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PARAMETRIC OPERATIONS BETWEEN 2-DIMENSIONAL TRIANGULAR FUZZY NUMBER AND TRAPEZOIDAL FUZZY SET

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Abstract

By defining parametric operations between two regions valued α-cuts, we get the parametric operations for two fuzzy numbers defined on $\mathbb{R}^2$. The results for the parametric operations generalized Zadeh's extended algebraic operations.

In this paper, we generate the trapezoidal fuzzy set on $\mathbb{R}$ to $\mathbb{R}^2$ and calculate the parametric operations between a 2-dimensional triangular fuzzy number and a trapezoidal fuzzy set.

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1. Introduction

We defined the parametric operations on $\mathbb{R}$. The results of parametric operations for two triangular fuzzy numbers were same as those of Zadeh's max-min operations [1]. We generated the triangular fuzzy numbers on $\mathbb{R}$ to $\mathbb{R}^2$ and calculated Zadeh's extension principle for 2-dimensional triangular fuzzy numbers on $\mathbb{R}^2$ [5]. By defining parametric operations between two regions valued α-cuts, we get the parametric operations for two triangular fuzzy numbers defined on $\mathbb{R}^2$. We proved that the results for the parametric operations generalized Zadeh's max-min composition operations [2]. We generalized the triangular fuzzy numbers on $\mathbb{R}^2$ and calculated parametric operations for two generalized 2-dimensional triangular fuzzy sets on $\mathbb{R}^2$ [4]. We calculated the operators for two quadratic fuzzy numbers [6]. We generated also the quadratic fuzzy numbers on $\mathbb{R}$ to $\mathbb{R}^2$. We calculated the Zadeh's max-min composition operator for two 2-dimensional quadratic fuzzy numbers [3].

In this paper, we generated the trapezoidal fuzzy numbers on $\mathbb{R}$ to $\mathbb{R}^2$. We calculate the parametric operations between a 2-dimensional triangular fuzzy number and a trapezoidal fuzzy set.

2. Generalized 2-dimensional Triangular Fuzzy Sets on $\mathbb{R}^2$

Let X be a set. We define α-cut and α-set of the fuzzy set A with the membership function $\mu_A(x)$.

Definition 2.1. An α-*cut* of the fuzzy number A is defined by $A_\alpha = \{x \in \mathbb{R} \mid \mu_A(x) \geq \alpha\}$ if $\alpha \in (0, 1]$ and $A_\alpha = cl\{x \in \mathbb{R} \mid \mu_A(x) > \alpha\}$ if $\alpha = 0$. For $\alpha \in (0, 1)$, the set $A^\alpha = \{x \in X \mid \mu_A(x) = \alpha\}$ is said to be the α-*set* of the fuzzy set A, A^0 is the boundary of $\{x \in \mathbb{R} \mid \mu_A(x) > \alpha\}$ and $A^1 = A_1$.

Definition 2.2 [7]**.** The extended addition $A(+)B$, extended subtraction $A(-)B$, extended multiplication $A(\cdot)B$ and extended division $A(/)B$ are fuzzy sets with membership functions as follows. For all $x \in A$ and $y \in B$,

$$\mu_{A(*)B}(z) = \sup_{z = x * y} \min\{\mu_A(x), \mu_B(y)\}, \quad * = +, -, \cdot, / .$$

We define the generalized 2-dimensional triangular fuzzy numbers on $\mathbb{R}^2$ as a generalization of generalized triangular fuzzy numbers on $\mathbb{R}$. Then we define the parametric operations between two 2-dimensional fuzzy numbers. For that, we have to calculate operations between α-cuts in $\mathbb{R}$. The α-cuts are intervals in $\mathbb{R}$ but in $\mathbb{R}^2$, the α-cuts are regions, which make the existing method of calculations between α-cuts unusable. We interpret the existing method from a different perspective and apply the method to the region valued α-cuts on $\mathbb{R}^2$.

