Far East Journal of Mathematical Sciences (FJMS)

http://www.pphmj.com
http://dx.doi.org/10.17654/MS102112551
Volume 102, Number 11, 2017, Pages 2551-2561 **ISSN: 0972-0871**

A NEW APPROACH FOR SOLUTION OF LINEAR TIME-DELAY INTEGRAL EQUATIONS VIA Z-DECOMMISSION METHOD

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Abstract

The main objective of the present research is to provide a new approach for solution of linear time-delay integral equations via Z-decommission method (ZDM). The selected method is applied to solve the equations of Fredholm and Volterra types with time-delay. This study is so important for various fields of sciences, engineering, space and time domain and population dynamics. We applied this approach to some examples of time-delay integral equations. In fact,

Received: May 19, 2017; Accepted: September 20, 2017

2010 Mathematics Subject Classification: 45A05, 45B05, 65D30, 45D05, 62P35, 62P30.

Keywords and phrases: Z-decommissions method (ZDM), exact solution, time-delay Volterra integral equations (TDVIEs), time-delay Fredholm integral equations (TDFIEs).

ZDM is easily applied for obtaining highly accurate (exact) solutions without requiring a large number of computations and complex numerical operations.

1. Introduction

Consider the type of integral equations with time-delay $\tau > 0$, shown below [1-3, 10]

$$h(t)g(t) = f(t) + \lambda\int_{t_0}^{b(t)} k(t, s)g(s-\tau)ds, \quad t \in [t_0, T], \tag{1}$$

where the functions $h(t)$, $f(t)$, $k(t, s)$, $b(t)$ are given and $g(t)$ is unknown, λ is constant and $\forall t \in [t_0, T]$, $k(t, s) \le M$.

Equation (1) can be determined by many types of equations which depend on $h(t)$, $f(t)$ and $b(t)$. If $h(t) = 0$ and $h(t) = 1$, then the *time-delay* is called to be of the first kind and second kind, respectively. If $f(t) = 0$ and $f(t) \neq 0$, then it is called *time-delay* of the homogeneous and nonhomogeneous equations, respectively. If $b(t) = b$, where b is fixed, then it is called *Fredholm* (*FIEs*) with time-delay. If $b(t) = t$, then it is called *Volterra* (*VIEs*) with time-delay.

These types of equations are so important for various fields of sciences, engineering, population dynamics, and space and time domain. Here we review some of the studies done [1-8] on this type of equations. To begin with [1], expansion methods have been used to solve VIEs and FIEs with time lags. In [2], Nouri and Maleknejad used block pulse functions to VIEs and FIEs with time-delay. Stability analysis for VIEs with delay has been discussed in [3]. In [4], trapezoidal direct quadrature method is introduced for VIEs with delay. Zarebnia and Shiri applied sinc-collocation method to VIEs with delay in [5]. Otadi and Mosleh [6] presented universal approximation method for VIEs with delay. Numerical solution of VIEs with delay by single term Walsh series method has been given in [7]. In [8], collocation methods are proposed for VIEs with delay. And also other papers (see [9-11]).

In general, integral equations whether Fredholm or Volterra have been studied by various methods (see [12-20] and references therein). This research focuses on linear time-delay integral equations which are essential in various sciences.

2. Description of the Algorithm

Below, we suggest a new algorithm for solution of equation (1) with non-homogeneous second kind $(h(t) = 1 \text{ and } f(t) \neq 0)$. To apply this algorithm, we split $f(t)$ as

$$f(t) = f_1(t) + f_2(t). \tag{2}$$

Thus, equation (1) becomes

$$g(t) = f_1(t) + f_2(t) + \lambda \int_{t_0}^{b(t)} k(t, s) g(s - \tau) ds, \quad \tau > 0. \tag{3}$$

Start with the following initial guess:

$$g_0(t) = f_1(t). \tag{4}$$

The basic of proposed algorithm is to construct a recurrence relation, which reads

$$Z_n(t) = g_0(t) + f_2(t) - g_n(t) + \lambda \int_{t_0}^{b(t)} k(t, s) g_n(s - \tau) ds, \quad n = 0, 1, 2, ..., \tag{5}$$

where $g_n(t)$ is obtained by

$$g_{n+1}(t) = g_n(t) + Z_n(t), \quad n = 0, 1, 2, \ldots. \tag{6}$$

We finally obtain the solution as

$$g(t) = \lim_{n\to\infty} g_n(t). \tag{7}$$

Theorem. *Equation* (1) *has a unique solution if* $\alpha = LMT$, *where L-Lipschitz and* $\alpha \in (0, 1)$.

