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ASYMPTOTIC BEHAVIOR OF PERTURBED NONLINEAR DIFFERENTIAL SYSTEMS

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Abstract

We study the asymptotic behavior of solutions of the following perturbed differential system:

$$y' = f(t, y) + \int_{t_0}^{t} g(s, y(s), T_1 y(s))\, ds + h(t, y(t), T_2 y(t)).$$

We impose the conditions on the perturbed part

$$\int_{t_0}^{t} g(s, y(s), T_1 y(s))\, ds,\ h(t, y(t), T_2 y(t))$$

and on the fundamental matrix of the unperturbed system $y' = f(t, y)$.

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1. Introduction and Preliminaries

Elaydi and Farran [8] introduced the notion of exponential asymptotic stability (EAS) which is a stronger notion than that of uniformly Lipschitz stable. They investigated some analytic criteria for an autonomous differential system and its perturbed systems to be EAS. In Pachpatte [16, 17], the stability and asymptotic behavior of the solutions of perturbed nonlinear systems under some suitable conditions on the perturbation term is studied. In Gonzalez and Pinto [9], the asymptotic behavior and boundedness of the solutions of nonlinear differential systems is investigated. In Choi et al. [6], Lipschitz and exponential asymptotic stability for nonlinear functional systems is proved. Also, in Goo [10-12] and Goo et al. [13, 14], Lipschitz and asymptotic stability for perturbed differential systems is studied.

In this paper, we investigate asymptotic behavior for solutions of perturbed nonlinear systems using integral inequalities. The method incorporating integral inequalities takes an important place among the methods developed for the qualitative analysis of solutions to linear and nonlinear system of differential equations.

Consider the unperturbed nonlinear system

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{1.1}$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^n$ is the Euclidean n-space. We assume that the Jacobian matrix $f_x = \partial f / \partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and $f(t, 0) = 0$. Also, consider the perturbed functional differential system of (1.1)

$$y' = f(t, y) + \int_{t_0}^{t} g(s, y(s), T_1 y(s))ds + h(t, y(t), T_2 y(t)), \quad y(t_0) = y_0, \tag{1.2}$$

where $g, h \in C(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $g(t, 0, 0) = h(t, 0, 0) = 0$, and $T_1, T_2 : C(\mathbb{R}^+, \mathbb{R}^n) \to C(\mathbb{R}^+, \mathbb{R}^n)$ are continuous operators. The symbol

$|\cdot|$ will be used to denote any convenient vector norm in $\mathbb{R}^+$. For an $n \times n$ matrix A, define the norm $|A|$ of A by $|A| = \sup_{|x| \leq 1} |Ax|$.

Let $x(t, t_0, x_0)$ denote the unique solution of (1.1) with $x(t_0, t_0, x_0) = x_0$, existing on $[t_0, \infty)$. Then we can consider the associated variational systems around the zero solution of (1.1) and around $x(t)$, respectively,

$$v'(t) = f_x(t, 0)v(t), \quad v(t_0) = v_0 \tag{1.3}$$

and

$$z'(t) = f_x(t, x(t, t_0, x_0))z(t), \quad z(t_0) = z_0. \tag{1.4}$$

The fundamental matrix $\Phi(t, t_0, x_0)$ of (1.4) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (1.3).

The following definitions are standard, we state them here for convenient [8].

Definition 1.1. The system (1.1) (the zero solution $x = 0$ of (1.1)) is called:

(S) *stable* if for any $\varepsilon > 0$ and $t_0 \geq 0$, there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that if $|x_0| < \delta$, then $|x(t)| < \varepsilon$ for all $t \geq t_0 \geq 0$,

(AS) *asymptotically stable* if it is stable and if there exists $\delta = \delta(t_0) > 0$ such that if $|x_0| < \delta$, then $|x(t)| \to 0$ as $t \to \infty$,

(ULS) *uniformly Lipschitz stable* if there exist $M > 0$ and $\delta > 0$ such that $|x(t)| \leq M|x_0|$ whenever $|x_0| \leq \delta$ and $t \geq t_0 \geq 0$,

(EAS) *exponentially asymptotically stable* if there exist constants $K > 0$, $c > 0$, and $\delta > 0$ such that

$$|x(t)| \le K|x_0|e^{-c(t-t_0)}, \quad 0 \le t_0 \le t$$

provided that $|x_0| < \delta$,

($EASV$) *exponentially asymptotically stable* in variation if there exist constants $K > 0$ and $c > 0$ such that

$$|\Phi(t, t_0, x_0)| \le Ke^{-c(t-t_0)}, \quad 0 \le t_0 \le t$$

provided that $|x_0| < \infty$.

