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CENTER OF THE SKEW POLYNOMIAL RING OVER QUATERNIONS

Amir Kamal Amir, Jusriani Djuddin, Mawardi Bahri, Nur Erawaty and Nurul Mukhlisah Abdal

Department of Mathematics
Faculty of Mathematics and Natural Sciences
Hasanuddin University
Makassar, Indonesia

Abstract

We determine the center of a skew polynomial ring over quaternions.

1. Definitions and Notations

The form of centers and ideals of skew polynomial rings over commutative ring have been investigated during the last few years. In [1], center of skew polynomial ring over a domain, has been considered. In [2, 3, 5], invertible, maximal and prime ideals of skew polynomial ring over commutative Dedekind domain were considered. In this paper, the form of

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the center of the skew polynomial ring over the noncommutative ring which is quaternion is found.

Recall the set of quaternion

$$\mathbb{H} = \{q = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3 \,|\, q_0, q_1, q_2, q_3 \in \mathbb{R}\},$$

where $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ should satisfy Hamilton multiplication rule as follows:

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1; \quad \boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k}; \quad \boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i}; \quad \boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j}.$$

Then the maps $\sigma_1 : \mathbb{H} \to \mathbb{H}$ and $\sigma_2 : \mathbb{H} \to \mathbb{H}$ defined, respectively, by

$$\sigma_1(a) = \sigma_1(a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}) = (a_0 - a_1\boldsymbol{i} - a_2\boldsymbol{j} + a_3\boldsymbol{k})$$

and

$$\sigma_2(a) = \sigma_2(a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}) = (a_0 + a_2\boldsymbol{i} + a_3\boldsymbol{j} - a_1\boldsymbol{k})$$

are endomorphisms.

Let σ be an endomorphism over $\mathbb{H}$. Then the skew polynomial ring over quaternions with an unknown variable x, denoted by $\mathbb{H}[x; \sigma]$, is a ring consisting of polynomials

$$q(x) = q_n x^n + q_{n-1}x^{n-1} + \cdots + q_2 x^2 + q_1 x + q_0$$

with $q_0, q_1, q_2, q_3 \in \mathbb{H}$ and the multiplication rule $xa = \sigma(a)x$, for each $a \in \mathbb{H}$.

2. The Main Results

In this part, we formulate the center's form of the skew polynomials ring over quaternion.

Theorem 1. *Let* $\mathbb{H}[x; \sigma_1]$ *be a skew polynomial ring over quaternions, where* σ_1 *is defined by*

$$\sigma_1(a) = \sigma_1(a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}) = a_0 - a_1\boldsymbol{i} - a_2\boldsymbol{j} + a_3\boldsymbol{k}.$$

Then its center is given by

$$\boldsymbol{Z}(\mathbb{H}[x;\ \sigma_1]) = \left\{\sum_{n=0}^{t}(p_{2n}x^{2n} + p_{2n+1}\boldsymbol{k}x^{2n+1}) \mid p_{2n},\ p_{2n+1} \in \mathbb{R}\right\}.$$

Proof. First, we show that

$$\left\{\sum_{n=0}^{t}(p_{2n}x^{2n} + p_{2n+1}\boldsymbol{k}x^{2n+1}) \mid p_{2n},\ p_{2n+1} \in \mathbb{R}\right\} \subseteq \boldsymbol{Z}(\mathbb{H}[x;\ \sigma_1]).$$

For this, choose

$$p(x) \in \left\{\sum_{n=0}^{t}(p_{2n}x^{2n} + p_{2n+1}\boldsymbol{k}x^{2n+1}) \mid p_{2n},\ p_{2n+1} \in \mathbb{R}\}\right\}.$$

We obtain that $p(x)q(x) = q(x)p(x),\ \ \forall q(x) \in \mathbb{H}[x;\ \sigma_1]$. Without loss of generality, the proof is given for $q(x) = a$, where $a \in \mathbb{H}$ and for $q(x) = x$.

