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ASYMPTOTICS OF THE FUNDAMENTAL SYSTEM SOLUTIONS OF DIFFERENTIAL EQUATION WITH BLOCK-TRIANGULAR OPERATOR POTENTIAL

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Abstract

In this paper, the asymptotics of the fundamental system of solutions for the Sturm-Liouville equation with block-triangular operator potential, increasing at infinity is established. One of the solutions is found decreasing at infinity, while the other one increasing.

1. Introduction

The theory of singular non-self-adjoint differential operators is relatively new. Results turning out in self-adjoint and non-self-adjoint cases differentiate substantially. In [3], Marchenko introduced a notion of generalized spectral function R for a Sturm-Liouville operator with arbitrary complex valued potential on the semi axis. For an operator with a triangular matrix potential decreasing at infinity the spectral structure was established

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in [4]. For differential equations with block-triangular matrix potential that increases at infinity, a spectrum structure is investigated in the paper [1].

In this paper, the asymptotics of fundamental system of solutions of differential equation with block-triangular operator potential that increases at infinity is set. In the monograph [6] for the scalar differential equation the Langer method in another form is set the asymptotics of the fundamental system of solutions by using Hankel functions.

2. Preliminary Notes

Let us designate H_k, $k = 1, 2, ..., r$ as finite-dimensional or infinite-dimensional separable Hilbert space with inner product $(\cdot, \cdot)$ and norm $|\cdot|$, $\dim H_k \le \infty$. Denoted by $\mathbf{H} = H_1 \oplus H_2 \oplus \ldots \oplus H_r$. Element $\bar{h} \in \mathbf{H}$ will be written in the form $\bar{h} = col(\bar{h}_1, \bar{h}_2, ..., \bar{h}_r)$, where $\bar{h}_k \in H_k$, $k = \overline{1, r}$, I_k, I-identity operators in H_k and $\mathbf{H}$ accordingly.

Let us consider the equation with block-triangular operator potential in $B(\mathbf{H})$:

$$l[\bar{y}] = -\bar{y}'' + V(x)\bar{y} = \lambda\bar{y}, \quad 0 \le x < \infty, \tag{1}$$

where

$$V(x) = v(x) \cdot I + U(x), \; U(x) = \begin{pmatrix} U_{11}(x) & U_{12}(x) & \ldots & U_{1r}(x) \\ 0 & U_{22}(x) & \ldots & U_{2r}(x) \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & U_{rr}(x) \end{pmatrix}, \tag{2}$$

$v(x)$ is a real scalar function, that $0 < v(x) \to \infty$ monotonically, as $x \to \infty$, and it has monotone absolutely continuous derivative. Also, $U(x)$ is a relatively small perturbation, e.g., $|U(x)| \cdot v^{-1}(x) \to 0$ as $x \to \infty$ or $|U| v^{-1} \in L^{\infty}(\mathbb{R}_+)$. The diagonal blocks $U_{kk}(x)$, $k = \overline{1, r}$ are assumed as bounded self-adjoint operators in H_k.

Let $v(x) = x^{2\alpha}$, $0 < \alpha \le 1$ and coefficients of equation (1) satisfy the condition

$$\int_a^\infty |U(t)| \cdot t^{-\alpha} dt < \infty, \quad a > 0. \tag{3}$$

Let us rewrite equation (1) in the form

$$-\bar{y}'' + (x^{2\alpha} - \lambda + q(x, \lambda))\bar{y} = (q(x, \lambda) \cdot I - U(x))\bar{y},$$

where $q(x, \lambda)$ is determined by a formula (cf. with [5])

$$q(x, \lambda) = \frac{5\alpha^2}{4}\left(\frac{x^{2\alpha-1}}{x^{2\alpha} - \lambda}\right)^2 - \frac{\alpha(2\alpha - 1)x^{2\alpha-2}}{2(x^{2\alpha} - \lambda)}.$$

Now let us denote

$$\gamma_0(x, \lambda) = \frac{1}{\sqrt[4]{4(x^{2\alpha} - \lambda)}} \cdot \exp\left(-\int_a^x \sqrt{u^{2\alpha} - \lambda}\, du\right),$$

$$\gamma_\infty(x, \lambda) = \frac{1}{\sqrt[4]{4(x^{2\alpha} - \lambda)}} \cdot \exp\left(\int_a^x \sqrt{u^{2\alpha} - \lambda}\, du\right).$$