Definition 2.3 [5]**.** A fuzzy set A with a membership function

$$\mu_A(x, y) = \begin{cases} h - \sqrt{\dfrac{(x - x_1)^2}{a^2} + \dfrac{(y - y_1)^2}{b^2}}, & b^2(x - x_1)^2 + a^2(y - y_1)^2 \le a^2b^2h^2, \\ 0, & \text{otherwise}, \end{cases}$$

where $a, b > 0$ and $0 < h < 1$ is called the *generalized* 2*-dimensional triangular fuzzy number* and denoted by $((a, x_1, h, b, y_1))^2$.

Note that $\mu_A(x, y)$ is a cone. The intersections of $\mu_A(x, y)$ and the horizontal planes $z = \alpha\ (0 < \alpha < h)$ are ellipses. The intersections of $\mu_A(x, y)$ and the vertical planes $y - y_1 = k(x - x_1)(k \in \mathbb{R})$ are symmetric triangular fuzzy numbers in those planes. If $a = b$, then ellipses become circles. The α-cut $(0 < \alpha < h)$ A_α of a generalized 2-dimensional triangular fuzzy number $A = ((a, x_1, h, b, y_1))^2$ is interior to an ellipse in the *xy*-plane including the boundary

$$A_\alpha = \{(x, y) \in \mathbb{R}^2 \mid b^2(x - x_1)^2 + a^2(y - y_1)^2 \le a^2b^2(h - \alpha)^2\}$$

$$= \left\{(x, y) \in \mathbb{R}^2 \mid \left(\frac{x - x_1}{a(h - \alpha)}\right)^2 + \left(\frac{y - y_1}{b(h - \alpha)}\right)^2 \le 1\right\}.$$

Definition 2.4. A 2-dimensional fuzzy number A defined on $\mathbb{R}^2$ is called *convex fuzzy number* if for all $\alpha \in (0, 1)$, the α-cuts $A_\alpha = \{(x, y) \in \mathbb{R}^2 \mid \mu_A(x, y) \ge \alpha\}$ are convex subsets in $\mathbb{R}^2$.

Theorem 2.5 [5]**.** *Let A be a convex fuzzy number defined on* $\mathbb{R}^2$ *and* $A^\alpha = \{(x, y) \in \mathbb{R}^2 \mid \mu_A(x, y) = \alpha\}$ *be the* α*-set of A. Then, for all* $\alpha \in (0, 1)$, *there exist piecewise continuous functions* $f_1^\alpha(t)$ *and* $f_2^\alpha(t)$ *defined on* $[0, 2\pi]$ *such that* $A^\alpha = \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\}$.

If A is a continuous convex fuzzy number defined on $\mathbb{R}^2$, *then the* α*-set* A^α *is a closed circular convex subset in* $\mathbb{R}^2$.

Corollary 2.6 [5]**.** *Let A be a continuous convex fuzzy number defined on* $\mathbb{R}^2$ *and* $A^\alpha = \{(x, y) \in \mathbb{R}^2 \mid \mu_A(x, y) = \alpha\}$ *be the* α*-set of A. Then, for all* $\alpha \in (0, 1)$, *there exist continuous functions* $f_1^\alpha(t)$ *and* $f_2^\alpha(t)$ *defined on* $[0, 2\pi]$ *such that* $A^\alpha = \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\}$.

Definition 2.7 [5]**.** Let A and B be convex fuzzy numbers defined on $\mathbb{R}^2$ and

$$A^\alpha = \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\},$$

$$B^\alpha = \{(g_1^\alpha(t), g_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\}$$

be the α-sets of A and B, respectively. For $\alpha \in (0, 1)$, we define that the parametric addition $A(+)_p B$, parametric subtraction $A(-)_p B$, parametric multiplication $A(\cdot)_p B$ and parametric division $A(/)_p B$ of two fuzzy numbers A and B are fuzzy numbers that have their α-sets as follows:

(1)

$$A(+)_p B : (A(+)_p B)^\alpha = \{(f_1^\alpha(t) + g_1^\alpha(t),\ f_2^\alpha(t) + g_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\},$$

