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# FIXED POINT THEOREMS FOR THE MULTIVALUED CONTRACTION MAPPING IN THE QUASI $\alpha b$-METRIC SPACE 

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#### Abstract

In this paper, we introduce the quasi $\alpha b$-metric space which is a generalization of the quasi $b$-metric space and show an existence and uniqueness of the fixed point for the multivalued contraction mapping in the complete quasi $\alpha b$-metric space.


## 1. Introduction

The notion of a $b$-metric was introduced by Bakhtin in 1989 [1]. Few years later, this concept was applied to generalize the Banach's fixed point theorem in $b$-metric space by Czerwik [2]. Furthermore, Nadler [3] introduced the fixed point for the multivalued contraction mappings in a metric space. Some authors have presented results for several generalized contractive multivalued functions in $b$-metric spaces [4-9]. While the quasi

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$\alpha b$-metric space is an extension of the quasi $b$-metric space that has been introduced by Nurwahyu [10]. This study is motivated by authors who have been working on the fixed point for several multivalued contraction mappings in the quasi $b$-metric spaces. Therefore, the aim of this study is to establish and prove existence and uniqueness of fixed point theorem for the multivalued contraction mapping in the complete quasi $\alpha b$-metric.

## 2. Preliminaries

Definition 2.1 [10]. Let $X$ be a nonempty set and let $0 \leq \alpha<1$ and $b \geq 1$ be a given real number.

Let $d: X \times X \rightarrow[0, \infty)$ be a self mapping on $X$ which satisfies the following conditions:
(1) $d(x, y)=d(y, x)=0$ if and only if $x=y$;
(2)

$$
\begin{equation*}
d(x, y) \leq \alpha d(y, x)+\frac{1}{2} b(d(x, z)+d(z, y)) \text { for all } x, y, z \in X . \tag{2.1}
\end{equation*}
$$

Then $d$ is called a quasi $\alpha b$-metric on $X$ and $(X, d)$ is called a quasi $\alpha b$-metric space.

From the definition of the quasi $b$-metric, we show that every quasi $b$-metric is a quasi $\alpha b$-metric, but the converse is not true.

Example 2.2 [10]. Let $X=\{0,1,2\}$. Define $d: X \times X \rightarrow R^{+} \cup\{0\}$ as follows: $d(0,0)=d(1,1)=d(2,2)=d(0,2)=d(2,1)=0, d(1,0)=4$, $d(2,0)=1, d(0,1)=2$ and $d(1,2)=3$.

It is clear that $d$ is a quasi $\alpha b$-metric with $\alpha=\frac{1}{2}$ and $b=4$. This is because $2=d(0,1) \leq \frac{1}{2} d(1,0)+2(d(0,2))+d(2,1)$. However, $2=d(0,1)$ $>c(d(0,2)+d(2,1))$ for every $c \geq 1, d$ is not a quasi $b$-metric.

Example 2.3 [10]. Let $X=R$ and define $d: X \times X \rightarrow R^{+}$as $d(x, y)$ $=\left\{\begin{array}{l}2 x^{2}+y^{2}, x \neq y, \\ 0, x=y\end{array}\right.$

As seen from the given function, it is clear that the first condition of a quasi $b$-metric is satisfied. However, the second condition has to be shown.

For $x \neq y$, and every $z \in X, d(x, y)$ can be written as

$$
\begin{aligned}
d(x, y) & =2 x^{2}+y^{2} \leq \frac{5}{2} x^{2}+2 y^{2}+3 z^{2} \\
& =\frac{1}{2}\left(2 y^{2}+x^{2}\right)+\left(\left(2 x^{2}+z^{2}\right)+\left(2 z^{2}+y^{2}\right)\right) \\
& =\frac{1}{2} d(y, x)+\frac{2}{2}(d(x, z)+d(z, y))
\end{aligned}
$$

This equation can be rewritten as

$$
d(x, y)=\frac{1}{2} d(y, x)+\frac{2}{2}(d(x, z)+d(z, y))
$$

Hence, $d$ is clearly seen as a quasi $\alpha b$-metric with $\alpha=\frac{1}{2}$ and $b=2$.
Definition 2.4 [3]. Let $X$ and $Y$ be nonempty sets, let $2^{Y}$ be the collection of all subsets of $Y$. A mapping $T: X \rightarrow 2^{Y}$ is said to be a multivalued function on $X$.