Remark 1.2 [9]**.** The last definition implies that for $|x_0| \le \delta$,

$$|x(t)| \le K|x_0|e^{-c(t-t_0)}, \quad 0 \le t_0 \le t.$$

Before proceeding to the statement of main results, we set forth some known results. We need Alekseev formula to compare between the solutions of (1.1) and the solutions of perturbed nonlinear system

$$y' = f(t, y) + g(t, y), \quad y(t_0) = y_0, \tag{1.5}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (1.5) passing through the point (t_0, y_0) in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

Lemma 1.3 [2]**.** *Let x and y be solutions of* (1.1) *and* (1.5), *respectively. If $y_0 \in \mathbb{R}^n$, then for all $t \ge t_0$ such that $x(t, t_0, y_0) \in \mathbb{R}^n$, $y(t, t_0, y_0) \in \mathbb{R}^n$,*

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s))g(s, y(s))ds.$$

Lemma 1.4 (Bihari-type inequality)**.** *Let* $u, \lambda \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ *and* $w(u)$ *be nondecreasing in* u. *Suppose that, for some* $c > 0$,

$$u(t) \leq c + \int_{t_0}^{t} \lambda(s) w(u(s)) ds, \quad t \geq t_0 \geq 0.$$

Then

$$u(t) \leq W^{-1}\left[W(c) + \int_{t_0}^{t} \lambda(s) ds\right],$$

where $t_0 \leq t < b_1$, $W(u) = \int_{u_0}^{u} \frac{ds}{w(s)}$, $W^{-1}(u)$ *is the inverse of* $W(u)$, *and*

$$b_1 = \sup\left\{t \geq t_0 : W(c) + \int_{t_0}^{t} \lambda(s) ds \in \operatorname{dom} W^{-1}\right\}.$$

Lemma 1.5 [3]**.** *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9, \lambda_{10} \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$, *and* $w(u)$ *be nondecreasing in* u, $u \leq w(u)$. *Suppose that for some* $c > 0$ *and* $0 \leq t_0 \leq t$,

$$\begin{aligned} u(t) \leq c &+ \int_{t_0}^{t} \lambda_1(s) u(s) ds + \int_{t_0}^{t} \lambda_2(s) w(u(s)) ds + \int_{t_0}^{t} \lambda_3(s) \int_{t_0}^{s} \Big(\lambda_4(\tau) u(\tau) \\ &+ \lambda_5(\tau) \int_{t_0}^{\tau} \lambda_6(r) u(r) dr + \lambda_7(\tau) \int_{t_0}^{\tau} \lambda_8(r) w(u(r)) dr\Big) d\tau ds \\ &+ \int_{t_0}^{t} \lambda_9(s) \int_{t_0}^{s} \lambda_{10}(\tau) w(u(\tau)) d\tau ds. \end{aligned}$$

Then

$$\begin{aligned} u(t) \leq W^{-1}\Big[W(c) &+ \int_{t_0}^{t} \Big(\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^{s} \Big(\lambda_4(\tau) + \lambda_5(\tau) \int_{t_0}^{\tau} \lambda_6(r) dr \\ &+ \lambda_7(\tau) \int_{t_0}^{\tau} \lambda_8(r) dr\Big) d\tau + \lambda_9(s) \int_{t_0}^{\tau} \lambda_{10}(\tau) d\tau\Big) ds\Big], \end{aligned}$$

where $t_0 \leq t < b_1$, W, W^{-1} *are the same functions as in Lemma* 1.4, *and*

$$b_1 = \sup\left\{t \geq t_0 : W(c) + \int_{t_0}^{t}\left(\lambda_1(s) + \lambda_2(s) + \lambda_3(s)\right.\right.$$

$$\cdot \int_{t_0}^{s}\left(\lambda_4(\tau) + \lambda_5(\tau)\int_{t_0}^{\tau}\lambda_6(r)dr + \lambda_7(\tau)\int_{t_0}^{\tau}\lambda_8(r)dr\right)d\tau$$

$$\left.\left. + \lambda_9(s)\int_{t_0}^{s}\lambda_{10}(\tau)d\tau\right)ds \in \operatorname{dom} W^{-1}\right\}.$$

We need the following corollary for the proof.