(2) $A(-)_p B : (A(-)_p B)^\alpha = \{(x_\alpha(t),\ y_\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\}$, where

$$x_\alpha(t) = \begin{cases} f_1^\alpha(t) - g_1^\alpha(t+\pi), & \text{if } 0 \le t \le \pi, \\ f_1^\alpha(t) - g_1^\alpha(t-\pi), & \text{if } \pi \le t \le 2\pi \end{cases}$$

and

$$y_\alpha(t) = \begin{cases} f_2^\alpha(t) - g_2^\alpha(t+\pi), & \text{if } 0 \le t \le \pi, \\ f_2^\alpha(t) - g_2^\alpha(t-\pi), & \text{if } \pi \le t \le 2\pi, \end{cases}$$

(3)

$$A(\cdot)_p B : (A(\cdot)_p B)^\alpha = \{(f_1^\alpha(t) \cdot g_1^\alpha(t),\ f_2^\alpha(t) \cdot g_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\},$$

(4) $A(/)_p B : (A(/)_p B)^\alpha = \{(x_\alpha(t),\ y_\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\}$, where

$$x_\alpha(t) = \frac{f_1^\alpha(t)}{g_1^\alpha(t+\pi)} \quad (0 \le t \le \pi), \quad x_\alpha(t) = \frac{f_1^\alpha(t)}{g_1^\alpha(t-\pi)} \quad (\pi \le t \le 2\pi)$$

and

$$y_\alpha(t) = \frac{f_2^\alpha(t)}{g_2^\alpha(t+\pi)} \quad (0 \le t \le \pi), \quad y_\alpha(t) = \frac{f_2^\alpha(t)}{g_2^\alpha(t-\pi)} \quad (\pi \le t \le 2\pi).$$

For $\alpha = 0$ and $\alpha = 1$, $(A(*)_p B)^0 = \lim_{\alpha \to 0^+} (A(*)_p B)^\alpha$ and $(A(*)_p B)^1 = \lim_{\alpha \to 1^-} (A(*)_p B)^\alpha$, where $* = +, -, \cdot, /$.

For $0 < h_1 < h_2 \le 1$, let

$$A = ((a_1, x_1, h_1, b_1, y_1))^2 \quad \text{and} \quad B = ((a_2, x_2, h_2, b_2, y_2))^2$$

be two generalized 2-dimensional triangular fuzzy numbers. If $0 \le \alpha < h_1$, then $(A(*)_p B)^\alpha$ can be defined same as Definition 2.7. If $\alpha = h_1$, then

$$(A(*)_p B)^{h_1} = \lim_{\alpha \to h_1^-} (A(*)_p B)^{\alpha}, \quad * = +, -, \cdot, / .$$

Thus, $(A(*)_p B)^{h_1}$ becomes an ellipse not a point. If $h_1 < \alpha \leq h_2$, then by the Zadeh's max-min principle operations, we have to define

$$(A(*)_p B)^{\alpha} = \varnothing, \quad * = +, -, \cdot, / .$$

Theorem 2.8 [3]**.** *Let*

$$A = ((a_1, x_1, h_1, b_1, y_1))^2 \quad \textit{and} \quad B = ((a_2, x_2, h_2, b_2, y_2))^2$$

be two generalized 2*-dimensional triangular fuzzy numbers. If* $0 < h_1 < h_2 \leq 1$, *then we have the following*:

(1) *For* $0 < \alpha < h_1$, *the* α*-set of* $A(+)_p B$ *is*

$$(A(+)_p B)^{\alpha} = \left\{ (x, y) \in \mathbb{R}^2 \,|\, \left(\frac{x - x_1 - x_2}{a_1(h_1 - \alpha) + a_2(h_2 - \alpha)} \right)^2 + \left(\frac{y - y_1 - y_2}{b_1(h_1 - \alpha) + b_2(h_2 - \alpha)} \right)^2 = 1 \right\}.$$

