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ON REGULARITY OF TRANSFORMATION SEMIGROUPS PRESERVING EQUIVALENCE WITH RESTRICTED CROSS-SECTION

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Abstract

Let X be a nonempty set and $T(X)$ be the full transformation semigroup on a set X. For an equivalence relation E on X and a cross-section R of the partition X/E induced by E, let

$$T_{E^*}(X) = \{\alpha \in T(X) : \forall x, y \in X, (x, y) \in E \Leftrightarrow (x\alpha, y\alpha) \in E\}$$

and

$$T_E(X, R)$$

$$= \{\alpha \in T(X) : R\alpha = R, \forall x, y \in X, (x, y) \in E \Rightarrow (x\alpha, y\alpha) \in E\}.$$

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Then $T_{E^*}(X)$ and $T_E(X, R)$ are subsemigroups of $T(X)$. In this paper, we show that the set of all regular elements of $T_E(X, R)$ becomes a regular semigroup. Also, we give a necessary and sufficient condition when semigroups $T_{E^*}(X)$ and $T_E(X, R)$ coincide.

1. Introduction and Preliminaries

An element x of a semigroup S is called *regular* if there exists y in S such that $x = xyx$. A semigroup S is called a *regular semigroup* if every element of S is regular. The set of all regular elements of S is denoted by $Reg(S)$.

The full transformation semigroup on a nonempty set X is denoted by $T(X)$, that is, $T(X)$ is the semigroup of all mappings $\alpha : X \to X$ under composition. The semigroup $T(X)$ is known to be regular in [5].

Let E be an equivalence relation on X. Pei [7] has introduced a family of subsemigroups of $T(X)$ defined by

$$T_E(X) = \{\alpha \in T(X) : \forall x, y \in X, (x, y) \in E \Rightarrow (x\alpha, y\alpha) \in E\}$$

and call it the *semigroup of transformations that preserve an equivalence on X*. He has studied Green's relations and regularity on $T_E(X)$. Recently, Deng et al. [4] introduced the subsemigroup of $T_E(X)$ as follows:

$$T_{E^*}(X) = \{\alpha \in T(X) : \forall x, y \in X, (x, y) \in E \Leftrightarrow (x\alpha, y\alpha) \in E\}.$$

The authors considered regularity of elements and Green's relations for $T_{E^*}(X)$.

Let R be a cross-section of the partition X/E induced by E. In [1], Araújo and Konieczny defined a subsemigroup of $T(X)$ as follows:

$$T(X, E, R)$$
$$= \{\alpha \in T(X) : R\alpha \subseteq R \text{ and } \forall x, y \in X, (x, y) \in E \Rightarrow (x\alpha, y\alpha) \in E\}.$$

Clearly, $T(X, E, R) \subseteq T_E(X)$. They have proved that the semigroups $T(X, E, R)$ are precisely the centralizers of idempotents of $T(X)$. After a year, they studied the structure and regularity of the semigroups $T(X, E, R)$. Moreover, they determined Green's relations in $T(X, E, R)$ in [2]. In this research, we examine a related subsemigroup of $T(X, E, R)$: namely, the *transformation semigroups that preserve an equivalence with restricted cross-section on X* defined by

$$T_E(X, R)$$

$$= \{\alpha \in T(X) : R\alpha = R \text{ and } \forall x, y \in X, (x, y) \in E \Rightarrow (x\alpha, y\alpha) \in E\}.$$

The aim of this paper is to prove that $Reg(T_E(X, R))$ is a regular semigroup. Also, we show that $T_{E^*}(X) = T_E(X, R)$ if and only if R is finite and E is the identity relation.

In the remainder, let E be an equivalence relation on a set X and R be a cross-section of the partition X/E. Denote by X/E the quotient set and E_r the E-class containing r for all $r \in R$.

