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STRUCTURES AND $\mathcal{D}$-HOMOTHETIC PRODUCT METRIC

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Abstract

The purpose of this paper is to determine some remarkable classes of the induced structures on the product of a locally conformally Kähler manifold with the real line and an almost contact metric manifold.

1. Introduction

From Chinea and Gonzalez [6], it is known that there are a large number of classes of almost contact structures. These manifolds are grouped in Sasakian, cosymplectic and Kenmotsu types. In 1985, using the warped product, Oubiña showed that there is a one-to-one correspondence between Sasakian and Kählerian structures [10]. In 2013, building on the work of Tanno [8], Blair [5] introduced the notion of $\mathcal{D}$-homothetic warping and he showed by another way this correspondence.

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Recently, Beldjilali and Belkhelfa [1] introduced the notion of $\mathcal{D}$-homothetic bi-warping and proved that every Sasakian manifold M generates a 1-parameter family of Kählerian manifolds, thereby generalizing the results of Oubiña and Blair. They investigated the conditions on the product of a cosymplectic or Kenmotsu manifold and the real line to be a family of conformal Kähler manifolds. Regarding this result, one can ask if it is possible to construct a Sasakian cone with the same reasoning.

On the other hand, in [7], Kenmotsu proved that a locally Kenmotsu manifold is a warped product $\mathbb{I} \times M$ of an interval $\mathbb{I}$ and a Kähler manifold M with warping function $f(t) = ce^t$, where c is a positive constant. Conversely, the conformal Kähler manifold M is supposed to be warped product of odd dimensional manifold M and the real line $\mathbb{R}$. Then the conditions in which the odd dimensional manifold M is an almost Kenmotsu manifold are investigated lately.

In [9], Tshikuna-Matamba examined the product of an almost Hermitian manifold with an almost contact metric manifold and completed the study of Oubiña [10].

Here, by deforming the canonical almost contact metric structure with some functions of the norm of vectors and a 1-form, we construct many geometric structures rely on a locally conformally Kähler structure.

This paper is organized in the following way.

Section 2 is devoted to the background of the structures which will be used in the sequel.

In Section 3, we examine the product of a locally conformally Kähler manifold with the real line. Since this product is Sasakian or Kenmotsu manifold, we give the conversely study of Oubiña [10] and Blair [5]. In Section 4, we construct a concrete example. Section 5 is devoted to the case of the product of an almost Hermitian manifold with an almost contact metric manifold. This completes the work of Tshikuna-Matamba [9].

2. Preliminaries

2.1. Almost contact metric structures and Sasakian structures

For more background on almost contact metric manifolds, we refer the reader to [3, 4].

An odd dimensional Riemannian manifold (M, g) is said to be an *almost contact metric manifold* if there exist on *M* a $(1, 1)$ tensor field φ, a vector field ξ (called the *structure vector field*) and a 1-form η such that

$$\eta(\xi) = 1, \quad \varphi^2(X) = -X + \eta(X)\xi \quad \text{and}$$

$$g(\varphi X, \varphi X) = g(X, Y) - \eta(X)\eta(Y), \tag{2.1}$$

for any vector fields *X*, *Y* on *M*. In particular, in an almost contact metric manifold, we also have $\varphi\xi = 0$ and $\eta \circ \varphi = 0$.

Such a manifold is said to be a *contact metric manifold* if $d\eta = \Phi$, where $\Phi(X, Y) = g(X, \varphi Y)$ is called the *fundamental* 2-*form* of *M*. If, in addition, ξ is a Killing vector field, then *M* is said to be a *K-contact manifold*. It is well-known that a contact metric manifold is a *K*-contact manifold if and only if $\nabla_X \xi = -\varphi X$, for any vector field *X* on *M*.