Definition 2.5 [3]. A point $x \in X$ is said to be a fixed point of the multivalued mapping of $T$ if $x \in T(x)$.

Definition 2.6. Let $(X, d)$ be a quasi $\alpha b$-metric space. Let $C B(X)$ be a collection of closed and bounded subsets of $X$. Define a Hausdorff metric on $C B(X)$ as follows:

$$
H(A, B)=\max \left\{\sup _{a \in A} d(a, B), \sup _{b \in B} d(b, A)\right\}
$$

for all $A, B \in C B(X)$, where $d(a, B)=\inf \{d(a, x) \mid x \in B\}$ for all $a \in A$ and $d(B, a)=\inf \{d(x, a) \mid x \in B\}$ for all $a \in A$. Similarly,

$$
H(B, A)=\max \left\{\sup _{a \in A} d(B, a), \sup _{b \in B} d(A, b)\right\}
$$

for all $A, B \in C B(X)$, where $d(B, a)=\inf \{d(x, a) \mid x \in B\}$ for all $a \in A$ and $d(A, b)=\inf \{d(x, b) \mid x \in A\}$ for all $b \in B$.

In general, $H(A, B) \neq H(B, A)$.
Let $A=\{0,2\}, B=\{1\}$. Then

$$
d(0, B)=\inf _{x \in B}\{d(0, x)\}=\inf \{d(0,1)\}=2
$$

and

$$
d(2, B)=\inf _{x \in B}\{d(2, x)\}=\inf \{d(2,1)\}=0
$$

As a result, $\sup _{a \in A} d(a, B)=2$.
Similarly,

$$
d(1, A)=\inf _{x \in A} d(1, A)=\inf \{d(1,0), d(1,2)\}=3
$$

then $\sup _{b \in B} d(b, A)=3$.
Therefore, $H(A, B)$ gives

$$
H(A, B)=\max \left\{\sup _{a \in A} d(a, B), \sup _{b \in B} d(b, A)\right\}=3
$$

Following a similar procedure to the previous one, $d(B, 0)$ and $d(B, 2)$ are given by

$$
\begin{aligned}
& d(B, 0)=\inf _{x \in B}\{d(x, 0)\}=\inf \{d(1,0)\}=4 \\
& d(B, 2)=\inf _{x \in B}\{d(x, 2)\}=\inf \{d(1,2)\}=3
\end{aligned}
$$

Hence, $\sup _{a \in A} d(B, a)=4$. Likewise,

$$
d(A, 1)=\inf _{x \in A}\{d(x, 1)\}=\inf \{d(0,1), d(2,1)\}=0
$$

and then $\sup _{b \in B} d(A, b)=0$ is obtained.
Since $H(A, B)=3$ and $H(B, A)$ is given by

$$
H(B, A)=\max \left\{\sup _{a \in A} d(B, a), \sup _{b \in B} d(A, b)\right\}=4
$$

it shows that $H(A, B) \neq H(B, A)$.
Lemma 2.7. Let $(X, d)$ be a quasi $\alpha b$-metric space. If $A, B \in C B(X)$ and $a \in A$, then for each $\varepsilon>0$, there exists $b \in B$ such that $d(a, b) \leq$ $H(A, B)+\varepsilon$.