When $q(x) = a$, let

$$p(x) = \sum\nolimits_{n=0}^{t}(p_{2n}x^{2n} + p_{2n+1}\boldsymbol{k}x^{2n+1}), \quad p_{2n},\ p_{2n+1} \in \mathbb{R}.$$

Then

$$p(x)q(x) = \left[\sum_{n=0}^{t}(p_{2n}x^{2n} + p_{2n+1}\boldsymbol{k}x^{2n+1})\right][a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}].$$

Since $\sigma_1^{2n}(a) = a = a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\sigma_1^{2n+1}(a) = \sigma_1(a) = a_0 - a_1\boldsymbol{i} - a_2\boldsymbol{j} + a_3\boldsymbol{k}$, we obtain

$$\begin{aligned} & p(x)q(x) \\ &= \left[\sum_{n=0}^{t}(p_{2n}(a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k})x^{2n} + p_{2n+1}\boldsymbol{k}(a_0 - a_1\boldsymbol{i} - a_2\boldsymbol{j} + a_3\boldsymbol{k})x^{2n+1})\right] \\ &= \sum_{n=0}^{t}[p_{2n}(a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k})x^{2n} + p_{2n+1}(-a_3 + a_2\boldsymbol{i} - a_1\boldsymbol{j} + a_0\boldsymbol{k})x^{2n+1}]. \end{aligned} \tag{1}$$

Next,

$$q(x)\,p(x)$$

$$= [a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}]\left[\sum_{n=0}^{t}(p_{2n}x^{2n} + p_{2n+1}\boldsymbol{k}x^{2n+1})\right]$$

$$= \left[\sum_{n=0}^{t}(p_{2n}(a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k})x^{2n} + p_{2n+1}(a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k})\boldsymbol{k}x^{2n+1})\right]$$

$$= \sum_{n=0}^{t}[p_{2n}(a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k})x^{2n} + p_{2n+1}(-a_3 + a_2\boldsymbol{i} - a_1\boldsymbol{j} + a_0\boldsymbol{k})x^{2n+1}]. \tag{2}$$

From equations (1) and (2), we see that $p(x)q(x) = q(x)\,p(x)$.

When $q(x) = x$, then

$$p(x)q(x) = \left[\sum_{n=0}^{t}(p_{2n}x^{2n} + p_{2n+1}\boldsymbol{k}x^{2n+1})\right]x$$

$$= \left[\sum\nolimits_{n=0}^{t}(p_{2n}x^{2n+1} + p_{2n+1}\boldsymbol{k}x^{2n+2})\right]. \tag{3}$$

On the other side,

$$q(x)\,p(x) = x\left[\sum_{n=0}^{t}(p_{2n}x^{2n} + p_{2n+1}\boldsymbol{k}x^{2n+1})\right].$$

Since $xp_i = p_i x$ and $x\boldsymbol{k} = \boldsymbol{k}x$, thus

$$q(x)\,p(x) = \left[\sum\nolimits_{n=0}^{t}(p_{2n}x^{2n+1} + p_{2n+1}\boldsymbol{k}x^{2n+2})\right]. \tag{4}$$

From equations (3) and (4), we get $p(x)q(x) = q(x)\,p(x)$.

Now, we show that

$$\boldsymbol{Z}(\mathbb{H}[x;\sigma_1]) \subseteq \left\{ \sum_{n=0}^{t} (p_{2n}x^{2n} + p_{2n+1}\boldsymbol{k}x^{2n+1}) \mid p_{2n},\ p_{2n+1} \in \mathbb{R} \right\}.$$

Let $r(x) \in \boldsymbol{Z}(\mathbb{H}[x;\sigma_1])$. Thus $r(x)$ satisfies $r(x)q(x) = q(x)r(x)$, $\forall q(x) \in \mathbb{H}[x;\sigma_1]$.