These solutions constitute a fundamental system of solutions of the scalar differential equation $-z'' + (x^{2\alpha} - \lambda + q(x, \lambda))z = 0$, in such a way that for all $x \in [0, \infty)$, one has

$$W(\gamma_0, \gamma_\infty) := \gamma_0(x, \lambda) \cdot \gamma'_\infty(x, \lambda) - \gamma'_0(x, \lambda) \cdot \gamma_\infty(x, \lambda) = 1.$$

3. Asymptotics of the Fundamental System of Solutions at Infinity

We are about to establish the asymptotics[1] of $\gamma_0(x, \lambda)$ as $x \to \infty$:

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[1]For $\alpha = 1$ and $\alpha = \frac{1}{2}$, i.e., for $v(x) = x^2$ and $v(x) = x$, the asymptotics of the functions $\gamma_0(x, \lambda)$ and $\gamma_\infty(x, \lambda)$ is known.

$$\gamma_0(x, \lambda) = (2x^{\alpha})^{-\frac{1}{2}} \cdot \left(1 - \frac{\lambda}{x^{2\alpha}}\right)^{-\frac{1}{4}} \cdot \exp\left(-\int_a^x u^{\alpha}\left(1 - \frac{\lambda}{u^{2\alpha}}\right)^{\frac{1}{2}} du\right).$$

After expanding here the integral, we obtain the exponential as follows:

$$\exp\left(-\int_a^x u^{\alpha} \cdot \left(1 - \frac{1}{2} \cdot \frac{\lambda}{u^{2\alpha}} - \sum_{k=2}^{\infty} \frac{1 \cdot 3 \cdot \ldots \cdot (2k-3)}{k! \cdot 2^k} \cdot \left(\frac{\lambda}{u^{2\alpha}}\right)^k\right) du\right).$$

In case $\frac{\alpha + 1}{2\alpha} = n \in N$, i.e., $\alpha = \frac{1}{2n-1}$, this expression after integration acquires the form:

$$c \cdot \exp\left(-\frac{x^{1+\alpha}}{1+\alpha} + \frac{\lambda}{2} \cdot \frac{x^{1-\alpha}}{1-\alpha} + \sum_{k=2}^{n-1} \frac{1 \cdot 3 \cdot \ldots \cdot (2k-3)}{k!} \cdot \left(\frac{\lambda}{2}\right)^k \cdot \frac{x^{1-(2k-1)\alpha}}{1-(2k-1)\alpha}\right)$$

$$\cdot \exp\left(\frac{1 \cdot 3 \cdot \ldots \cdot (2n-3)}{n!} \cdot \left(\frac{\lambda}{2}\right)^n \cdot \ln x + o(1)\right)$$

$$= c \cdot \exp\left(-\frac{x^{1+\alpha}}{1+\alpha} + \frac{\lambda}{2} \cdot \frac{x^{1-\alpha}}{1-\alpha} + \sum_{k=2}^{n-1} \frac{1 \cdot 3 \cdot \ldots \cdot (2k-3)}{k!} \cdot \left(\frac{\lambda}{2}\right)^k \cdot \frac{x^{1-(2k-1)\alpha}}{1-(2k-1)\alpha}\right)$$

$$\cdot\, x^{\frac{1 \cdot 3 \cdot \ldots \cdot (2n-3)}{n!} \cdot \left(\frac{\lambda}{2}\right)^n} \cdot (1 + o(1)).$$

The asymptotics of $\gamma_0(x, \lambda)$ as $x \to \infty$ is as follows:

$$\gamma_0(x, \lambda) = c \cdot \exp\left(-\frac{x^{1+\alpha}}{1+\alpha} + \frac{\lambda}{2} \cdot \frac{x^{1-\alpha}}{1-\alpha} + \sum_{k=2}^{n-1} \frac{1 \cdot 3 \cdot \ldots \cdot (2k-3)}{k!} \cdot \left(\frac{\lambda}{2}\right)^k \cdot \frac{x^{1-(2k-1)\alpha}}{1-(2k-1)\alpha}\right)$$

$$\cdot\, x^{\frac{1 \cdot 3 \cdot \ldots \cdot (2n-3)}{n!} \cdot \left(\frac{\lambda}{2}\right)^n - \frac{\alpha}{2}} \cdot (1 + o(1)).$$