(2) *For* $0 < \alpha < h_1$, *the* α*-set of* $A(-)_p B$ *is*

$$(A(-)_p B)^{\alpha} = \left\{ (x, y) \in \mathbb{R}^2 \,|\, \left(\frac{x - x_1 + x_2}{a_1(h_1 - \alpha) + a_2(h_2 - \alpha)} \right)^2 + \left(\frac{y - y_1 + y_2}{b_1(h_1 - \alpha) + b_2(h_2 - \alpha)} \right)^2 = 1 \right\}.$$

(3) $(A(\cdot)_p B)^{\alpha} = \{(x_\alpha(t), y_\alpha(t)) \,|\, 0 \leq t \leq 2\pi\}$, *where*

$$x_\alpha(t) = x_1 x_2 + (x_1 a_2(h_2 - \alpha) + x_2 a_1(h_1 - \alpha)) \cos t + a_1 a_2 (h_1 - \alpha)(h_2 - \alpha) \cos^2 t, \quad 0 < \alpha < h_1,$$

$$y_\alpha(t) = y_1 y_2 + (y_1 b_2(h_2 - \alpha) + y_2 b_1(h_1 - \alpha))\sin t$$

$$+ b_1 b_2 (h_1 - \alpha)(h_2 - \alpha)\sin^2 t, \quad 0 < \alpha < h_1.$$

(4) $(A(/)_p B)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, *where*

$$x_\alpha(t) = \frac{x_1 + a_1(h_1 - \alpha)\cos t}{x_2 - a_2(h_2 - \alpha)\cos t}, \quad y_\alpha(t) = \frac{y_1 + b_1(h_1 - \alpha)\sin t}{y_2 - b_2(h_2 - \alpha)\cos t}, \quad 0 < \alpha < h_1.$$

3. Parametric Operations between 2-dimensional Triangular Fuzzy Number and Trapezoidal Fuzzy Set

In this section, we define the 2-dimensional trapezoidal fuzzy sets on $\mathbb{R}^2$ as a generalization of trapezoidal fuzzy sets on $\mathbb{R}$. Then we want to calculate the parametric operations between 2-dimensional triangular fuzzy number and trapezoidal fuzzy set on $\mathbb{R}^2$.

Definition 3.1 [5]**.** A fuzzy set A with a membership function

$$\mu_A(x, y) = \begin{cases} 1 - \sqrt{\dfrac{(x - x_1)^2}{a^2} + \dfrac{(y_1 - y_1)^2}{b^2}}, & b^2(x - x_1)^2 + a^2(y - y_1)^2 \le a^2 b^2, \\ 0, & \text{otherwise}, \end{cases}$$

where $a, b > 0$ is called the 2-*dimensional triangular fuzzy number* and denoted by $A = (a, x_1, b, y_1)^2$.

Note that $\mu_A(x, y)$ is a cone. The intersections of $\mu_A(x, y)$ and the horizontal planes $z = \alpha$ $(0 < \alpha < 1)$ are ellipses. The intersections of $\mu_A(x, y)$ and the vertical planes $y - y_1 = k(x - x_1)(k \in \mathbb{R})$ are symmetric triangular fuzzy numbers in those planes. If $a = b$, then ellipses become circles. The α-cut A_α of a 2-dimensional triangular fuzzy number $A = (a, x_1, b, y_1)^2$ is interior to an ellipse in the xy-plane including the boundary

$$A_\alpha = \{(x, y) \in \mathbb{R}^2 \mid b^2(x - x_1)^2 + a^2(y - y_1)^2 \le a^2b^2(1 - \alpha)^2\}$$

$$= \left\{(x, y) \in \mathbb{R}^2 \mid \left(\frac{x - x_1}{a(1 - \alpha)}\right)^2 + \left(\frac{y - y_1}{b(1 - \alpha)}\right)^2 \le 1\right\}.$$

Definition 3.2. A fuzzy set B with a membership function

$$\mu_B(x, y) = \begin{cases} h - \sqrt{\dfrac{(x - x_1)^2}{a^2} + \dfrac{(y - y_1)^2}{b^2}}, & h - 1 \le \sqrt{\dfrac{(x - x_1)^2}{a^2} + \dfrac{(y - y_1)^2}{b^2}} \le h, \\ 1, & 0 \le \sqrt{\dfrac{(x - x_1)^2}{a^2} + \dfrac{(y - y_1)^2}{b^2}} \le h - 1, \\ 0, & \text{otherwise}, \end{cases}$$

where $a, b > 0$ and $1 < h$ is called the 2-*dimensional trapezoidal fuzzy set* and denoted by $B = ((a, x_1, h, b, y_1))^2$.