2. Main Results

A characterization of the regularity for elements in $T_{E^*}(X)$ as follows [4]:

Theorem 2.1 [4]. *Let* $\alpha \in T_{E^*}(X)$. *Then* $\alpha \in Reg(T_{E^*}(X))$ *if and only if* $A \cap X\alpha \neq \varnothing$ *for all* $A \in X/E$.

The nature of regular elements in $T_E(X, R)$ was considered in [6].

Theorem 2.2 [6]. *Let* $\alpha \in T_E(X, R)$. *Then* $\alpha \in Reg(T_E(X, R))$ *if and only if* $\alpha|_R$ *is an injection.*

By Theorem 2.2, we obtain the following interesting result.

Proposition 2.3. $Reg(T_E(X, R)) \subseteq Reg(T_{E^*}(X))$.

Proof. Let $\alpha \in Reg(T_E(X, R))$. Then $\alpha \in T_E(X)$. Let $x, y \in X$ be such that $(x\alpha, y\alpha) \in E$. Then there exist $r, s \in R$ such that $x \in E_r$ and $y \in E_s$. Since $(x\alpha, y\alpha) \in E$ and $\alpha \in T_E(X)$, we deduce that $r\alpha = s\alpha$. Since α is regular in $T_E(X, R)$, it follows from Theorem 2.2 that $r = s$. Hence, $(x, y) \in E$, so $\alpha \in T_{E^*}(X)$. Since $R\alpha = R$, we then have $E_r \cap X\alpha \neq \emptyset$ for all $r \in R$. By Theorem 2.1, $\alpha \in Reg(T_{E^*}(X))$. Hence, $Reg(T_E(X, R)) \subseteq Reg(T_{E^*}(X))$, as desired. □

The following example shows that $Reg(T_E(X, R))$ may not be equal to $Reg(T_{E^*}(X))$.

Example 2.4. Let $X = \mathbb{N}$, $X/E = \{\{x, x+1\} : x \text{ is odd}\}$ and $R = \{x \in X : x \text{ is odd}\}$. Let $\alpha \in T(X)$ be defined by

$$x\alpha = \begin{cases} 2 & \text{if } x \leq 2, \\ x & \text{otherwise.} \end{cases}$$

Then $\alpha \in T_{E^*}(X)$. By Theorem 2.1, we have that α is a regular element of $T_{E^*}(X)$. Since $r\alpha \neq 1$ for all $r \in R$ and $1 \in R$, we have $R\alpha \neq R$. This implies that $\alpha \notin T_E(X, R)$, hence $\alpha \notin Reg(T_E(X, R))$.

We first prove lemma needed to determine if $Reg(T_E(X, R))$ is regular.

Lemma 2.5. *Let* $\alpha \in Reg(T_E(X, R))$. *If* $r \in R$, *then* $E_r \cap X\alpha = E_{r'}\alpha$ *for some* $r' \in R$.

Proof. Assume that $r \in R$. Since $R\alpha = R$, there exists $r' \in R$ such that $r = r'\alpha$. Since $\alpha \in T_E(X)$, it then follows that $E_{r'}\alpha \subseteq E_r \cap X\alpha$. For the reverse inclusion, if $y \in E_r \cap X\alpha$, then $y = x\alpha$ for some $x \in X$. This implies that $x \in E_s$ for some $s \in R$ and so $s\alpha = r$. Since $\alpha \in$

$Reg(T_E(X, R))$ by Theorem 2.2, $\alpha|_R$ is injective. Since $s\alpha = r'\alpha$, we have $s = r'$. Hence, $y \in E_{r'}\alpha$. This shows that $E_r \cap X\alpha \subseteq E_{r'}\alpha$ and so equality holds. □