In particular, for $\alpha = 1$ $(n = 1)$, $\gamma_0(x, \lambda)$ has the following asymptotics at infinity: $\gamma_0(x, \lambda) = c \cdot x^{\frac{\lambda - 1}{2}} \cdot \exp\left(-\frac{x^2}{2}\right)(1 + o(1)).$

In case $\frac{\alpha+1}{2\alpha} \notin \mathbb{N}$, we set $n = \left[\frac{\alpha+1}{2\alpha}\right] + 1$, with $[\beta]$ being the integral part of β, to obtain the following asymptotics for $\gamma_0(x, \lambda)$ at infinity:

$$\gamma_0(x, \lambda) = c \cdot x^{-\frac{\alpha}{2}} \exp\left(-\frac{x^{1+\alpha}}{1+\alpha} + \frac{\lambda}{2} \cdot \frac{x^{1-\alpha}}{1-\alpha} + \sum_{k=2}^{n-1} \frac{1 \cdot 3 \cdot \ldots \cdot (2k-3)}{k!} \cdot \left(\frac{\lambda}{2}\right)^k \cdot \frac{x^{1-(2k-1)\alpha}}{1-(2k-1)\alpha} \right)$$

$$\cdot \exp\left(-\frac{1 \cdot 3 \cdot \ldots \cdot (2n-3)}{n!} \cdot \left(\frac{\lambda}{2}\right)^n \cdot \frac{x^{-\alpha}}{\alpha} \right) \cdot (1 + o(x^{-\alpha})).$$

In particular, with $\alpha = \frac{1}{2}$ $(n = 2)$, one has

$$\gamma_0(x, \lambda) = cx^{-\frac{1}{4}} \cdot \exp\left(-\frac{2}{3} x^{\frac{3}{2}} + \lambda x^{\frac{1}{2}} - \left(\frac{\lambda}{2}\right)^2 x^{-\frac{1}{2}} \right) \cdot \left(1 + o\left(x^{-\frac{1}{2}} \right) \right).$$

A similar procedure allows to establish the asymptotics of $\gamma_\infty(x)$ as $x \to \infty$. If $\frac{\alpha+1}{2\alpha} = n \in N$, i.e., $\alpha = \frac{1}{2n-1}$, then

$$\gamma_\infty(x, \lambda) = c \cdot \exp\left(\frac{x^{1+\alpha}}{1+\alpha} - \frac{\lambda}{2} \cdot \frac{x^{1-\alpha}}{1-\alpha} - \sum_{k=2}^{n-1} \frac{1 \cdot 3 \cdot \ldots \cdot (2k-3)}{k!} \cdot \left(\frac{\lambda}{2}\right)^k \cdot \frac{x^{1-(2k-1)\alpha}}{1-(2k-1)\alpha} \right)$$

$$\cdot x^{-\left(\frac{1 \cdot 3 \cdot \ldots \cdot (2n-3)}{n!} \cdot \left(\frac{\lambda}{2}\right)^n + \frac{\alpha}{2} \right)} \cdot (1 + o(1)).$$

With $\alpha = 1 (n = 1)$, this becomes $\gamma_\infty(x, \lambda) = c \cdot x^{-\frac{\lambda+1}{2}} \cdot \exp\left(\frac{x^2}{2}\right)(1 + o(1))$.

In case $\frac{\alpha+1}{2\alpha} \notin N$, we set $n = \left[\frac{\alpha+1}{2\alpha}\right] + 1$ to get the asymptotics

$$\gamma_\infty(x, \lambda) = c \cdot x^{-\frac{\alpha}{2}} \exp\left(\frac{x^{1+\alpha}}{1+\alpha} - \frac{\lambda}{2} \cdot \frac{x^{1-\alpha}}{1-\alpha} - \sum_{k=2}^{n-1} \frac{1 \cdot 3 \cdot \ldots \cdot (2k-3)}{k!} \cdot \left(\frac{\lambda}{2}\right)^k \cdot \frac{x^{1-(2k-1)\alpha}}{1-(2k-1)\alpha} \right)$$

$$\cdot \exp\left(\frac{1 \cdot 3 \cdot \ldots \cdot (2n-3)}{n!} \cdot \left(\frac{\lambda}{2}\right)^n \cdot \frac{x^{-\alpha}}{\alpha} \right) \cdot (1 + o(x^{-\alpha})).$$