$\mu_B(x, y)$ is a truncated cone. The intersections of $\mu_B(x, y)$ and the horizontal planes $z = \alpha$ $(0 < \alpha < 1)$ are ellipses. The intersections of $\mu_B(x, y)$ and the vertical planes $y - y_1 = k(x - x_1)(k \in \mathbb{R})$ are symmetric trapezoidal fuzzy sets in those planes. If $a = b$, then ellipses become circles. The α-cut B_α of a 2-dimensional trapezoidal fuzzy number $B = ((a, x_1, h, b, y_1))^2$ is interior to an ellipse in the *xy*-plane including the boundary

$$B_\alpha = \{(x, y) \in \mathbb{R}^2 \mid b^2(x - x_1)^2 + a^2(y - y_1)^2 \le a^2b^2(h - \alpha)^2\}$$

$$= \left\{(x, y) \in \mathbb{R}^2 \mid \left(\frac{x - x_1}{a(h - \alpha)}\right)^2 + \left(\frac{y - y_1}{b(h - \alpha)}\right)^2 \le 1\right\}.$$

Note that if $0 < h < 1$, then $((a, x_1, h, b, y_1))^2$ becomes a generalized 2-dimensional triangular fuzzy number and if $1 < h$, then $((a, x_1, h, b, y_1))^2$ becomes a 2-dimensional trapezoidal fuzzy set.

Theorem 3.3. *Let* $A = (a_1, x_1, b_1, y_1)^2$ *be a* 2-*dimensional triangular fuzzy number and* $B = ((a_2, x_2, h, b_2, y_2))^2$ *be a* 2-*dimensional trapezoidal fuzzy set*. *Then we have the following*:

(1) *For* $0 < \alpha < 1$, *the* α-*set of* $A(+)_p B$ *is*

$$(A(+)_p B)^{\alpha} = \left\{(x, y) \in \mathbb{R}^2 \,|\left(\frac{x - x_1 - x_2}{a_1(1-\alpha) + a_2(h-\alpha)}\right)^2 + \left(\frac{y - y_1 - y_2}{b_1(1-\alpha) + b_2(h-\alpha)}\right)^2 = 1\right\}.$$

(2) *For* $0 < \alpha < 1$, *the* α-*set of* $A(-)_p B$ *is*

$$(A(-)_p B)^{\alpha} = \left\{(x, y) \in \mathbb{R}^2 \,|\left(\frac{x - x_1 + x_2}{a_1(1-\alpha) + a_2(h-\alpha)}\right)^2 + \left(\frac{y - y_1 + y_2}{b_1(1-\alpha) + b_2(h-\alpha)}\right)^2 = 1\right\}.$$

(3) $(A(\cdot)_p B)^{\alpha} = \{(x_{\alpha}(t), y_{\alpha}(t)) \,|\, 0 \le t \le 2\pi\}$, *where*

$$x_{\alpha}(t) = x_1x_2 + (x_1a_2(h-\alpha) + x_2a_1(1-\alpha))\cos t + a_1a_2(1-\alpha)(h-\alpha)\cos^2 t, \quad 0 < \alpha < 1,$$

$$y_{\alpha}(t) = y_1y_2 + (y_1b_2(h-\alpha) + y_2b_1(1-\alpha))\sin t + b_1b_2(1-\alpha)(h-\alpha)\sin^2 t, \quad 0 < \alpha < 1.$$