Proof. Suppose that there exists $\varepsilon>0$ such that

$$
d(a, b)>H(A, B)+\varepsilon
$$

for every $b \in B$. So $H(A, B)+\varepsilon$ is a lower bound of $\{d(a, b) \mid b \in B\}$.

$$
d(a, B)=\inf _{b \in B}\{d(a, b)\}, \quad \text { then we have } d(a, B) \geq H(A, B)+\varepsilon
$$ However, from the definition of $H(A, B)$, we have

$$
H(A, B) \geq \sup _{a \in A} d(a, B) \geq d(a, B)
$$

for every $a \in A$.
Since $(a, b)>H(A, B)+\varepsilon$ for every $b \in B$, we can obtain

$$
H(A, B) \geq d(a, B) \geq H(A, B)+\varepsilon
$$

Here, we obtain $\varepsilon \leq 0$ which is a contradiction.
Lemma 2.8. Let $(X, d)$ be a quasi $\alpha b$-metric space. If $A, B \in C B(X)$ and $a \in A$, then for each $\varepsilon>0$, there exists $b \in B$ such that $d(b, a) \leq$ $H(B, A)+\varepsilon$.

Proof. The proof is similar to the proof of Lemma 2.7.
Definition 2.9 [10]. Let $(X, d)$ be a quasi $\alpha b$-metric space. A sequence $\left\{x_{n}\right\}$ in $(X, d)$ converges to $x \in X$ if $\lim _{n \rightarrow \infty} d\left(x_{n}, x\right)=\lim _{n \rightarrow \infty} d\left(x, x_{n}\right)$ $=0$, we write $\lim _{n \rightarrow \infty} x_{n}=x$.

Definition 2.10 [10]. Let $\left\{x_{n}\right\}$ be a sequence in a quasi $\alpha b$-metric space $(X, d)$. Then $\left\{x_{n}\right\}$ is called a Cauchy sequence if $\lim _{n, m \rightarrow \infty} d\left(x_{n}, x_{m}\right)$ $=\lim _{n \rightarrow \infty} d\left(x_{m}, x_{n}\right)=0$.

Definition 2.11 [10]. Let ( $X, d$ ) be a quasi $\alpha b$-metric space. Then ( $X, d$ ) is called complete if every Cauchy sequence in $X$ is convergent in $X$.

Definition 2.12 [3]. Let ( $X, d$ ) be a metric space. Then a mapping $T: X \rightarrow C B(X)$ is said to be a multivalued contraction if there exists $0 \leq \lambda<1$ such that $H(T x, T y) \leq \lambda d(x, y)$, for all $x, y \in X$.

## 3. Main Results

Theorem 3.1. Let $(X, d)$ be a quasi $\alpha b$-metric space with $0 \leq \alpha<1$ and $b \geq 1$, let $\left\{x_{n}\right\}$ be a sequence in $X$ such that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} d\left(x_{n}, x_{n+1}\right)=\lim _{n \rightarrow \infty} d\left(x_{n+1}, x_{n}\right)=0 \tag{3.1}
\end{equation*}
$$

Then $\left\{x_{n}\right\}$ is a Cauchy sequence in $X$.
Proof. Based on the second condition of Definition 2.1, $d\left(x_{n}, x_{n+2}\right)$ is given by

$$
\begin{aligned}
& d\left(x_{n}, x_{n+2}\right) \\
\leq & \alpha d\left(x_{n+2}, x_{n}\right)+\frac{b}{2}\left(d\left(x_{n}, x_{n+1}\right)+d\left(x_{n+1}, x_{n+2}\right)\right) \\
\leq & \alpha\left[\alpha d\left(x_{n}, x_{n+2}\right)+\frac{b}{2}\left(d\left(x_{n+2}, x_{n+1}\right)+d\left(x_{n+1}, x_{n}\right)\right)\right] \\
& +\frac{b}{2}\left(d\left(x_{n}, x_{n+1}\right)+d\left(x_{n+1}, x_{n+2}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
& d\left(x_{n}, x_{n+2}\right) \\
\leq & \frac{\frac{1}{2} \alpha b\left(d\left(x_{n+2}, x_{n+1}\right)+d\left(x_{n+1}, x_{n}\right)\right)+\frac{b}{2}\left(d\left(x_{n}, x_{n+1}\right)+d\left(x_{n+1}, x_{n+2}\right)\right)}{1-\alpha^{2}}
\end{aligned}
$$