In case $\alpha = \frac{1}{2}(n = 2)$, one has

$$\gamma_\infty(x, \lambda) = cx^{-\frac{1}{4}} \cdot \exp\left(\frac{2}{3}x^{\frac{3}{2}} - \lambda x^{\frac{1}{2}} + \left(\frac{\lambda}{2}\right)^2 x^{-\frac{1}{2}}\right) \cdot \left(1 + o\left(x^{-\frac{1}{2}}\right)\right).$$

In paper [2] for equation (1) with operator potential the fundamental system of solutions is built, one of that is decreasing and at infinity has asymptotics $\gamma_0(x, \lambda)$, and the second is increasing with asymptotics $\gamma_\infty(x, \lambda)$.

Theorem 1. *Under condition* (3) *equation* (1) *has a unique decreasing at infinity operator solution* $\Phi(x, \lambda) \in B(\mathbf{H})$, *satisfying the conditions*

$$\lim_{x\to\infty} \frac{\Phi(x, \lambda)}{\gamma_0(x, \lambda)} = I \quad \textit{and} \quad \lim_{x\to\infty} \frac{\Phi'(x, \lambda)}{\gamma_0'(x, \lambda)} = I. \tag{4}$$

Also, there exists increasing at infinity operator solution $\Psi(x, \lambda) \in B(\mathbf{H})$ *satisfying the conditions* $\lim_{x\to\infty} \frac{\Psi(x, \lambda)}{\gamma_\infty(x, \lambda)} = I$ *and* $\lim_{x\to\infty} \frac{\Psi'(x, \lambda)}{\gamma_\infty'(x, \lambda)} = I$.

Corollary 1. *If* $\alpha = 1$, *i.e., the coefficient* $v(x) = x^2$, *then, under condition* (3), *equation* (1) *has a unique operator solution* $\Phi(x, \lambda)$ *decreasing at infinity and satisfying the relation* (4), *where* $\gamma_0(x, \lambda) = x^{\frac{\lambda-1}{2}} \cdot \exp\left(-\frac{x^2}{2}\right)$.

Also, this equation has operator solution $\Psi(x, \lambda)$ *growing at infinity and satisfying the relation* $\lim_{x\to\infty} \frac{\Psi(x, \lambda)}{\gamma_\infty(x, \lambda)} = I$, *such that* $\lim_{x\to\infty} \frac{\Psi'(x, \lambda)}{\gamma_\infty'(x, \lambda)} = I$, *where* $\gamma_\infty(x, \lambda) = x^{-\frac{\lambda+1}{2}} \cdot \exp\left(\frac{x^2}{2}\right)$.

Remark 1. In the monograph [5], it was shown that the scalar equation

$$-\varphi'' + x^2\varphi = \lambda\varphi, \tag{5}$$

with $\lambda = 2n + 1$, has a solution $\varphi_n(x) = H_n(x)\exp\left(-\frac{x^2}{2}\right)$, where $H_n(x)$ is the Chebyshev-Hermite polynomial. Note that this polynomial has the following asymptotics as $x \to \infty : H_n(x) = (2x)^n(1 + o(1))$, and thus the asymptotics of the solution $\varphi_n(x)$ as $x \to \infty$ is $\varphi_n(x) = (2x)^n \cdot \exp\left(-\frac{x^2}{2}\right) \cdot (1 + o(1))$.

If in equation (2), one has $U(x) = 0$, $v(x) = x^2$, then in the case of m-dimensional Hilbert space matrix equation (1) splits into scalar equations of the form (5). The matrix solution $\Phi(x, \lambda)$ in this case appears to be diagonal. Denote by $\varphi(x, \lambda)$ the diagonal elements of the matrix $\Phi(x, \lambda)$. Then by virtue (4), the solution $\varphi(x, \lambda)$ has the following asymptotics at infinite $\varphi(x, \lambda) = (x)^{\frac{\lambda-1}{2}} \cdot \exp\left(-\frac{x^2}{2}\right)(1 + o(1))$. In particular, with $\lambda = 2n + 1$, this allows one to derive a solution which is a scalar multiple of $\varphi_n(x)$.

4. Conclusion

In this paper, the asymptotics of solutions of the auxiliary scalar differential equation is obtained using the asymptotics of the fundamental system of solutions of differential equation with block-triangular operator potential.

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