Corollary 1.6. *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ *and* $w(u)$ *be nondecreasing in* u, $u \leq w(u)$. *Suppose that for some* $c > 0$,

$$u(t) \leq c + \int_{t_0}^{t}\lambda_1(s)u(s)ds + \int_{t_0}^{t}\lambda_2(s)w(u(s))ds$$

$$+ \int_{t_0}^{t}\lambda_3(s)\int_{t_0}^{s}\lambda_4(\tau)u(\tau)d\tau ds$$

$$+ \int_{t_0}^{t}\lambda_5(s)\int_{t_0}^{s}\lambda_6(\tau)w(u(\tau))d\tau ds, \quad 0 \leq t_0 \leq t.$$

Then

$$u(t) \leq W^{-1}\left[W(c) + \int_{t_0}^{t}\left(\lambda_1(s) + \lambda_2(s)\right.\right.$$

$$\left.\left. + \lambda_3(s)\int_{t_0}^{s}\left(\lambda_4(\tau)d\tau + \lambda_5(\tau)\int_{t_0}^{s}\lambda_6(\tau)d\tau\right)ds\right)\right],$$

where $t_0 \leq t < b_1$, W, W^{-1} *are the same functions as in Lemma* 1.4, *and*

$$b_1 = \sup\left\{t \geq t_0 : W(c) + \int_{t_0}^{t}\left(\lambda_1(s) + \lambda_2(s) + \lambda_3(s)\int_{t_0}^{s}\lambda_4(\tau)d\tau\right.\right.$$

$$\left.\left. + \lambda_5(s)\int_{t_0}^{s}\lambda_6(\tau)d\tau\right)ds \in \operatorname{dom} W^{-1}\right\}.$$

2. Main Results

In this section, we study the asymptotic behavior for solutions of the perturbed functional differential systems.

To obtain the asymptotic behavior, the following assumptions are needed:

(H1) The solution $x = 0$ of (1.1) is EASV.

(H2) $w(u)$ is nondecreasing in u such that $u \leq w(u)$ and $\frac{1}{v} w(u) \leq w\left(\frac{u}{v}\right)$ for some $v > 0$.

Theorem 2.1. *Assume that* (H1), (H2), *and that the perturbing term* $g(t, y, T_1 y)$ *satisfies*

$$|g(t, y(t), T_1 y(t))| \leq e^{-\alpha t}(a(t)|y(t)| + b(t)w(|y(t)|) + |T_1 y(t)|), \quad (2.1)$$

$$|T_1 y(t)| \leq b(t)\int_{t_0}^{t} k(s)|y(s)|ds + d(t)\int_{t_0}^{t} p(s)w(|y(s)|)ds, \quad (2.2)$$

$$|h(t, y(t), T_2 y(t))| \leq e^{-\alpha t}c(t)w(|y(t)|) + |T_2 y(t)|, \quad (2.3)$$

and

$$|T_2 y(t)| \leq e^{-\alpha t}m(t)|y(t)| + \int_{t_0}^{t} e^{-\alpha s}q(s)|y(s)|ds, \quad (2.4)$$

where $\alpha > 0$, $a, b, c, d, k, m, p, q, w \in C(R^+)$, $a, b, c, d, k, m, p, q \in L^1(\mathbb{R}^+)$, *and* $T_1, T_2 : C(\mathbb{R}^+, \mathbb{R}^n) \to C(\mathbb{R}^+, \mathbb{R}^n)$ *are continuous operators. If*