(4) $(A(/)_p B)^{\alpha} = \{(x_{\alpha}(t), y_{\alpha}(t)) \,|\, 0 \le t \le 2\pi\}$, *where*

$$x_{\alpha}(t) = \frac{x_1 + a_1(1-\alpha)\cos t}{x_2 - a_2(h-\alpha)\cos t}, \quad y_{\alpha}(t) = \frac{y_1 + b_1(1-\alpha)\sin t}{y_2 - b_2(h-\alpha)\sin t}, \quad 0 < \alpha < 1.$$

Proof. Since A and B are convex fuzzy numbers defined on $\mathbb{R}^2$, by Theorem 2.5, there exist $f_i^\alpha(t),\ g_i^\alpha(t)(i = 1,\ 2)$ such that for $0 < \alpha < 1$,

$$A^\alpha = \{(f_1^\alpha(t),\ f_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\},$$

$$B^\alpha = \{(g_1^\alpha(t),\ g_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\}.$$

Since $A = (a_1,\ x_1,\ b_1,\ y_1)^2$ and $B = ((a_2,\ x_2,\ h,\ b_2,\ y_2))^2$, we have

$$f_1^\alpha(t) = x_1 + a_1(1-\alpha)\cos t, \quad f_2^\alpha(t) = y_1 + b_1(1-\alpha)\sin t, \quad 0 < \alpha < 1,$$

$$g_1^\alpha(t) = x_2 + a_2(h-\alpha)\cos t, \quad g_2^\alpha(t) = y_2 + b_2(h-\alpha)\sin t, \quad 0 < \alpha < 1.$$

(1) If $0 < \alpha < 1$, since

$$f_1^\alpha(t) + g_1^\alpha(t) = x_1 + x_2 + (a_1(1-\alpha) + a_2(h-\alpha))\cos t,$$

$$f_2^\alpha(t) + g_2^\alpha(t) = y_1 + y_2 + (b_1(1-\alpha) + b_2(h-\alpha))\sin t,$$

we have

$$(A(+)_p B)^\alpha = \left\{(x,\ y) \in \mathbb{R}^2 \,\middle|\, \left(\frac{x - x_1 - x_2}{a_1(1-\alpha) + a_2(h-\alpha)}\right)^2 + \left(\frac{y - y_1 - y_2}{b_1(1-\alpha) + b_2(h-\alpha)}\right)^2 = 1\right\}.$$

Furthermore, we have

$$(A(+)_p B)^0 = \lim_{\alpha \to 0^+} (A(+)_p B)^\alpha$$
$$= \left\{(x,\ y) \in \mathbb{R}^2 \,\middle|\, \left(\frac{x - x_1 - x_2}{a_1 + a_2 h}\right)^2 + \left(\frac{y - y_1 - y_2}{b_1 + b_2 h}\right)^2 = 1\right\},$$

$$(A(+)_p B)^1 = \lim_{\alpha \to 0^-} (A(+)_p B)^\alpha$$

$$= \left\{ (x,\ y) \in \mathbb{R}^2 \,|\, \left(\frac{x - x_1 - x_2}{a_2(h-1)} \right)^2 + \left(\frac{y - y_1 - y_2}{b_2(h-1)} \right)^2 = 1 \right\}.$$

(2) If $0 \le t \le \pi$ and $0 < \alpha < 1$, then we have

$$f_1^\alpha(t) - g_1^\alpha(t+\pi) = x_1 - x_2 + (a_1(1-\alpha) + a_2(h-\alpha))\cos t,$$

$$f_2^\alpha(t) - g_2^\alpha(t+\pi) = y_1 - y_2 + (b_1(1-\alpha) + b_2(h-\alpha))\sin t.$$

In the case of $\pi \le t \le 2\pi$, we have

$$f_1^\alpha(t) - g_1^\alpha(t-\pi) = f_1^\alpha(t) - g_1^\alpha(t+\pi),$$

$$f_2^\alpha(t) - g_2^\alpha(t-\pi) = f_2^\alpha(t) - g_2^\alpha(t+\pi).$$

Thus,

$$(A(-)_p B)^\alpha = \left\{ (x,\ y) \in \mathbb{R}^2 \,|\, \left(\frac{x - x_1 + x_2}{a_1(1-\alpha) + a_2(h-\alpha)} \right)^2 + \left(\frac{y - y_1 + y_2}{b_1(1-\alpha) + b_2(h-\alpha)} \right)^2 = 1 \right\}.$$