Theorem 2.6. *$Reg(T_E(X, R))$ is a regular semigroup.*

Proof. It is easy to see that $Reg(T_E(X, R))$ contains the identity mapping on X, hence $Reg(T_E(X, R)) \neq \varnothing$. Let $\alpha, \beta \in Reg(T_E(X, R))$. To show that $\alpha\beta$ is regular, let $r, s \in R$ be such that $r\alpha\beta = s\alpha\beta$. Since $r\alpha, s\alpha \in R$ and $\beta \in Reg(T_E(X, R))$, we get $r\alpha = s\alpha$ by Theorem 2.2. Similarly, we have $r = s$. This verifies that $\alpha\beta|_R$ is injective. From Theorem 2.2, $\alpha\beta \in Reg(T_E(X, R))$. Hence, $Reg(T_E(X, R))$ is a subsemigroup of $T_E(X, R)$.

Let $\alpha \in Reg(T_E(X, R))$. We construct $\beta \in Reg(T_E(X, R))$ such that $\alpha = \alpha\beta\alpha$. For each $r \in R$, we choose $r' \in R$ such that $E_r \cap X\alpha = E_{r'}\alpha$ by Lemma 2.5. It follows that $r = r'\alpha$. Let $a_r = r'$. For each $y \in (E_r \cap X\alpha)\backslash\{r\}$, we choose $a_y \in E_{r'}$ such that $a_y\alpha = y$. Define $\beta_r : E_r \to E_{r'}$ by

$$x\beta_r = \begin{cases} a_x & \text{if } x \in X\alpha, \\ r' & \text{otherwise.} \end{cases}$$

Then β_r is well-defined, $E_r\beta_r \subseteq E_{r'}$ and $r\beta_r = a_r = r' \in R$. Let $\beta \in T(X)$ be defined by $\beta|_{E_r} = \beta_r$ for all $r \in R$. Since R is a cross-section of the partition X/E induced by E, β is well-defined. Obviously, $\beta \in T_E(X)$ and $R\beta \subseteq R$. Let $r \in R$. Then $r\alpha = s$ for some $s \in R$. Thus, $s\beta_s = a_s = s'$ for some $s' \in R$ with $s'\alpha = s$. Therefore, $s'\alpha = r\alpha$. By assumption, we have that $s' = r$ and thus $s\beta = s\beta|_{E_s} = s\beta_s = a_s = s' = r$. It follows that $R\beta = R$ and therefore $\beta \in T_E(X, R)$. If $x \in X$, then $x\alpha \in E_r$ for some $r \in R$. Thus,

$$x\alpha\beta\alpha = (x\alpha)\beta|_{E_r}\alpha = (x\alpha)\beta_r\alpha = a_{x\alpha}\alpha = x\alpha$$

and therefore $\alpha = \alpha\beta\alpha$. It remains to show that $\beta \in Reg(T_E(X, R))$. To do this, let $r, s \in R$ be such that $r\beta = s\beta$. Then $r\beta = a_r = r'$ and $s\beta = a_s = s'$ for some $r', s' \in R$ with $r = r'\alpha$ and $s = s'\alpha$. Thus, $r = r'\alpha = s'\alpha = s$. By Theorem 2.2, $\beta \in Reg(T_E(X, R))$.

Hence, the theorem is completely proved. □

Next, we describe the complement of $Reg(T_E(X, R))$ in $T_E(X, R)$.

Theorem 2.7. *If* $T_E(X, R)\setminus Reg(T_E(X, R))$ *is a nonempty set, then it is an ideal of* $T_E(X, R)$.