$$M(t_0) = W^{-1}\Bigg[W(c) + M\int_{t_0}^{\infty}\Bigg(c(s) + m(s) + e^{\alpha s}\int_{t_0}^{s}\Bigg(a(\tau) + b(\tau) + q(\tau) + b(\tau)\int_{t_0}^{\tau} k(r)dr + d(\tau)\int_{t_0}^{\tau} p(r)dr\Bigg)d\tau\Bigg)ds\Bigg] < \infty, \quad (2.5)$$

where $t \leq t_0$, $c = |y_0| M e^{\alpha t_0}$, *and* W, W^{-1} *are the same functions as in Lemma* 1.4, *then all solutions of* (1.2) *approach zero as* $t \to \infty$.

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. By the assumption (H1), it is EAS by Remark 1.2. Using Lemma 1.3, together with (2.1)-(2.4), we obtain

$$
\begin{aligned}
&|y(t)| \\
&\leq M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha(t-s)}\Bigg(\int_{t_0}^{s} e^{-\alpha\tau}\Bigg((a(\tau)+q(\tau))|y(\tau)| \\
&\quad + b(\tau)w(|y(\tau)|) + b(\tau)\int_{t_0}^{\tau} k(r)|y(r)|dr \\
&\quad + d(\tau)\int_{t_0}^{\tau} p(r)w(|y(r)|)dr\Bigg)d\tau + e^{-\alpha s}m(s)|y(s)| + e^{-\alpha s}c(s)w(|y(s)|)\Bigg)ds.
\end{aligned}
$$

Applying the assumption (H2), we have

$$
\begin{aligned}
&|y(t)| \\
&\leq M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha t}\Bigg(e^{\alpha s}\int_{t_0}^{s}\Bigg((a(\tau)+q(\tau))|y(\tau)|e^{\alpha\tau} \\
&\quad + b(\tau)w(|y(\tau)|e^{\alpha\tau}) + b(\tau)\int_{t_0}^{\tau} k(r)|y(r)|e^{\alpha r}dr \\
&\quad + d(\tau)\int_{t_0}^{\tau} p(r)w(|y(r)|e^{\alpha r})dr\Bigg)d\tau + m(s)|y(s)|e^{\alpha s} + c(s)w(|y(s)|)e^{\alpha s}\Bigg)ds.
\end{aligned}
$$

Set $u(t) = |y(t)|e^{\alpha t}$. An application of Lemma 1.5 and (2.5) obtains

$$
\begin{aligned}
|y(t)| \leq e^{-\alpha t}W^{-1}\Bigg[W(c) + M\int_{t_0}^{t}\Bigg(c(s) + m(s) + e^{\alpha s}\int_{t_0}^{s}\Bigg(a(\tau) + b(\tau) \\
+ q(\tau) + b(\tau)\int_{t_0}^{\tau} k(r)dr + d(\tau)\int_{t_0}^{\tau} p(r)dr\Bigg)d\tau\Bigg)ds\Bigg] \leq e^{-\alpha t}M(t_0),
\end{aligned}
$$

where $t \geq t_0$ and $c = M|y_0|e^{\alpha t_0}$. Hence, all solutions of (1.2) approach zero as $t \to \infty$. This completes the proof. □

Remark 2.2. Letting $c(t) = m(t) = q(t) = 0$ in Theorem 2.1, we obtain the same result as that of Theorem 3.5 in [4].

Theorem 2.3. *Assume that* $(H1)$, $(H2)$, *and that the perturbing term* $g(t, y, T_1 y)$ *satisfies*

$$\int_{t_0}^{t} |g(s, y(s), T_1 y(s))| ds \leq e^{-\alpha t}(a(t)|y(t)| + b(t)w(|y(t)|) + |T_1 y(t)|), \quad (2.6)$$

$$|T_1 y(t)| \leq b(t)\int_{t_0}^{t} k(s)|y(s)|ds + d(t)\int_{t_0}^{t} p(s)w(|y(s)|)ds, \quad (2.7)$$

$$|h(t, y(t), T_2 y(t))| \leq e^{-\alpha t}\left(b(t)\int_{t_0}^{t} c(s)|y(s)|ds + |T_2 y(t)|\right), \quad (2.8)$$