On the other hand, the almost contact metric structure of *M* is said to be *normal* if

$$N_\varphi(X, Y) = [\varphi, \varphi](X, Y) + 2d\eta(X, Y)\xi = 0, \tag{2.2}$$

for any *X*, *Y*, where $[\varphi, \varphi]$ denotes the Nijenhuis torsion of φ, given by

$$[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y].$$

An almost contact metric structure (φ, ξ, η, g) on *M* is said to be [3, 4, 7]

$$\begin{cases} (\mathrm{a}) : \textit{Sasaki} \Leftrightarrow \Phi = d\eta \text{ and } (\varphi, \xi, \eta) \text{ is normal,} \\ (\mathrm{b}) : \textit{Cosymplectic} \Leftrightarrow d\Phi = d\eta = 0 \text{ and } (\varphi, \xi, \eta) \text{ is normal,} \\ (\mathrm{c}) : \textit{Kenmotsu} \Leftrightarrow d\eta = 0,\ d\Phi = 2\Phi \wedge \eta \text{ and } (\varphi, \xi, \eta) \text{ is normal,} \end{cases} \tag{2.3}$$

where *d* denotes the exterior derivative. These manifolds can be characterized through their Levi-Civita connection, by requiring

$$\begin{cases} (1) : \textit{Sasaki} \Leftrightarrow (\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \\ (2) : \textit{Cosymplectic} \Leftrightarrow \nabla\varphi = 0, \\ (3) : \textit{Kenmotsu} \Leftrightarrow (\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X \end{cases} \tag{2.4}$$

(see [3, 4, 11]).

2.2. Almost complex structures and Kählerian structures

For more background on almost complex structure manifolds, we recommend [11].

An almost complex manifold with a Hermitian metric is called an *almost Hermitian manifold*. For an almost Hermitian manifold (M, J, g), we thus have

$$J^2 = -1, \quad g(JX, JY) = g(X, Y).$$

An almost complex structure J is integrable, and hence the manifold is a complex manifold, if and only if its Nijenhuis tensor N_j vanishes, with

$$N_j(X, Y) = [JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]. \tag{2.5}$$

For an almost Hermitian manifold (M, J, g), we define the fundamental Kähler form as:

$$\Omega(X, Y) = g(X, JY).$$

(M, J, g) is then called *almost Kähler* if is closed, i.e., $d\Omega = 0$. It can be shown that this condition for (M, J, g) to be almost Kähler is equivalent to

$$g((\nabla_X J)Y, Z) + g((\nabla_Y J)Z, X) + g((\nabla_Z J)X, Y) = 0.$$

An almost Kähler manifold with integrable J is called a *Kähler manifold*, and thus is characterized by the conditions: $d\Omega = 0$ and $N_J = 0$. One can prove that both these conditions combined are equivalent with the single condition

$$\nabla J = 0.$$

For brevity, we call a Kählerian manifold with exact Kähler form $(\text{i.e., } \Omega = d\theta)$ for some 1-form θ an exact Kählerian manifold.

A locally conformally Kähler structure, or shortly l.c.K. structure on a differentiable manifold M is a Hermitian structure on M with its associated fundamental form Ω satisfying $d\Omega = \omega \wedge \Omega$ for some closed 1-form ω (which is so-called Lee form). A differentiable manifold M is called a *locally conformally Kähler manifold*, or shortly *l.c.K. manifold* if M admits a l.c.K. structure. Note that l.c.K. structure Ω is globally conformally Kähler (or Kähler) if and only if ω is exact (or 0, respectively).

3. Product $\mathbb{R} \times M^{2n}$

Let (M, J, g) be an almost Hermitian manifold of dimension $2n$. On the product $\tilde{M} = \mathbb{R} \times M$, one can define an almost contact structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta})$ by setting

$$\tilde{\varphi}X = JX - \theta(JX)\partial_r, \quad \tilde{\xi} = \partial_r, \quad \tilde{\eta} = dr + \theta, \tag{3.1}$$

and a Riemannian metric $\tilde{g}$ given by

$$\tilde{g} = \alpha g + \tilde{\eta} \otimes \tilde{\eta}, \tag{3.2}$$

for any vector field X of M and ∂_r denotes the unit tangent field to $\mathbb{R}$, where θ is a 1-form on M and α is a positive function on $\mathbb{R}$.