Making use of equation (3.1), it is clearly seen that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} d\left(x_{n}, x_{n+2}\right)=0 . \tag{3.2}
\end{equation*}
$$

Following a similar procedure to later one, we also have

$$
\begin{aligned}
& d\left(x_{n+2}, x_{n}\right) \\
\leq & \frac{\frac{1}{2} \alpha b\left(d\left(x_{n}, x_{n+1}\right)+d\left(x_{n+1}, x_{n+2}\right)\right)+\frac{b}{2}\left(d\left(x_{n+2}, x_{n+1}\right)+d\left(x_{n+1}, x_{n}\right)\right)}{1-\alpha^{2}}
\end{aligned}
$$

in which

$$
\begin{equation*}
\lim _{n \rightarrow \infty} d\left(x_{n+2}, x_{n}\right)=0 \tag{3.3}
\end{equation*}
$$

Repeating this procedure for $d\left(x_{n}, x_{n+3}\right)$, one will get the following:

$$
\begin{aligned}
& d\left(x_{n}, x_{n+3}\right) \\
\leq & \frac{\frac{1}{2} \alpha b\left(d\left(x_{n+3}, x_{n+2}\right)+d\left(x_{n+2}, x_{n}\right)\right)+\frac{b}{2}\left(d\left(x_{n}, x_{n+2}\right)+d\left(x_{n+2}, x_{n+3}\right)\right)}{1-\alpha^{2}}
\end{aligned}
$$

and hence using (3.2) and (3.3), we also get

$$
\begin{equation*}
\lim _{n \rightarrow \infty} d\left(x_{n}, x_{n+3}\right)=0 \tag{3.4}
\end{equation*}
$$

Moreover,

$$
\begin{aligned}
& d\left(x_{n+3}, x_{n}\right) \\
\leq & \frac{\frac{1}{2} \alpha b\left(d\left(x_{n+2}, x_{n+3}\right)+d\left(x_{n}, x_{n+2}\right)\right)+\frac{b}{2}\left(d\left(x_{n+2}, x_{n}\right)+d\left(x_{n+3}, x_{n+2}\right)\right)}{1-\alpha^{2}}
\end{aligned}
$$

which also yields the following result by using (3.2) and (3.4)

$$
\lim _{n \rightarrow \infty} d\left(x_{n+3}, x_{n}\right)=0
$$

Thus, by using induction, we have

$$
\begin{aligned}
& d\left(x_{n}, x_{n+k}\right) \\
\leq & \frac{\frac{1}{2} \alpha b\left(d\left(x_{n+k}, x_{n+k-1}\right)+d\left(x_{n+k-1}, x_{n}\right)\right)+\frac{b}{2}\left(d\left(x_{n}, x_{n+k-1}\right)+d\left(x_{n+k-1}, x_{n+k}\right)\right)}{1-\alpha^{2}}
\end{aligned}
$$

and

$$
\begin{aligned}
& d\left(x_{n+k}, x_{n}\right) \\
\leq & \frac{\frac{1}{2} \alpha b\left(d\left(x_{n+k-1}, x_{n+k}\right)+d\left(x_{n}, x_{n+k-1}\right)\right)+\frac{b}{2}\left(d\left(x_{n+k-1}, x_{n}\right)+d\left(x_{n+k}, x_{n+k-1}\right)\right)}{1-\alpha^{2}} .
\end{aligned}
$$

Therefore, we obtain

$$
\lim _{n \rightarrow \infty} d\left(x_{n}, x_{n+k}\right)=0 \quad \text { and } \quad \lim _{n \rightarrow \infty} d\left(x_{n+k}, x_{n}\right)=0 .
$$

