} g(t-\tau)dtd\tau,$$

and when t varies τ to ∞, τ varies 0 to t. Hence,

$$G_{-2}(f)G_{-2}(g) = \frac{1}{u^4}\int_0^\infty e^{-\frac{t}{u}}\int_0^t f(\tau)g(t-\tau)d\tau dt$$

$$= \frac{1}{u^4}\int_0^\infty e^{-\frac{t}{u}}(f*g)(t)dt$$

$$= \frac{1}{u^2}G_{-2}(f*g). \qquad \square$$

As we have seen above, for G_{-2}-integral transform, the integration of a function $f(t)$ corresponds to multiplication of $G_{-2}(f)$ by u, whereas the differentiation of $f(t)$ corresponds to division of $G_{-2}(f)$ by u. Using $G_{-2}(f) = \frac{1}{u^2}\cdot F\left(\frac{1}{u}\right)$ for Laplace transform $\pounds(f) = F(s)$, we can obtain the table of G_{-2}-integral transforms as follows.

It is a well-known fact that convolution helps in solving integral equations of certain type, mainly Volterra integral equation. Hence, we would like to explain the idea in terms of examples appearing in [17]. Next, we would like to check an example related to semi-infinite string.

Example 1. Solve the Volterra integral equation of the second kind

$$y(t) - \int_0^t y(\tau)\sin(t-\tau)d\tau = t.$$

Solution. This is rewritten as a convolution

$$y(t) - y*\sin t = t.$$

Taking G_{-2}-integral transform on both sides and applying (C) of Theorem 1, we have

$$Y(u) - u^2Y(u)\frac{1}{1+u^2} = Y(u)\left(1-\frac{u^2}{1+u^2}\right) = 1$$

for $Y = G_{-2}(y)$. The solution is

$$Y(u) = 1 + u^2$$

and gives the answer

$$y(t) = t + \frac{1}{6}t^3$$

by Table 1.

Table 1. Table of G_{-2}-integral transforms

	$f(t)$	$G_{-2}(f)$
1	1	$1/u$
2	t	1
3	t^n	$n!,\ u^{n-1}$
4	e^{at}	$\frac{1}{u(1-au)}$
5	$\sin at$	$\frac{a}{1+u^2a^2}$
6	$\cos at$	$\frac{1}{u(1+u^2a^2)}$
7	$\sinh at$	$\frac{a}{1-u^2a^2}$
8	$\cosh at$	$\frac{1}{u(1-u^2a^2)}$
9	$e^{at}\cos at$	$\frac{1-au}{u^3\left[\left(\frac{1}{u}-a\right)^2+a^2\right]}$
10	$e^{at}\sin at$	$\frac{a}{u^2\left[\left(\frac{1}{u}-a\right)^2+a^2\right]}$

Example 2. Solve the Volterra integral equation of the second kind:

$$y(t) - \int_0^t (1+\tau)\,y(t-\tau)\,d\tau = 1 - \sinh t.$$

Solution. In a similar way as in Example 1, the given equation is same to $y-(1+t)*y=1-\sinh t$. Taking G_{-2}-transform, we have

$$Y(u)-u^2\left(\frac{1}{u}+1\right)Y(u)=\frac{1}{u}-\frac{1}{1-u^2},$$

hence

$$Y(u)[1-u-u^2]=\frac{1-u^2-u}{u(1-u^2)}.$$

Simplification gives

$$Y(u)=\frac{1}{u(1-u^2)}$$

and so we obtain the answer

$$y(t)=\cosh t$$

by Table 1.

Example 3. Find the solution of

$$y(t)+\int_0^t (t-\tau)\,y(\tau)\,d\tau=1.$$

Solution. Taking G_{-2}-transform on both sides, we have

$$Y+u^2(Y\cdot 1)=\frac{1}{u}$$

for $Y=G_{-2}(y)$. Thus,

$$Y=\frac{1}{u(1+u^2)}$$

and so, we obtain the solution $y=\cos t$.

Let us check this by the direct calculation. Expanding the given equation, we have

$$y(t) + t \cdot \int_0^t y(\tau)d\tau - \int_0^t \tau y(\tau)d\tau = 1.$$

Differentiating both sides twice with respect to t, we have $y''(t) + y(t) = 0$. Thus, from $y(0) = 1$ and $y'(0) = 0$ obtained by calculating course, we have the solution $y = \cos t$.

Similarly, we can easily obtain the solution of integral equations by using G_{-2}-integral transform. For example, let us consider

$$y(t) - \int_0^t y(\tau)d\tau = 1.$$

By G_{-2}-transform, we have $Y - uY = 1/u$ and so, we have the solution $y = e^t$ for $Y = G_{-2}(y)$. Of course, this result is same to the result $y - y * 1 = 1$ by using convolution, and the result $y'(t) - y(t) = 0$ of the direct calculation is same as well. Similarly, since

$$G_{-2}(te^{at}) = \frac{1}{(1-au)^2},$$

the solution of

$$y(t) + 2e^t \int_0^t e^{-\tau} y(\tau)d\tau = te^t$$

is $y = \sinh t$. Here, we note that the G_{-2}-transform of $y + 2(y * e^t) = te^t$ is

$$Y + 2Y \cdot \frac{1}{u(1-u)} = \frac{1}{(1-u)^2}$$

for $Y = G_{-2}(y)$.

Example 4. Heaviside function and Dirac's delta function, shifted data problems.