Proposition 3.1. *The structure* $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ *constructed on the product* $\tilde{M}$ *is an almost contact metric structure.*

Proof. The proof follows by a routine calculation, we shall omit it. □

We denote by $N_{\tilde{\varphi}}$ the tensor field of type $(1, 2)$ on $\tilde{M}$. Using definition of $\tilde{\varphi}$ in (3.1) with formulas (2.2) and (2.5), the only non-zero component of $N_{\tilde{\varphi}}$ is

$$\begin{aligned} N_{\tilde{\varphi}} &= ((0, X), (0, Y)) \\ &= (2(d\theta(X, Y) - d\theta(JX, JY)) - \theta(N_J(X, Y)); N_J(X, Y)), \end{aligned}$$

for all X, Y are vector fields on M.

We note that $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta})$ is normal if and only if J is integrable and $d\theta = d\theta \circ J$.

On the other hand, the fundamental 2-form $\tilde{\Phi}$ of $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is

$$\tilde{\Phi}\left(\left(a\frac{\partial}{\partial t}, X\right), \left(b\frac{\partial}{\partial t}, Y\right)\right) = \tilde{g}\left(\left(a\frac{\partial}{\partial t}, X\right), \tilde{\varphi}\left(b\frac{\partial}{\partial t}, Y\right)\right).$$

Easily, it follows that

$$\tilde{\Phi} = \alpha\Omega, \tag{3.3}$$

and hence

$$\begin{cases} \tilde{\Phi} = \alpha\Omega \\ \tilde{\eta} = dr + \theta \end{cases} \Rightarrow \begin{cases} d\tilde{\Phi} = \alpha' dr \wedge \Omega + \alpha d\Omega \\ d\tilde{\eta} = d\theta. \end{cases}$$

For the special cases, we have the following:

(1) Almost Kähler: $\begin{cases} d\tilde{\Phi} = \alpha' dr \wedge \Omega, \\ d\tilde{\eta} = d\theta = \tilde{\Phi} - \alpha\Omega + d\theta, \end{cases}$

(2) L.c.K.: $\begin{cases} d\tilde{\Phi} = 2\tilde{\eta} \wedge \tilde{\Phi} + ((\alpha' - 2\alpha)dr + \alpha(\Omega - 2\theta)) \wedge \Omega, \\ d\tilde{\eta} = d\theta. \end{cases}$

We can claim the following first main theorem:

Theorem 3.1. (1) *The almost contact metric structure on* $\tilde{M}$ *is an almost cosymplectic if and only if the almost Hermitian structure* (g, J) *is almost Kähler with* $d\theta = 0$. *In addition, the structure on* $\tilde{M}$ *is cosymplectic if and only if the structure* (g, J) *is exact Kählerian.*

(2) *The almost contact metric structure on* $\tilde{M}$ *is a contact metric if and only if the almost Hermitian structure* (g, J) *is almost Kähler with* $d\theta = \alpha\Omega$ *and* α *is a positive constant. In addition, the structure on* $\tilde{M}$ *is* α*-Sasakian if and only if the structure* (g, J) *is exact Kählerian with* $\Omega = \frac{1}{\alpha} d\theta$.

(3) *The almost contact metric structure on* $\tilde{M}$ *is a Kenmotsu if and only if the almost Hermitian structure* (g, J) *is locally conformally Kähler with* $d\Omega = 2\theta \wedge \Omega$ *and* $\alpha = ce^{2r}$, *where* $c > 0$.

Proof. The necessity was observed above for both cases.