Finally, for $m>n \geq 0$, we have

$$
\lim _{n \rightarrow \infty} d\left(x_{n}, x_{m}\right)=0 \quad \text { and } \quad \lim _{n \rightarrow \infty} d\left(x_{m}, x_{n}\right)=0
$$

Hence, $\left\{x_{m}\right\}$ is a Cauchy sequence in $X$.
Theorem 3.2. Let $(X, d)$ be a complete quasi $\alpha b$-metric space with $0 \leq \alpha<1$ and $b \geq 1$ and let $C B(X)$ be a collection of closed and bounded subsets of $X$.

Let $T: X \rightarrow C B(X)$ be a map that satisfies the following condition:

$$
\begin{equation*}
H(T x, T y) \leq \frac{d(y, T y) d(T x, x)}{1+p(d(y, T y)+d(T x, x))} \tag{3.5}
\end{equation*}
$$

for $x, y \in X, p>2$.
Then $T$ has a unique fixed point in $X$.

Proof. Let $x_{0} \in X$, and $x_{1} \in T x_{0}$. By Lemma 2.7, there exists $x_{2} \in T x_{1}$ such that

$$
d\left(x_{1}, x_{2}\right) \leq H\left(T x_{0}, T x_{1}\right)+\frac{1}{p^{2}} d\left(x_{0}, x_{1}\right)
$$

Then, using (3.5), we obtain

$$
\begin{aligned}
& d\left(x_{1}, x_{2}\right) \leq \frac{d\left(x_{1}, T x_{1}\right) d\left(T x_{0}, x_{0}\right)}{1+p\left(d\left(x_{1}, T x_{1}\right)+d\left(T x_{0}, x_{0}\right)\right)}+\frac{1}{p^{2}} d\left(x_{0}, x_{1}\right) \\
& d\left(x_{1}, x_{2}\right) \leq \frac{1}{p} d\left(x_{1}, x_{2}\right)+\frac{1}{p^{2}} d\left(x_{0}, x_{1}\right)
\end{aligned}
$$

Simplifying this equation yields

$$
d\left(x_{1}, x_{2}\right) \leq \frac{1}{p(p-1)} d\left(x_{0}, x_{1}\right)
$$

Furthermore, based on Lemma 2.7, there exists $x_{3} \in T x_{2}$ such that

$$
d\left(x_{2}, x_{3}\right) \leq H\left(T x_{1}, T x_{2}\right)+\frac{1}{p^{2}} d\left(x_{1}, x_{2}\right)
$$

which gives

$$
\begin{aligned}
& d\left(x_{2}, x_{3}\right) \leq \frac{d\left(x_{2}, T x_{2}\right) d\left(T x_{1}, x_{1}\right)}{1+p\left(d\left(x_{2}, T x_{2}\right)+d\left(T x_{1}, x_{1}\right)\right)}+\frac{1}{p^{2}} d\left(x_{1}, x_{2}\right) \\
& d\left(x_{2}, x_{3}\right) \leq \frac{1}{p} d\left(x_{2}, x_{3}\right)+\frac{1}{p^{2}} d\left(x_{1}, x_{2}\right)
\end{aligned}
$$

This equation can be rewritten as

$$
d\left(x_{2}, x_{2}\right) \leq \frac{1}{p(p-1)} d\left(x_{1}, x_{2}\right) \leq\left(\frac{1}{p(p-1)}\right)^{2} d\left(x_{0}, x_{1}\right)
$$

Continuing this process, we obtain a sequence $\left(x_{n}\right)$, where $x_{n+1} \in T x_{n}$ such that

$$
d\left(x_{n}, x_{n+1}\right) \leq\left(\frac{1}{p(p-1)}\right)^{n} d\left(x_{0}, x_{1}\right) .
$$