Proof. Let equation (1) has two solutions φ and φ^*. Then

$$
\begin{aligned}
|\varphi - \varphi^*| &= \left| \lambda \int_{t_0}^{b(t)} k(t, s)\varphi(s-\tau)ds - \lambda \int_{t_0}^{b(t)} k(t, s)\varphi^*(s-\tau)ds \right| \\
&= \left| \lambda \int_{t_0}^{b(t)} k(t, s)(\varphi(s-\tau) - \varphi^*(s-\tau))ds \right| \\
&\leq |\lambda| \int_{t_0}^{b(t)} |k(t, s)| |(\varphi(s-\tau) - \varphi^*(s-\tau))| ds
\end{aligned}
$$

Using Lipschitz condition, we have

$$
\begin{aligned}
&\leq LM|\varphi - \varphi^*||\lambda| \int_{t_0}^{b(t)} ds \\
&\leq LMT|\varphi - \varphi^*| \\
&\leq \alpha|\varphi - \varphi^*| \\
&(1-\alpha)|\varphi - \varphi^*| \leq 0.
\end{aligned}
$$

Since $\alpha \in (0, 1)$, therefore $|\varphi - \varphi^*| = 0$ and hence $\varphi = \varphi^*$. □

3. Applications

To show the performance of ZDM in obtaining highly accurate (true) solutions of FIEs and VIEs types with time-delay, we examine in depth three illustrative examples that are solved using ZDM. The computations are performed by Maple 17 software. Examples 1 and 2 take VIEs form with time-delay. Also, Example 3 takes FIEs form with time-delay.

Volterra type with time-delay

Example 1. Consider equation (1) with the constants $\lambda = 1$, $\tau = 1$ and $t_0 = 0$, and the functions $k(t, s) = t$, $h(t) = 1$, $f(t) = e^t - te^{t-1} + te^{-1}$ and $b(t) = t$. Then true solution $g(t) = e^t$, $t \in [0, 1]$, i.e., [10]

$$g(t) = e^t - te^{t-1} + te^{-1} + \int_0^t tg(s-1)ds. \tag{8}$$

To apply the algorithm in Section 2, we split $f(t)$ as

$$f(t) = f_1(t) + f_2(t),$$

where $f_1(t) = e^t$ and $f_2(t) = -te^{t-1} + te^{-1}$.

Apply equation (5) with above equations to get

$$Z_n(t) = g_0(t) - te^{t-1} + te^{-1} - g_n(t) + \int_0^t tg_n(s-1)ds, \quad n = 0, 1, 2, ..., \tag{9}$$

where $g_0(t) = f_1(t)$.

Calculate $g_n(t)$ as follows:

$$g_{n+1}(t) = g_n(t) + Z_n(t), \quad n = 0, 1, 2, \ldots.$$

The use of above steps, yields:

Zeroth-order problem

$$\begin{cases} g_0(t) = f_1(t) \\ \qquad\;\; = e^t. \end{cases}$$

First-order problem

$$\begin{cases} Z_0(t) = g_0(t) - te^{t-1} + te^{-1} - g_0(t) + \int_0^t tg_0(s-1)ds \\ \qquad\;\; = 0, \\ g_1(t) = g_0(t) + Z_0(t) \\ \qquad\;\; = e^t. \end{cases}$$

Second-order problem

$$\begin{cases} Z_1(t) = g_0(t) - te^{t-1} + te^{-1} - g_1(t) + \int_0^t tg_1(s-1)ds \\ \qquad = 0, \\ g_2(t) = g_1(t) + Z_1(t) \\ \qquad = e^t. \end{cases}$$

Third-order problem

$$\begin{cases} Z_2(t) = g_0(t) - te^{t-1} + te^{-1} - g_2(t) + \int_0^t tg_2(s-1)ds \\ \qquad = 0, \\ g_3(t) = g_2(t) + Z_2(t) \\ \qquad = e^t. \end{cases}$$

So, true solution is obtained.