For the sufficiency, first note that

$$\begin{cases} d\tilde{\Phi}((\partial_r, 0), (0, X), (0, Y)) = \alpha'\Omega(X, Y), \\ d\tilde{\Phi}((0, X), (0, Y), (0, Z)) = \alpha d\Omega(X, Y, Z), \\ d\tilde{\eta}((0, X), (0, Y)) = d\theta(X, Y). \end{cases} \tag{3.4}$$

(1) Suppose that $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is an almost cosymplectic structure on $\tilde{M}$, i.e., we have $d\tilde{\Phi} = d\tilde{\eta} = 0$. Then (3.4) gives

$$\alpha' = 0, \quad d\Omega = 0 \quad \text{and} \quad d\theta = 0,$$

i.e., J is an almost Kählerian structure with θ as an exact 1-form on M and α is a strictly positive constant.

(2) Suppose that $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a contact metric structure on $\tilde{M}$, i.e., we have $d\tilde{\eta} = \tilde{\Phi}$ which gives $d\theta = \alpha\Omega$ implying that $d\Omega = 0$ since $d\theta = 0$

which confirms that J is an almost Kählerian structure with exact Kähler form $\Omega = \frac{1}{\alpha} d\theta$ and α is a strictly positive constant.

(3) Suppose that $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a Kenmotsu structure on $\tilde{M}$, i.e., we have $d\tilde{\Phi} = 2\tilde{\eta} \wedge \tilde{\Phi}$, and $d\tilde{\eta} = 0$. From equations (3.4), we obtain

$$\alpha = ce^{2r}, \; d\Omega = 2\theta \wedge \Omega, \; d\theta = 0$$

which shows that J is a locally conformally Kähler with $d\Omega = 2\theta \wedge \Omega$ and $\alpha = ce^{2r}$, where $c > 0$. $\square$

4. Construction of an Example

For this construction, we use our example in [1]. We denote the Cartesian coordinates in a 4-dimensional Euclidean space E^4 by (t, x, y, z) and let (J, g) be an almost Hermitian structure defined by

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & f^2(\rho^2 + f'^2\tau^2) & 0 & -\tau f^2 f'^2 \\ 0 & 0 & f^2\rho^2 & 0 \\ 0 & -\tau f^2 f'^2 & 0 & f^2 f'^2 \end{pmatrix},$$

$$J = \begin{pmatrix} 1 & -\tau ff' & 0 & ff' \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{ff'} & 0 & -\tau & 0 \end{pmatrix},$$

where $f = f(t)$, $\rho = \rho(x, y, z)$ and $\tau = \tau(x, y, z)$ are three functions on E^4.

We know that (J, g) has two cases:

(1) Kählerian structure when $\tau_2 = -2\rho^2$ and $\rho_3 = \tau_3 = 0$,

(2) locally conformally Kählerian structure when $\rho_3 = \tau_2 = \tau_3 = 0$, where $\rho_i = \frac{\partial \rho}{\partial x_i}$ and $\tau_i = \frac{\partial \tau}{\partial x_i}$.

On the other hand, we have

$$\Omega = 2ff'dt \wedge (dz - \tau dx) - 2f^2\rho^2 dx \wedge dy,$$

which implies

$$d\Omega = -4ff'\rho^2 dt \wedge dx \wedge dy + 2ff'dt \wedge dx \wedge \tau - 4f^2\rho dx \wedge dy \wedge d\rho.$$

So, for the first case, we get

$$\Omega = d(f^2(dz - \tau dx)),$$

and for the second case, we obtain

$$d\Omega = d \ln f^2 \wedge \Omega.$$

Using the above cases and Theorem 3.1, the manifold $(\mathbb{R} \times E^4, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is:

(1) Sasakian if $\tau_2 = -2\rho^2$, $\rho_3 = \tau_3 = 0$ and $\theta = f^2(dz - \tau dx)$,

(2) Kenmotsu if $\rho_3 = \tau_2 = \tau_3 = 0$, $\alpha = ce^{2r}$ and $\omega = d \ln f^2$.