Proof. Suppose that $T_E(X, R)\setminus Reg(T_E(X, R)) \neq \emptyset$. Let $\alpha \in T_E(X, R)\setminus Reg(T_E(X, R))$ and $\beta, \gamma \in T_E(X, R)$. Suppose that $\beta\alpha\gamma \in Reg(T_E(X, R))$. Claim that $\alpha\gamma|_R$ is injective. Let $r, s \in R$ be such that $r\alpha\gamma = s\alpha\gamma$. Since $R\beta = R$, $r'\beta = r$ and $s'\beta = s$ for some $r', s' \in R$. Thus, $r'\beta\alpha\gamma = s'\beta\alpha\gamma$. By the regularity of $\beta\alpha\gamma$, we get that $r' = s'$ and hence $r = r'\beta = s'\beta = s$. So we have the claim. From Theorem 2.2, $\alpha\gamma \in Reg(T_E(X, R))$. Since $\alpha \notin Reg(T_E(X, R))$, there are distinct elements $r, s \in R$ such that $r\alpha = s\alpha$. Thus, $r\alpha\gamma = s\alpha\gamma$. But $\alpha\gamma \in Reg(T_E(X, R))$, we have that r and s must be equal, contradicting the supposition. Hence, $\beta\alpha\gamma \notin Reg(T_E(X, R))$. Consequently, $T_E(X, R)\setminus Reg(T_E(X, R))$ is an ideal of $T_E(X, R)$, as required. □

The next theorems provide conditions under which semigroups $T_{E^*}(X)$ and $T_E(X, R)$ are regular [4, 6].

Theorem 2.8 [4]. $T_{E^*}(X)$ *is a regular semigroup if and only if* X/E *is finite*.

Theorem 2.9 [6]. $T_E(X, R)$ *is a regular semigroup if and only if* R *is finite*.

Finally, we investigate the equality of the semigroups $T_{E^*}(X)$ and $T_E(X, R)$.

Theorem 2.10. *$T_E(X, R) = T_{E^*}(X)$ if and only if R is finite and E is the identity relation.*

Proof. Suppose that $T_E(X, R) = T_{E^*}(X)$. To show that $T_{E^*}(X)$ is regular, let $\alpha \in T_{E^*}(X)$. Then $\alpha \in T_E(X, R)$ and hence $R\alpha = R$. It follows that $E_r \cap X\alpha \neq \varnothing$ for all $r \in R$. From Theorem 2.1, we have $\alpha \in Reg(T_{E^*}(X))$. Hence, $T_{E^*}(X)$ is regular. By Theorem 2.8, X/E is finite and this implies that R is finite. Next, to show that E is the identity relation on X, let $a, b \in X$ be such that $(a, b) \in E$. Then there exists $r \in R$ such that $a, b \in E_r$. Let $\alpha, \beta \in T(X)$ be defined by

$$x\alpha = \begin{cases} a & \text{if } x \in E_r, \\ x & \text{otherwise} \end{cases}$$

and

$$x\beta = \begin{cases} b & \text{if } x \in E_r, \\ x & \text{otherwise.} \end{cases}$$

Obviously, $\alpha, \beta \in T_{E^*}(X)$. By assumption, $\alpha, \beta \in T_E(X, R)$. It then follows that $a = r\alpha = r = r\beta = b$. Hence, E is the identity relation, as desired.

Conversely, assume that R is finite and E is the identity relation. Since R is finite, X/E is also finite. Then Theorems 2.8, 2.9 and Proposition 2.3 imply that $T_E(X, R) \subseteq T_{E^*}(X)$. For the reverse inclusion, let $\alpha \in T_{E^*}(X)$. It suffices to show that $R\alpha = R$. Let $r \in R$. Then $r\alpha \in E_s$ for some $s \in R$. Since E is the identity relation, $r\alpha = s$ and hence $R\alpha \subseteq R$. On the other hand, let $r \in R$. Since X/E is finite, by Theorem 2.8, we have α is a regular element of $T_{E^*}(X)$. By Theorem 2.1, $E_r \cap X\alpha \neq \varnothing$. Then there exists

$r' \in R$ such that $E_{r'}\alpha \subseteq E_r$. Consequently, $(r'\alpha, r) \in E$. By assumption, we have $r'\alpha = r$. Thus, $R \subseteq R\alpha$, the equality holds. Thus, $\alpha \in T_E(X, R)$ and hence $T_E(X, R) = T_{E^*}(X)$.

Therefore, the proof is complete. $\square$

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