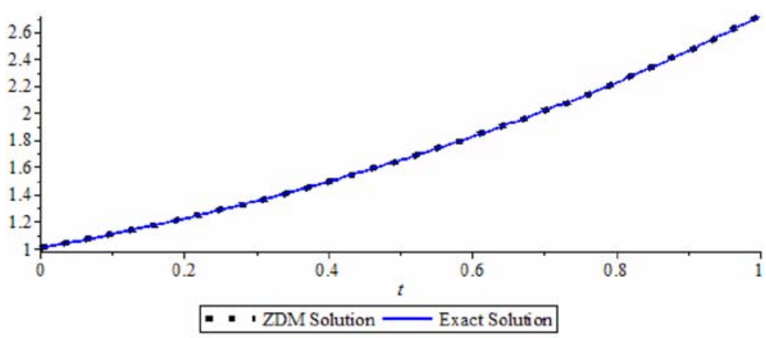

Figure 1. Graph solution for Example 1.

Example 2. Here, we solve equation (1) with $\lambda = 1$, $\tau = 1$, $t_0 = 0$, $k(t, s) = t^2$, $h(t) = 1$, $b(t) = t$ and $f(t) = \sin(t) + t^2\cos(t-1) - t^2\cos(-1)$ with true solution $g(t) = \sin(t)$, $t \in [0, 1]$. So, we have [10]

$$g(t) = \sin(t) + t^2\cos(t-1) - t^2\cos(-1) + \int_0^t t^2 g(s-1)ds. \tag{10}$$

Let $f_1(t) = \sin(t)$ and $f_2(t) = t^2\cos(t-1) - t^2\cos(-1)$.

Now, $Z_n(t)$ is obtained as

$$Z_n(t) = g_0(t) + t^2 \cos(t-1) - t^2 \cos(-1) - g_n(t) + \int_0^t t^2 g_n(s-1)ds,$$

$$n = 0, 1, 2, \ldots. \qquad (11)$$

Choose $g_0(t) = f_1(t)$ and calculate $g_n(t)$ as follows:

$$g_{n+1}(t) = g_n(t) + Z_n(t), \quad n = 0, 1, 2, \ldots.$$

Consequently, we have:

Zeroth-order problem

$$\begin{cases} g_0(t) = f_1(t) \\ \qquad\quad = \sin(t). \end{cases}$$

First-order problem

$$\begin{cases} Z_0(t) = g_0(t) + t^2 \cos(t-1) - t^2 \cos(-1) - g_0(t) + \int_0^t t^2 g_0(s-1)ds \\ \qquad\quad = 0, \\ g_1(t) = g_0(t) + Z_0(t) \\ \qquad\quad = \sin(t). \end{cases}$$

Second-order problem

$$\begin{cases} Z_1(t) = g_0(t) + t^2 \cos(t-1) - t^2 \cos(-1) - g_1(t) + \int_0^t t^2 g_1(s-1)ds \\ \qquad\quad = 0, \\ g_2(t) = g_1(t) + Z_1(t) \\ \qquad\quad = \sin(t). \end{cases}$$

Third-order problem

$$\begin{cases} Z_2(t) = g_0(t) + t^2 \cos(t-1) - t^2 \cos(-1) - g_2(t) + \int_0^t t^2 g_2(s-1)ds \\ \qquad\quad = 0, \\ g_3(t) = g_2(t) + Z_2(t) \\ \qquad\quad = \sin(t). \end{cases}$$

So, the true solution is $g(t) = \sin(t)$.