5. Product $M_1^{2m+1} \times M_2^{2n}$

Let $(M_1, \varphi_1, \xi_1, \eta_1, g_1)$ be an almost contact metric manifold of dimension $2n_1 + 1$ and (M_2, J, g_2) be an almost Hermitian manifold of dimension $2n_2$. It is known that the product $M = M_1 \times M_2$ is a differentiable manifold of dimension $2(n_1 + n_2) + 1$. One can put $n = n_1 + n_2$ so that the dimension of M is $2n + 1$. On the product $M = M_1 + M_2$, one defines an almost contact structure (φ, ξ, η, g) by setting

$$\begin{cases} \varphi(X_1, X_2) = (\varphi_1 X_1 - \theta(JX_2)\xi_1, JX_2), \\ \eta(X_1, X_2) = \eta_1(X_1) + \theta(X_2), \; \xi = (\xi_1, 0), \\ g((X_1, X_2), (Y_1, Y_2)) = g_1(X_1, Y_1) + \alpha g_2(X_2, Y_2) \\ \qquad\qquad + \eta_1(X_1)\theta(Y_2) + \theta(X_2)\eta_1(Y_1) + \theta(X_2)\theta(Y_2). \end{cases} \tag{5.1}$$

for any vector fields X_1, Y_1 of M_1 and X_2, Y_2 of M_2, where θ is a 1-form on M_2 and α is a positive function on M_1.

Proposition 5.1. *The structure* (φ, ξ, η, g) *constructed on the product* $M = M_1 \times M_2$ *is an almost contact metric structure.*

Proof. It follows from (2.1). □

We denote by N_φ the tensor of the almost contact metric structure of M_1 (see (2.2)) and N_J the Nijenhuis tensor of the almost complex structure J. Then from the almost contact metric structure of M defined in (5.1) and formula (2.2), we get

Proposition 5.2. *The almost contact metric structure* (φ, ξ, η, g) *on M is normal if and only if the almost contact metric structure* $(\varphi_1, \xi_1, \eta_1, g_1)$ on M_1 *is normal and the almost Hermitian structure J on* M_2 *is integrable and* $d\theta = d\theta \circ J$.

Proof. Since N_φ is a tensor field of type $(1, 2)$ on M, it suffices to compute it on pairs of vector fields of the forms (X_1, Y_1), (X_1, Y_2) and (X_2, Y_2), where X_1, Y_1 and X_2, Y_2 are vector fields on M_1 and M_2, respectively. Using definition of φ in (5.1) with formulas (2.2) and (2.5), we have

$$\begin{cases} N_\varphi(X_1, Y_1) = N_{\varphi_1}(X_1, Y_1), \\ N_\varphi(X_1, Y_2) = \theta(JY_2)(\mathcal{L}_{\xi_1}\varphi_1)(X_1), \\ N_\varphi(X_2, Y_2) = N_J(X_2, Y_2) - \theta(N_J(X_2, Y_2))\xi_1 \\ \qquad\qquad + 2(d\theta(X_2, Y_2) - d\theta(JX_2, JY_2))\xi_1. \end{cases} \tag{5.2}$$

Suppose that (φ, ξ, η, g) on M is normal, i.e., $N_\varphi = 0.$ From the first equation of (5.2), we get $N_{\varphi_1} = 0.$ From the third equation, we get

$$N_J(X_2, Y_2) - \theta(N_J(X_2, Y_2))\xi_1 + 2(d\theta(X_2, Y_2) - d\theta(JX_2, JY_2))\xi_1 = 0,$$

which implies $N_J = 0$ and $d\theta = d\theta \circ J.$

Conversely, suppose that $(\varphi_1, \xi_1, \eta_1, g_1)$ is normal and J is integrable with θ exact, i.e., we have $N_{\varphi_1} = 0,$ $N_J = 0$ and $d\theta = 0.$ From (5.2), we get

$$\begin{cases} N_\varphi(X_1, Y_1) = 0, \\ N_\varphi(X_1, Y_2) = \theta(JY_2)(\mathcal{L}_{\xi_1}\varphi_1)(X_1), \\ N_\varphi(X_2, Y_2) = 0 \end{cases}$$