and

$$|T_2 y(t)| \leq m(t)|y(t)| + d(t)\int_{t_0}^{t} q(s)w(|y(s)|)ds, \quad (2.9)$$

where $\alpha > 0$, $a, b, c, d, k, m, p, q, w \in C(\mathbb{R}^+)$, $a, b, c, d, k, m, p, q \in L^1(\mathbb{R}^+)$, *and* $T_1, T_2 : C(\mathbb{R}^+, \mathbb{R}^n) \to C(\mathbb{R}^+, \mathbb{R}^n)$ *are continuous operators. If*

$$M(t_0) = W^{-1}\Bigg[W(c) + M\int_{t_0}^{\infty}\Bigg(a(s) + b(s) + m(s) + b(s)$$

$$\cdot \int_{t_0}^{s}(c(\tau) + k(\tau))d\tau + d(s)\int_{t_0}^{s}(p(\tau) + q(\tau))d\tau\Bigg)ds\Bigg] < \infty, \quad (2.10)$$

where $b_1 = \infty$, $c = M|y_0|e^{\alpha t_0}$, *and* W, W^{-1} *are the same functions as in Lemma* 1.4, *then all solutions of* (1.2) *approach zero as* $t \to \infty$.

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. By the assumption (H1), it is EAS. Applying Lemma 1.3, together with (2.6)-(2.9), we have

$$
\begin{aligned}
|y(t)| \le M|y_0|e^{-\alpha(t-t_0)} \\
&+ \int_{t_0}^{t} Me^{-\alpha(t-s)}\Big(e^{-\alpha s}(a(s)+m(s))|y(s)| + b(s)w(|y(s)|) \\
&+ b(s)\int_{t_0}^{s}(c(\tau)+k(\tau))|y(\tau)|d\tau \\
&+ d(s)\int_{t_0}^{s}(p(\tau)+q(\tau))w(|y(\tau)|)d\tau\Big)ds.
\end{aligned}
$$

It follows from (H2) that

$$
\begin{aligned}
|y(t)| \le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha t}\Big((a(s)+m(s))|y(s)|e^{\alpha s} \\
&+ b(s)w(|y(s)|e^{\alpha s}) + b(s)\int_{t_0}^{s}(c(\tau)+k(\tau))|y(\tau)|e^{\alpha\tau}d\tau \\
&+ d(s)\int_{t_0}^{s}(p(\tau)+q(\tau))w(|y(\tau)|e^{\alpha\tau})d\tau\Big)ds.
\end{aligned}
$$

Define $u(t) = |y(t)|e^{\alpha t}$. Then, by Corollary 1.6 and (2.10), we have

$$
\begin{aligned}
|y(t)| &\le e^{-\alpha t}W^{-1}\Big[W(c) + M\int_{t_0}^{t}\Big(a(s)+b(s)+m(s) \\
&\quad + b(s)\int_{t_0}^{s}(c(\tau)+k(\tau))d\tau + d(s)\int_{t_0}^{s}(p(\tau)+q(\tau))d\tau\Big)ds\Big] \\
&\le e^{-\alpha t}M(t_0),
\end{aligned}
$$

where $c = M|y_0|e^{\alpha t_0}$. Hence, all solutions of (1.2) approach zero as $t \to \infty$, and so the proof is complete.

Remark 2.4. Letting $c(t) = k(t) = m(t) = q(t) = 0$ in Theorem 2.3, we obtain the same result as that of Theorem 3.7 in [4].