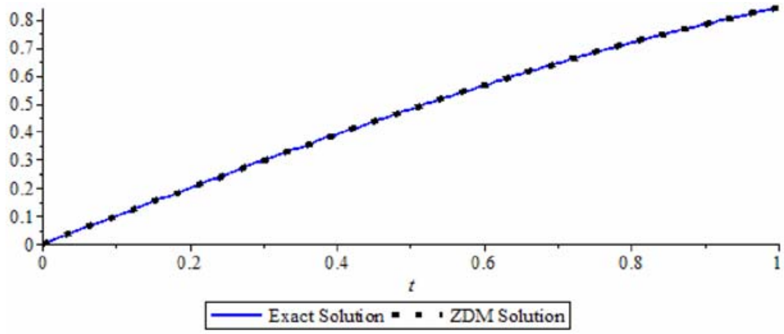

Figure 2. Graph solution for Example 2.

Fredholm type with time-delay

Example 3. In equation (1), let $\lambda = 1$, $\tau = 1$, $t_0 = -1$, $k(t, s) = st$, $h(t) = 1$, $b(t) = 1$ and $f(t) = t^2 + \frac{5}{3}t$. Then $g(t) = t^2 + t$ [9]

$$g(t) = t^2 + \frac{5}{3}t + \int_{-1}^{1} st\ g(s-1)ds. \tag{12}$$

Consider $f(t) = f_1(t) + f_2(t)$.

Next step, choose $g_0(t) = f_1(t) = t^2 + t$ and $f_2(t) = \frac{2}{3}t$.

Using equations (5)-(7), we have:

Zeroth-order problem

$$\begin{cases} g_0(t) = f_1(t) \\ \qquad\quad = t^2 + t. \end{cases}$$

First-order problem

$$\begin{cases} Z_0(t) = g_0(t) + \dfrac{2}{3}t - g_0(t) + \displaystyle\int_{-1}^{1} st\, g_0(s-1)ds \\ \qquad = 0, \\ g_1(t) = g_0(t) + Z_0(t) \\ \qquad = t^2 + t. \end{cases}$$

Second-order problem

$$\begin{cases} Z_1(t) = g_0(t) + \dfrac{2}{3}t - g_1(t) + \displaystyle\int_{-1}^{1} st\, g_1(s-1)ds \\ \qquad = 0, \\ g_2(t) = g_1(t) + Z_1(t) \\ \qquad = t^2 + t. \end{cases}$$

Third-order problem

$$\begin{cases} Z_2(t) = g_0(t) + \dfrac{2}{3}t - g_2(t) + \displaystyle\int_{-1}^{1} st\, g_2(s-1)ds \\ \qquad = 0, \\ g_3(t) = g_2(t) + Z_2(t) \\ \qquad = t^2 + t. \end{cases}$$

This is a true solution.

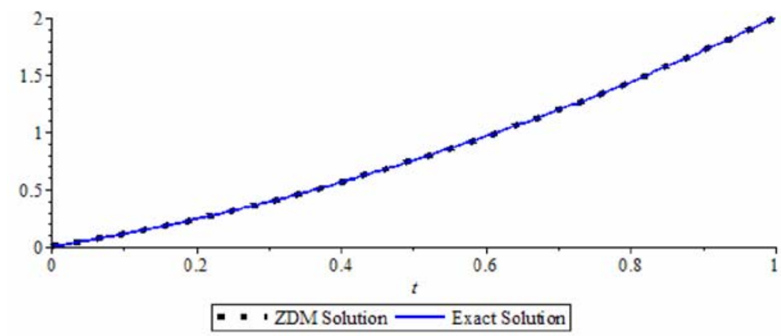

Figure 3. Graph solution for Example 3.

4. Conclusions

We have provided a new approach for solution of linear TDVIEs and TDFIEs, namely ZDM. We have applied the selected method for solving some examples of TDVIEs and TDFIEs. Clearly, ZDM is easily applied for obtaining highly accurate (exact) solutions without requiring a large number of computations and complex numerical operations.

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