and knowing that

$$N_{\varphi_1}(\varphi_1 X_1, \xi_1) = [\xi_1, \varphi, X_1] - \varphi_1[\xi_1, X_1] = (\mathcal{L}_{\xi_1}\varphi_1)(X_1),$$

then we obtain $N_\varphi = 0.$ □

The manifold $(M, \varphi, \xi, \eta, g)$ possesses a fundamental 2-form, ϕ, defined by

$$\phi((X_1, X_2), (Y_1, Y_2)) = g((X_1, X_2), \varphi(Y_1, Y_2)).$$

From the definitions of g and φ (see (5.1)), we get

$$\phi((X_1, X_2), (Y_1, Y_2)) = \phi_1(X_1, Y_1) + \alpha\Omega(X_2, Y_2), \tag{5.3}$$

where Ω is the fundamental Kähler form of M_2 given by $\Omega(X_2, Y_2) = g_2(X_2, JY_2).$

We have immediately that

$$d\phi = d\phi_1 + d\alpha \wedge \Omega + \alpha d\Omega. \tag{5.4}$$

For our motivation, we consider M_2 is a locally conformally Kähler manifold with $d\Omega = \omega \wedge \Omega.$ From (5.4), we get

$$d\phi = d\phi_1 + \alpha(d \ln \alpha + \omega) \wedge \Omega. \tag{5.5}$$

For the special cases of M_1, we have the following:

(1) *Almost cosymplectic*: $\begin{cases} d\phi = \alpha(d\ln\alpha + \omega) \wedge \Omega, \\ d\eta = d\theta. \end{cases}$

(2) *Almost Kenmotsu*:

$$\begin{cases} d\phi = 2\eta \wedge \phi - 2\theta \wedge \phi_1 + \alpha(d\ln\alpha - 2\eta_1 + \omega - 2\theta) \wedge \Omega, \\ d\eta = d\theta. \end{cases}$$

(3) *Contact metric*: $\begin{cases} d\phi = \alpha(d\ln\alpha + \theta) \wedge \Omega, \\ d\eta = \phi + d\theta - \alpha\Omega. \end{cases}$

We note that:

(1) ϕ is closed only in the case of almost cosymplectic if and only if $\omega = 0$ and α is a constant.

(2) $d\phi = 2\eta \wedge \phi$ only in the case of almost Kenmotsu if and only if $\omega = 2\theta = 0$ and $d\ln\alpha = 2\eta_1$.

(3) $d\eta = \phi$ only in the case of contact metric if and only if $\omega = 0$, $d\theta = \alpha\Omega$ and α is a constant.

Therefore, summing up the arguments above, we have the following second main theorem:

Theorem 5.1. (1) *The almost contact metric structure on M is almost cosymplectic if and only if the almost contact metric structure* $(\varphi_1, \xi_1, \eta_1, g_1)$ *on* M_1 *is almost cosymplectic and the almost Hermitian structure* (J, g_2) *on* M_2 *is almost Kähler with* θ *being exact*.

In addition, the structure on M is cosymplectic if and only if the structures on M_1 *and* M_2 *are cosymplectic and Kählerian, respectively, with* θ *being exact*.

(2) *The almost contact metric structure on M is almost Kenmotsu if and only if the almost contact metric structure* $(\varphi_1, \xi_1, \eta_1, g_1)$ *on* M_1 *is almost*

Kenmotsu and the almost Hermitian structure (J, g_2) *on* M_2 *is almost Kähler with* $\theta = 0$ *and* $d \ln \alpha = 2\eta_1$.

In addition, the structure on M is Kenmotsu if and only if the structures on M_1 *and* M_2 *are Kenmotsu and Kählerian, respectively, with* $\theta = 0$ *and* $d \ln \alpha = 2\eta_1$.

(3) *The almost contact metric structure on M is a contact metric structure if and only if the almost contact metric structure* $(\varphi_1, \xi_1, \eta_1, g_1)$ *on* M_1 *is a contact metric structure and the almost Hermitian structure* (J, g_2) *on* M_2 *is almost Kähler exact with* $d\theta = \alpha\Omega$, *where* α *is a constant*.