Since $p>2, p(p-1)>1$ which is equivalent to $0<\frac{1}{p(p-1)}<1$.
As a result, it is clearly seen that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} d\left(x_{n}, x_{n+1}\right)=0 . \tag{3.6}
\end{equation*}
$$

Following a similar procedure to the previous one but making use of Lemma 2.8, $d\left(x_{2}, x_{1}\right)$ is given by

$$
d\left(x_{2}, x_{1}\right) \leq H\left(T x_{1}, T x_{0}\right)+\frac{1}{p^{2}} d\left(x_{1}, x_{0}\right) .
$$

Using again (3.5) yields

$$
\begin{aligned}
& d\left(x_{2}, x_{1}\right) \leq \frac{d\left(x_{0}, T x_{0}\right) d\left(T x_{1}, x_{1}\right)}{1+p\left(d\left(x_{0}, T x_{0}\right)+d\left(T x_{1}, x_{1}\right)\right)}+\frac{1}{p^{2}} d\left(x_{1}, x_{0}\right) \\
& d\left(x_{2}, x_{1}\right) \leq \frac{1}{p} d\left(T x_{1}, x_{1}\right)+\frac{1}{p^{2}} d\left(x_{1}, x_{0}\right) .
\end{aligned}
$$

This equation can be written as

$$
d\left(x_{2}, x_{1}\right) \leq \frac{1}{p(p-1)} d\left(x_{1}, x_{0}\right) .
$$

Moreover, based on Lemma 2.8, there exists $x_{3} \in T x_{2}$ such that

$$
d\left(x_{3}, x_{2}\right) \leq H\left(T x_{2}, T x_{1}\right)+\frac{1}{p^{2}} d\left(x_{2}, x_{1}\right)
$$

which gives

$$
\begin{aligned}
& d\left(x_{3}, x_{2}\right) \leq \frac{d\left(x_{1}, T x_{1}\right) d\left(T x_{2}, x_{2}\right)}{1+p\left(d\left(x_{1}, T x_{1}\right)+d\left(T x_{1}, x_{1}\right)\right)}+\frac{1}{p^{2}} d\left(x_{2}, x_{1}\right) \\
& d\left(x_{3}, x_{2}\right) \leq \frac{1}{p} d\left(x_{3}, x_{2}\right)+\frac{1}{p^{2}} d\left(x_{1}, x_{2}\right) .
\end{aligned}
$$

Hence

$$
d\left(x_{3}, x_{2}\right) \leq \frac{1}{p(p-1)} d\left(x_{2}, x_{1}\right) \leq\left(\frac{1}{p(p-1)}\right)^{2} d\left(x_{1}, x_{2}\right) .
$$

Continuing this process, we obtain a sequence $\left(x_{n}\right)$, where $x_{n+1} \in T x_{n}$ such that

$$
d\left(x_{n+1}, x_{n}\right) \leq\left(\frac{1}{p(p-1)}\right)^{n} d\left(x_{1}, x_{0}\right) .
$$

Since $p>2$, we have $p(p-1)>1$ and hence $0<\frac{1}{p(p-1)}<1$.
It is clearly shown that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} d\left(x_{n+1}, x_{n}\right)=0 \tag{3.7}
\end{equation*}
$$

which gives

$$
\lim _{n \rightarrow \infty} d\left(x_{n}, x_{n+1}\right)=\lim _{n \rightarrow \infty} d\left(x_{n+1}, x_{n}\right)=0
$$

Based on Theorem 3.1, we obtain $\left\{x_{n}\right\}$ which is a Cauchy sequence in complete $X$.

Therefore, we conclude that there exists $x^{*} \in X$ such that $\lim _{n \rightarrow \infty} x_{n}=x^{*}$.