Furthermore, we have

$$(A(-)_p B)^0 = \lim_{\alpha \to 0^+} (A(-)_p B)^\alpha$$

$$= \left\{ (x,\ y) \in \mathbb{R}^2 \,|\, \left(\frac{x - x_1 + x_2}{a_1 + a_2 h} \right)^2 + \left(\frac{y - y_1 + y_2}{b_1 + b_2 h} \right)^2 = 1 \right\},$$

$$(A(-)_p B)^1 = \lim_{\alpha \to 1^-} (A(-)_p B)^\alpha$$

$$= \left\{ (x,\ y) \in \mathbb{R}^2 \,|\, \left(\frac{x - x_1 + x_2}{a_2(h-1)} \right)^2 + \left(\frac{y - y_1 + y_2}{b_2(h-1)} \right)^2 = 1 \right\}.$$

(3) Let $(A(\cdot)_p B)^\alpha = \{(x_\alpha(t),\ y_\alpha(t)) \mid 0 \le t \le 2\pi\}$. Since

$$f_1^\alpha(t) = x_1 + a_1(1-\alpha)\cos t, \quad f_2^\alpha(t) = y_1 + b_1(1-\alpha)\sin t,$$

$$g_1^\alpha(t) = x_2 + a_2(1-\alpha)\cos t, \quad g_2^\alpha(t) = y_2 + b_2(h-\alpha)\sin t,$$

we have

$$\begin{aligned} x_\alpha(t) &= f_1^\alpha(t)\cdot g_1^\alpha(t) \\ &= x_1x_2 + (x_1a_2(h-\alpha) + x_2a_1(1-\alpha))\cos t \\ &\quad + a_1a_2(1-\alpha)(h-\alpha)\cos^2 t, \quad 0 < \alpha < 1, \\ y_\alpha(t) &= f_2^\alpha(t)\cdot g_2^\alpha(t) \\ &= y_1y_2 + (y_1b_2(h-\alpha) + y_2b_1(1-\alpha))\sin t \\ &\quad + b_1b_2(1-\alpha)(h-\alpha)\sin^2 t, \quad 0 < \alpha < 1. \end{aligned}$$

Furthermore, we have

$$x_0(t) = \lim_{\alpha\to 0^+} x_\alpha(t) = x_1x_2 + (x_1a_2h + x_2a_1)\cos t + a_1a_2h\cos^2 t,$$

$$y_0(t) = \lim_{\alpha\to 0^+} y_\alpha(t) = y_1y_2 + (y_1b_2h + y_2b_1)\sin t + b_1b_2h\sin^2 t,$$

$$x_1(t) = \lim_{\alpha\to 1^-} x_\alpha(t) = x_1x_2 + x_1a_2(h-1)\cos t,$$

$$y_1(t) = \lim_{\alpha\to 1^-} y_\alpha(t) = y_1y_2 + y_1b_2(h-1)\sin t.$$

(4) Let $(A(/)_p B)^\alpha = \{(x_\alpha(t),\ y_\alpha(t)) \mid 0 \le t \le 2\pi\}$. Similarly, we have

$$x_\alpha(t) = \frac{x_1 + a_1(1-\alpha)\cos t}{x_2 - a_2(h-\alpha)\cos t}, \quad y_\alpha(t) = \frac{y_1 + b_1(1-\alpha)\sin t}{y_2 - b_2(h-\alpha)\sin t}, \quad 0 < \alpha < 1.$$

Furthermore, we have

$$x_0(t) = \frac{x_1 + a_1 \cos t}{x_2 - a_2 h \cos t}, \quad y_0(t) = \frac{y_1 + b_1 \sin t}{y_2 - b_2 h \sin t}.$$

The proof is complete.

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