Solution. First, let us check the G_{-2}-transform of Heaviside function $h(t-a)$:

$$G_{-2}[h(t-a)] = \frac{1}{u^2}\int_0^\infty e^{-\frac{t}{u}} h(t-a)\,dt = \frac{1}{u^2}\int_a^\infty e^{-\frac{t}{u}} dt$$

$$= -\frac{1}{u}[e^{-\frac{t}{u}}]_a^\infty = \frac{1}{u} e^{-\frac{a}{u}}.$$

Next, we consider the function $f_k(t-a) = 1/k$ if $a \le t \le a+k$, and 0 otherwise. Taking G_{-2}-transform, we have

$$G_{-2}[f_k(t-a)] = \frac{1}{u^2}\int_0^\infty e^{-\frac{t}{u}} f_k(t-a)\,dt = -\frac{1}{uk}[e^{-\frac{t}{u}}]_a^{a+k}$$

$$= -\frac{1}{uk}(e^{-\frac{a+k}{u}} - e^{-\frac{a}{u}}) = -e^{-\frac{a}{u}} \cdot \frac{e^{-\frac{k}{u}} - 1}{uk}.$$

If we denote the limit of f_k as $\delta(t-a)$, then

$$\delta(t-a) = \lim_{k \to 0} f_k(t-a) = \frac{1}{u^2} e^{-\frac{a}{u}}.$$

Finally, let us see shifted data problems. For a given differential equation $y'' + ay' + by = r(t)$, $y(a) = c_0$, $y'(a) = c_1$, where $a \ne 0$ and a and b are constants, we can set $t = t_1 + a$. This gives $t_1 = 0$ and so, we have

$$y_1'' + ay_1' + by_1 = r(t_1 + a), \quad y_1(0) = c_0, \quad y_1'(0) = c_1$$

for input $r(t)$. Taking the transform, we can obtain the output $y(t)$.

Example 5 (Semi-infinite string)**.** Find the displacement $w(x, t)$ of an elastic string subject to the following conditions:

(a) The string is initially at rest on the x-axis from $x = 0$ to ∞.

(b) For $t > 0$, the left end of the string is moved in a given manner, namely, according to a single sine wave $w(0, t) = f(t) = \sin t$ if $0 \le t \le 2\pi$, and zero otherwise.

(c) Furthermore, $\lim w(x, t) = 0$ as $x \to \infty$ for $t \ge 0$.

Of course, there is no infinite string, but our model describes a long string or rope (of negligible weight) with its right end fixed far out on the x-axis [17].

Solution. It is well-known fact that the equation of semi-infinite string can be expressed by

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$$

subject to $w(0, t) = f(t)$, $\lim w(x, t) = 0$ as $x \to \infty$, $w(x, 0) = 0$ and $w_t(x, 0) = 0$. Taking G_{-2}-transform with respect to t, and by (C) of Theorem 1, we have

$$G_{-2}\left[\frac{\partial^2 w}{\partial t^2}\right] = \frac{1}{u^2} W - \frac{1}{u^3} w(x, 0) - \frac{1}{u^2} w_t(x, 0) = \frac{1}{u^2} W$$

for $Y = G_{-2}(f)$. Writing $W(x, u) = G_{-2}[w(x, t)]$, we have

$$G_{-2}\left[\frac{\partial^2 w}{\partial x^2}\right] = \frac{1}{u^2} \int_0^\infty e^{-\frac{t}{u}} \frac{\partial^2 w}{\partial x^2} dt$$

$$= \frac{\partial^2}{\partial x^2} \frac{1}{u^2} \int_0^\infty e^{-\frac{t}{u}} w(x, t) dt = \frac{\partial^2}{\partial x^2} G_{-2}[w(x, t)] = \frac{\partial^2 W}{\partial x^2}.$$

Thus,

$$\frac{\partial^2 W}{\partial x^2} - \frac{1}{c^2 u^2} W = 0.$$

Since this equation contains only a derivative with respect to x, it may be regarded as an ODE, where $W(x, u)$ is considered as a function of x. This implies that a general solution can be represented by

$$W(x, u) = A(u)e^{x/cu} + B(u)e^{-x/cu}.$$

From the initial conditions, we have $W(0, u) = G_{-2}[w(0, t)] = G_{-2}[f(t)] = F(u)$. In [5, 12], we have dealt with the validity on exchangeability of integral and limit in the solving process of PDEs by using Lebesgue dominated convergence theorem [5]. Hence, we have

$$\lim_{x\to\infty} W(x, u) = \lim_{x\to\infty} \frac{1}{u^2}\int_0^\infty e^{-\frac{t}{u}} w(x, t)dt$$

$$= \frac{1}{u^2}\int_0^\infty e^{-\frac{t}{u}} \lim_{x\to\infty} w(x, t)dt = 0.$$

This implies $A(u) = 0$ and so, $W(0, u) = B(u) = F(u)$. Thus,

$$W(x, u) = F(u)e^{-x/cu}.$$

By the t-shifting theorem, we obtain the inverse transform

$$w(x, t) = f\left(t - \frac{x}{c}\right)h\left(t - \frac{x}{c}\right) = \sin\left(t - \frac{x}{c}\right)$$

for $\frac{x}{c} < t < \frac{x}{c} + 2\pi$ and zero otherwise, where h is a Heaviside function.

3. Conclusion

The form and the properties of an integral transform that has strengths in transforms of integrals have been proposed. This result is obtained from a generalized integral transform G and is applicable to $\alpha = -2$ in (1). This gives some help for solving integral equations by means of its simplicity for transform of integration. Additionally, some examples related to Volterra integral equation and semi-infinite string have been presented as well.

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