In addition, the structure on M is α*-Sasakian if and only if the structures on* M_1 *and* M_2 *are Sasakian and Kählerian exact, respectively*.

Proof. The necessity was observed above for both cases.

For the sufficiency, first note that

$$d\varphi((X_1, 0), (Y_1, 0), (Z_1, 0)) = d\varphi_1(X_1, Y_1, Z_1), \tag{5.6}$$

$$d\varphi((X_1, 0), (0, Y_2), (0, Z_2)) = X_1(\alpha)\Omega(Y_2, Z_2), \tag{5.7}$$

$$d\varphi((0, X_2), (0, Y_2), (0, Z_2)) = \alpha d\Omega(X_2, Y_2, Z_2), \tag{5.8}$$

for all X_1, Y_1, Z_1 vector fields on M_1 and X_2, Y_2, Z_2 vector fields on M_2.

(1) Suppose that (φ, ξ, η, g) is an almost cosymplectic structure. Then we have $d\varphi = 0$ and $d\eta = 0$. Equations (5.6), (5.7) and (5.8) give $\varphi_1 = 0$, $\alpha = constant$ and $d\Omega = 0$, respectively, and knowing that $\eta = \eta_1 + \theta$, we get $d\eta = d\theta = 0$. So $(\varphi_1, \xi_1, \eta, g)$ and (J, g_2) are cosymplectic and Kählerian structures, respectively.

(2) Suppose that (φ, ξ, η, g) is an almost Kenmotsu structure. Then we have $d\varphi = 2\eta \wedge \varphi$ and $d\eta = 0$. Equations (5.6), (5.7) and (5.8) give

$$\begin{cases} 2\eta_1(X_1)\varphi_1(Y_1, Z_1) = d\varphi_1(X_1, Y_1, Z_1), \\ 2\alpha\eta_1(X_1)\Omega(Y_2, Z_2) = X_1(\alpha)\Omega(Y_2, Z_2), \\ 2\alpha\theta(X_2)\varphi(Y_2, Z_2) = \alpha d\Omega(X_2, Y_2, Z_2), \end{cases}$$

with $d\eta = d\eta_1 + d\theta = 0,$ which implies

$$\begin{cases} d\varphi_1 = 2\eta_1 \wedge \varphi_1, \ d\eta_1 = 0, \\ d \ln \alpha = 2\eta_1, \\ d\Omega = 2\theta \wedge \Omega, \ d\theta = 0. \end{cases}$$

In addition, the equation

$$d\varphi((0, X_2), (Y_1, 0), (Z_1, 0)) = 2(\eta \wedge \varphi)((0, X_2), (Y_1, 0), (Z_1, 0)),$$

gives $0 = 2\theta(X_2)\varphi_1(Y_1, Z_1)$ which give $\theta = 0.$

So $(\varphi_1, \xi_1, \eta_1, g_1)$ and (J, g_2) are almost Kenmotsu and almost Kähler structures, respectively.

(3) Suppose that (φ, ξ, η, g) is a contact metric structure, i.e., $d\eta = \varphi.$ Then, for all X_1, Y_1 vector fields on M_1 and X_2, Y_2 vector fields on $M_2,$ we get

$$\begin{cases} d\eta(X_1, Y_1) = \varphi(X_1, Y_1) \\ d\eta(X_2 X_2) = \varphi(X_2 X_2) \end{cases} \Leftrightarrow \begin{cases} d\eta_1(X_1, Y_1) = \varphi_1(X_1, Y_1) \\ d\theta(X_2 X_2) = \alpha\Omega(X_2 X_2). \end{cases}$$

On the other hand, $d\eta = \varphi$ implies $d\varphi = 0$ which gives $\varphi_1 = 0,$ $\alpha =$ *constant* and $d\Omega = 0.$

So, $(\varphi_1, \xi_1, \eta_1, g_1)$ and (J, g_2) are contact metric and almost Kähler structures, respectively. □

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