Suppose

$$r(x) = r_0 + r_1x + \cdots + r_tx^t = \sum\nolimits_{n=0}^{t} r_nc^n, \quad r_n \in \mathbb{H}.$$

For $q(x) = \boldsymbol{i}$,

$$r(x)q(x) = r(x)\boldsymbol{i} = \boldsymbol{i}r(x) = q(x)r(x).$$

Since $\sigma_1^{2n}(\boldsymbol{i}) = \boldsymbol{i}$ and $\sigma_1^{2n+1}(\boldsymbol{i}) = -\boldsymbol{i}$,

$$r(x)\boldsymbol{i} = \left[\sum\nolimits_{n=0}^{t} r_nx^n\right]\boldsymbol{i} = \sum\nolimits_{n=0}^{t} (-1)^n r_n\boldsymbol{i}x^n \tag{5}$$

and

$$\boldsymbol{i}r(x) = \boldsymbol{i}\left[\sum\nolimits_{n=0}^{t} r_nx^n\right] = \left[\sum\nolimits_{n=0}^{t} \boldsymbol{i}r_nx^n\right]. \tag{6}$$

Since $r(x)\boldsymbol{i} = \boldsymbol{i}r(x)$, from equations (5) and (6), we obtain $r_{2n}\boldsymbol{i} = \boldsymbol{i}r_{2n}$ and $-r_{2n+1}\boldsymbol{i} = \boldsymbol{i}r_{2n+1}$. Then, suppose that $r_{2n} = a_{0n} + a_{1n}\boldsymbol{i} + a_{2n}\boldsymbol{j} + a_{3n}\boldsymbol{k}$, from the equality of $r_{2n}\boldsymbol{i} = \boldsymbol{i}r_{2n}$, we get $a_{2n} = a_{3n} = 0$, thus $r_{2n} = a_{0n} + a_{1n}\boldsymbol{i}$. Furthermore, let $r_{2n+1} = b_{0n} + b_{1n}\boldsymbol{i} + b_{2n}\boldsymbol{j} + b_{3n}\boldsymbol{k}$, from the equality of $-r_{2n+1}\boldsymbol{i} = \boldsymbol{i}r_{2n+1}$, we obviously obtain $b_{0n} = b_{1n} = 0$. Then $r_{2n+1} = b_{2n}\boldsymbol{j} + b_{3n}\boldsymbol{k}$. Therefore, $r(x)$ can be written as

$$r(x) = \sum_{n=0}^{t}(r_{2n}x^{2n} + r_{2n+1}x^{2n+1}) = \sum_{n=0}^{t}((h_{n0} + h_{n1}\boldsymbol{i})x^{2n} + (h_{n2}\boldsymbol{j} + h_{n3}\boldsymbol{k})x^{2n+1}),$$

where $h_{uv} \in \mathbb{R}$.

Next, choose $q(x) = \boldsymbol{j}$,

$$r(x)g(x) = r(x)\boldsymbol{j}$$

$$= \left[\sum_{n=0}^{t}((h_{n0} + h_{n1}\boldsymbol{i})x^{2n} + (h_{n2}\boldsymbol{j} + h_{n3}\boldsymbol{k})x^{2n+1})\right]\boldsymbol{j}$$

$$= \left[\sum_{n=0}^{t}((h_{n0} + h_{n1}\boldsymbol{i})\sigma_1^{2n}(\boldsymbol{j})x^{2n} + (h_{n2}\boldsymbol{j} + h_{n3}\boldsymbol{k})\sigma_1^{2n+1}(\boldsymbol{j})x^{2n+1})\right].$$

Since $\sigma_1^{2n}(\boldsymbol{j}) = \boldsymbol{j}$ and $\sigma_1^{2n+1}(\boldsymbol{j}) = -\boldsymbol{j}$,

$$r(x)q(x) = \left[\sum_{n=0}^{t}((h_{n0} + h_{n1}\boldsymbol{i})(\boldsymbol{j})x^{2n} + (h_{n2}\boldsymbol{j} + h_{n3}\boldsymbol{k})(-\boldsymbol{j})x^{2n+1})\right]$$

$$= \left[\sum_{n=0}^{t}((h_{n0}\boldsymbol{j} + h_{n1}\boldsymbol{k})x^{2n} + (h_{n2} + h_{n3}\boldsymbol{i})x^{2n+1})\right], \tag{7}$$

$$q(x)r(x) = (\boldsymbol{j})\left[\sum_{n=0}^{t}((h_{n0} + h_{n1}\boldsymbol{i})x^{2n} + (h_{n2}\boldsymbol{j} + h_{n3}\boldsymbol{k})x^{2n+1})\right]$$

$$= \sum_{n=0}^{t}((h_{n0}\boldsymbol{j} - h_{n1}\boldsymbol{k})x^{2n} + (-h_{n2} + h_{n3}\boldsymbol{i})x^{2n+1}). \tag{8}$$