Theorem 2.5. *Assume that* $(H1)$, $(H2)$, *and that the perturbing term* $g(t, y, T_1 y)$ *satisfies*

$$|g(t, y(t), T_1 y(t))| \le e^{-\alpha t}(a(t)|y(t)| + b(t)w(|y(t)|) + |T_1 y(t)|), \quad (2.11)$$

$$|T_1 y(t)| \le b(t)\int_{t_0}^{t} k(s)|y(s)|ds + c(t)\int_{t_0}^{t} p(s)w(|y(s)|)ds, \quad (2.12)$$

$$|h(t, y(t), T_2 y(t))| \le e^{-\alpha t} n(t)|y(t)| + |T_2 y(t)|, \quad (2.13)$$

and

$$|T_2 y(t)| \le e^{-\alpha t} m(t) w(|y(t)|) + \int_{t_0}^{t} e^{-\alpha s} d(s) w(|y(s)|)ds, \quad (2.14)$$

where $\alpha > 0$, $a, b, c, d, k, m, n, p, w \in C(\mathbb{R}^+)$, $a, b, c, d, k, m, n, p \in L^1(\mathbb{R}^+)$, *and* $T_1, T_2 : C(\mathbb{R}^+, \mathbb{R}^n) \to C(\mathbb{R}^+, \mathbb{R}^n)$ *are continuous operators. If*

$$M(t_0) = W^{-1}\Bigg[W(c) + M\int_{t_0}^{\infty}\Bigg(m(s) + n(s) + e^{\alpha s}\int_{t_0}^{s}\Bigg(a(\tau) + b(\tau)$$

$$+ d(\tau) + b(\tau)\int_{t_0}^{\tau} k(r)dr + c(\tau)\int_{t_0}^{\tau} p(r)dr\Bigg)d\tau\Bigg)ds\Bigg] < \infty, \quad (2.15)$$

where $t \ge t_0$, $c = |y_0| M e^{\alpha t_0}$, *and* W, W^{-1} *are the same functions as in Lemma* 1.4, *then all solutions of* (1.2) *approach zero as* $t \to \infty$.

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. By the assumption (H1), it is EAS by Remark 1.2. Using Lemma 1.3, together with (2.11)-(2.14), we obtain

$$|y(t)|$$

$$\le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} M e^{-\alpha(t-s)}\Bigg(\int_{t_0}^{s} e^{-\alpha\tau}\Bigg(a(\tau)|y(\tau)|$$

$$+ (b(\tau) + d(\tau))w(|y(\tau)|) + b(\tau)\int_{t_0}^{\tau} k(r)|y(r)|dr$$

$$+ c(\tau)\int_{t_0}^{\tau} p(r)w(|y(r)|)dr\Bigg)d\tau + e^{-\alpha s}n(s)|y(s)| + e^{-\alpha s}m(s)w(|y(s)|)\Bigg)ds.$$

Applying the assumption (H2), we have

$$\begin{aligned} &|y(t)| \\ &\le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha t}\Bigg(e^{\alpha s}\int_{t_0}^{s}\Bigg(a(\tau)|y(\tau)|e^{\alpha\tau} \\ &\quad + (b(\tau) + d(\tau))w(|y(\tau)|e^{\alpha\tau}) + b(\tau)\int_{t_0}^{\tau} k(r)|y(r)|e^{\alpha r}dr \\ &\quad + c(\tau)\int_{t_0}^{\tau} p(r)w(|y(r)|e^{\alpha r})dr\Bigg)d\tau \\ &\quad + m(s)w(|y(s)|e^{\alpha s}) + n(s)|y(s)|e^{\alpha s}\Bigg)ds. \end{aligned}$$

Set $u(t) = |y(t)|e^{\alpha t}$. An application of Lemma 1.5 and (2.15) obtains

$$\begin{aligned} |y(t)| &\le e^{-\alpha t}W^{-1}\Bigg[W(c) + M\int_{t_0}^{t}\Bigg(m(s) + n(s) + e^{\alpha s}\int_{t_0}^{s}\Bigg(a(\tau) + b(\tau) \\ &\quad + d(\tau) + b(\tau)\int_{t_0}^{\tau} k(r)dr + c(\tau)\int_{t_0}^{\tau} p(r)dr\Bigg)d\tau\Bigg)ds\Bigg] \le e^{-\alpha t}M(t_0), \end{aligned}$$

where $t \ge t_0$ and $c = M|y_0|e^{\alpha t_0}$. Therefore, all solutions of (1.2) approach zero as $t \to \infty$. Hence, the theorem is proved. □

Remark 2.6. Letting $d(t) = m(t) = n(t) = 0$ in Theorem 2.5, we obtain the same result as that of Theorem 3.5 in [4].