Since $r(x)q(x) = q(x)r(x)$, from equations (7) and (8), we obtain $h_{n1} = h_{n2} = 0$, so

$$r(x) = \sum_{n=0}^{t}(h_{n0}x^{2n} + h_{n3}\boldsymbol{k}x^{2n+1}).$$

Since $h_{n0}, h_{n3} \in \mathbb{R}$, it can be written in general form as

$$r(x) = \sum_{n=0}^{t}(h_{2n}x^{2n} + h_{2n+1}\boldsymbol{k}x^{2n+1}), \text{ where } h_{2n}, h_{2n+1} \in \mathbb{R}.$$

Accordingly, $r(x) \in \left\{\sum_{n=0}^{t}(p_{2n}x^{2n} + p_{2n+1}\boldsymbol{k}x^{2n+1}) \mid p_{2n}, p_{2n+1} \in \mathbb{R}\right\}$, it is shown that

$$\boldsymbol{Z}(\mathbb{H}[x;\ \sigma_1])$$

$$\subseteq \left\{ p(x) = \sum_{n=0}^{t} (p_{2n}x^{2n} + p_{2n+1}\boldsymbol{k}x^{2n+1}) \mid p_{2n},\ p_{2n+1} \in \mathbb{R},\ t \in \mathbb{Z}_+ \right\}. \quad \square$$

Theorem 2. *Let* $\mathbb{H}[x;\ \sigma_2]$ *be a skew polynomial ring over quaternions,* σ_2 *be an endomorphism such that*

$$\sigma_2(a) = \sigma_2(a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}) = a_0 + a_2\boldsymbol{i} + a_3\boldsymbol{j} + a_1\boldsymbol{k}$$

for each $a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k} \in \mathbb{H}$*. The center of ring* $\mathbb{H}[x;\ \sigma_2]$ *is*

$$\boldsymbol{Z}(\mathbb{H}[x;\ \sigma_2]) = \left\{ \sum\nolimits_{n=0}^{t} (p_{3n}x^{3n} + p_{3n+1}(1 + \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})x^{3n+1} \right.$$

$$\left. + p_{3n+2}(1 - \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k})x^{3n+2}) \mid p_i \in \mathbb{R} \right\}.$$

Proof. It is easily shown that

$$\sigma_2^{3n}(a) = a, \quad \sigma_2^{3n+1}(a) = \sigma_2(a) \quad \text{and} \quad \sigma_2^{3n+2}(a) = \sigma_2^2(a). \tag{9}$$

Further, the proof process is divided into two parts:

(a) We will show that

$$\left\{ \sum\nolimits_{n=0}^{t} (p_{3n}x^{3n} + p_{3n+1}(1 + \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})x^{3n+1} \right.$$

$$\left. + p_{3n+2}(1 - \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k})x^{3n+2}) \mid p_i \in \mathbb{R} \right\} \subseteq \boldsymbol{Z}(\mathbb{H}[x;\ \sigma_2]).$$

Let

$$r(x) \in \left\{ \sum\nolimits_{n=0}^{t} (p_{3n}x^{3n} + p_{3n+1}(1 + \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})x^{3n+1} \right.$$

$$\left. + p_{3n+2}(1 - \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k})x^{3n+2}) \mid p_i \in \mathbb{R} \right\}.$$

We will show that $r(x)q(x) = q(x)r(x),\ \forall q(x) \in \mathbb{H}[x;\ \sigma_2]$.

Without loss of generality, the proof will be done just for two types of $q(x)$. Firstly, for $q(x) = a$, where $a \in \mathbb{H}$ and secondly, for $q(x) = x$.

With methods similar to the proof of Theorem 1, we get

$$q(x)r(x) = r(x)q(x).$$

(b) We will show that

$$\boldsymbol{Z}(\mathbb{H}[x; \sigma_2])$$

$$\subseteq \left\{ \sum_{n=0}^{t} (p_{3n}x^{3n} + p_{3n+1}(1 + \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})x^{3n+1} + p_{3n+2}(1 - \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k})x^{3n+2}) | \, p_i \in \mathbb{R} \right\}.$$

Let $r(x) \in \boldsymbol{Z}(\mathbb{H}[x; \sigma_1])$. It means that $r(x)$ satisfies $r(x)q(x) = q(x)\,p(x)$, $\forall q(x) \in \mathbb{H}[x; \sigma_1]$. Let $r(x) = \sum_{n=0}^{t} r_n x^n$, $r_n \in \mathbb{H}$.

When $q(x) = \boldsymbol{i}$,

$$r(x)q(x) = r(x)\boldsymbol{i} = \boldsymbol{i}r(x) = q(x)r(x).$$

Since $\sigma_2^{3n}(\boldsymbol{i}) = \boldsymbol{i}$, $\sigma_2^{3n+1}(\boldsymbol{i}) = \boldsymbol{k}$ and $\sigma_2^{3n+2}(\boldsymbol{i}) = \boldsymbol{j}$, with a simple calculation, we obtain

$$\begin{aligned} r(x) &= \sum_{n=0}^{t} (r_{3n}x^{3n} + r_{3n+1}x^{3n+1} + r_{3n+2}x^{3n+2}) \\ &= \sum_{n=0}^{t} ((c_{3n}^0 + c_{3n}^1\boldsymbol{i})x^{3n} + (c_{3n+1}^0 + c_{3n+1}^1\boldsymbol{i} + c_{3n+1}^0\boldsymbol{j} + c_{3n+1}^1\boldsymbol{k})x^{3n+1}) \\ &\quad + (c_{3n+2}^0 + c_{3n+2}^1\boldsymbol{i} + c_{3n+2}^1\boldsymbol{j} - c_{3n+2}^0\boldsymbol{k})x^{3n+2} \end{aligned}$$

with $c_i^j \in \mathbb{R}$.

Next, when $q(x) = \boldsymbol{j}$, we have

$$r(x)q(x) = r(x)\boldsymbol{j}$$

$$= \Big[\sum\nolimits_{n=0}^{t}((c_{3n}^{0} + c_{3n}^{1}\boldsymbol{i})x^{3n} + (c_{3n+1}^{0} + c_{3n+1}^{1}\boldsymbol{i} + c_{3n+1}^{0}\boldsymbol{j} + c_{3n+1}^{1}\boldsymbol{k})x^{3n+1}$$

$$+ (c_{3n+2}^{0} + c_{3n+2}^{1}\boldsymbol{i} + c_{3n+2}^{1}\boldsymbol{j} - c_{3n+2}^{0}\boldsymbol{k})x^{3n+2})\Big]\boldsymbol{j}.$$

Since $r(x)\boldsymbol{j} = \boldsymbol{j}r(x)$, with a simple calculation, we get

$$c_{3n}^{1} = 0, \quad c_{3n+1}^{0} = c_{3n+1}^{1} \quad \text{and} \quad c_{3n+2}^{0} = -c_{3n+2}^{1}.$$

So,

$$r(x) = \sum_{n=0}^{t}[c_{3n}^{0}x^{3n} + c_{3n+1}^{0}(1 + \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})x^{3n+1} + (c_{3n+2}^{0}(1 - \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k})x^{3n+2}].$$

Thus, we have

$$r(x)$$

$$\in \left\{\sum_{n=0}^{t}(p_{3n}x^{3n} + p_{3n+1}(1 + \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})x^{3n+1} + p_{3n+2}(1 - \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k})x^{3n+2}) \mid p_i \in \mathbb{R}\right\}.$$

$\square$

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