Theorem 2.7. *Assume that* (*H*1), (*H*2), *and that the perturbing term* $g(t, y, T_1 y)$ *satisfies*

$$\int_{t_0}^{t} |g(s, y(s), T_1 y(s))| ds \le e^{-\alpha t}(a(t)|y(t)| + b(t)w(|y(t)|) + |T_1 y(t)|), \quad (2.16)$$

$$|T_1 y(t)| \le b(t)\int_{t_0}^{t} k(s)|y(s)|ds + m(t)\int_{t_0}^{t} p(s)w(|y(s)|)ds, \quad (2.17)$$

$$|h(t, y(t), T_2 y(t))| \le e^{-\alpha t}\left(m(t)\int_{t_0}^{t} c(s)w(|y(s)|)ds + |T_2 y(t)|\right), \quad (2.18)$$

and

$$|T_2 y(t)| \le d(t)w(|y(t)|) + b(t)\int_{t_0}^{t} q(s)|y(s)|ds, \quad (2.19)$$

where $\alpha > 0$, $a, b, c, d, k, m, p, q, w \in C(\mathbb{R}^+)$, $a, b, c, d, k, m, p, q \in L^1(\mathbb{R}^+)$, *and* $T_1, T_2 : C(\mathbb{R}^+, \mathbb{R}^n) \to C(\mathbb{R}^+, \mathbb{R}^n)$ *are continuous operators. If*

$$M(t_0) = W^{-1}\Bigg[W(c) + M\int_{t_0}^{\infty}\Bigg(a(s) + b(s) + d(s)$$

$$+ b(s)\int_{t_0}^{s}(k(\tau) + q(\tau))d\tau + m(s)\int_{t_0}^{s}(c(\tau) + p(\tau))d\tau\Bigg)ds\Bigg] < \infty, \quad (2.20)$$

where $b_1 = \infty$, $c = M|y_0|e^{\alpha t_0}$, *and* W, W^{-1} *are the same functions as in Lemma* 1.4, *then all solutions of* (1.2) *approach zero as* $t \to \infty$.

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. By the assumption (H1), it is EAS. Applying Lemma 1.3, together with (2.16)-(2.19), we have

$$|y(t)| \le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha(t-s)}$$

$$\cdot\Bigg(e^{-\alpha s}\Bigg(a(s)|y(s)| + b(s)w(|y(s)|) + b(s)\int_{t_0}^{s}(k(\tau) + q(\tau))|y(\tau)|d\tau$$

$$+ m(s)\int_{t_0}^{s}(c(\tau) + p(\tau))w(|y(\tau)|)d\tau + d(s)w(|y(s)|)\Bigg)\Bigg)ds.$$

It follows from (H2) that

$$|y(t)| \leq M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha t}\Big(a(s)|y(s)|e^{\alpha s}$$

$$+ (b(s)+d(s))w(|y(s)|e^{\alpha s}) + b(s)\int_{t_0}^{s}(k(\tau)+q(\tau))|y(\tau)|e^{\alpha\tau}d\tau$$

$$+ m(s)\int_{t_0}^{s}(c(\tau)+p(\tau))w(|y(\tau)|e^{\alpha\tau})d\tau\Big)ds.$$

Let $u(t) = |y(t)|e^{\alpha t}$. Then, by Corollary 1.6 and (2.20), we obtain

$$|y(t)| \leq e^{-\alpha t}W^{-1}\Big[W(c) + M\int_{t_0}^{t}\Big(a(s)+b(s)+d(s)$$

$$+ b(s)\int_{t_0}^{s}(k(\tau)+q(\tau))d\tau + m(s)\int_{t_0}^{s}(c(\tau)+p(\tau))d\tau\Big)ds\Big]$$

$$\leq e^{-\alpha t}M(t_0),$$

where $c = M|y_0|e^{\alpha t_0}$. Therefore, all solutions of (1.2) approach zero as $t \to \infty$. Hence, the proof is complete. □

Remark 2.8. Letting $a(t) = d(t) = m(t) = q(t) = 0$ in Theorem 2.7, we obtain the same result as that of Theorem 3.3 in [14].

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