Furthermore, it will be shown that $x^{*}$ is a fixed point of $T$, that is,

$$
\begin{aligned}
d\left(x^{*}, T x^{*}\right) \leq & \alpha d\left(T x^{*}, x^{*}\right)+\frac{b}{2}\left(d\left(x^{*}, x_{n}\right)+d\left(x_{n}, T x^{*}\right)\right) \\
\leq & \alpha\left(\alpha d\left(x^{*}, T x^{*}\right)+\frac{b}{2}\left(d\left(T x^{*}, x_{n}\right)+d\left(x_{n}, x^{*}\right)\right)\right) \\
& +\frac{b}{2}\left(d\left(x^{*}, x_{n}\right)+d\left(x_{n}, T x^{*}\right)\right)
\end{aligned}
$$

which gives

$$
d\left(x^{*}, T x^{*}\right) \leq \frac{\frac{a b}{2}\left(d\left(T x^{*}, x_{n}\right)+d\left(x_{n}, x^{*}\right)\right)+\frac{b}{2}\left(d\left(x^{*}, x_{n}\right)+d\left(x_{n}, T x^{*}\right)\right)}{1-\alpha^{2}} .
$$

Since $x_{n} \in T x_{n-1}$, we get the following equations:

$$
d\left(T x^{*}, x_{n}\right) \leq H\left(T x^{*}, T x_{n-1}\right) \leq \frac{d\left(x_{n-1}, T x_{n-1}\right) d\left(T x^{*}, x^{*}\right)}{1+p\left(d\left(x_{n-1}, T x_{n-1}\right)+d\left(T x^{*}, x^{*}\right)\right)} \leq \frac{d\left(x_{n-1}, T x_{n-1}\right)}{p}
$$

and

$$
d\left(x_{n}, T x^{*}\right) \leq H\left(T x_{n-1}, T x^{*}\right) \leq \frac{d\left(T x_{n-1}, x_{n-1}\right) d\left(x^{*}, T x^{*}\right)}{1+p\left(d\left(T x_{n-1}, x_{n-1}\right)+d\left(x^{*}, T x^{*}\right)\right)} \leq \frac{d\left(T x_{n-1}, x n_{-1}\right)}{p} .
$$

Substituting these equations into the previous one, we then get

$$
\begin{aligned}
d\left(x^{*}, T x^{*}\right) & \leq \frac{\frac{\alpha b}{2}\left(H\left(T x^{*}, T x_{n-1}\right)+d\left(x_{n}, x^{*}\right)\right)+\frac{b}{2}\left(d\left(x^{*}, x_{n}\right)+H\left(T x_{n-1}, T x^{*}\right)\right.}{1-\alpha^{2}} \\
& \leq \frac{\frac{\alpha b}{2}\left(\frac{d\left(x_{n-1}, T x_{n-1}\right)}{p}+d\left(x_{n}, x^{*}\right)\right)+\frac{b}{2}\left(d\left(x^{*}, x_{n}\right)+\frac{d\left(T x_{n-1}, x_{n-1}\right)}{p}\right)}{1-\alpha^{2}} .
\end{aligned}
$$

Moreover, since $x_{n} \in T x_{n-1}$, we have

$$
d\left(x_{n-1}, T x_{n-1}\right) \leq d\left(x_{n-1}, x_{n}\right) \quad \text { and } \quad d\left(T x_{n-1}, x_{n-1}\right) \leq d\left(x_{n}, x_{n-1}\right) .
$$

Finally, $d\left(x^{*}, T x^{*}\right)$ can be written as

$$
d\left(x^{*}, T x^{*}\right) \leq \frac{\frac{\alpha b}{2}\left(\frac{d\left(x_{n-1}, x_{n}\right)}{p}+d\left(x_{n}, x^{*}\right)\right)+\frac{b}{2}\left(d\left(x^{*}, x_{n}\right)+\frac{d\left(x_{n}, x_{n-1}\right)}{p}\right)}{1-\alpha^{2}}
$$

In addition, since $\lim _{n \rightarrow \infty} x_{n}=x^{*}$, we obtain $\lim _{n \rightarrow \infty} d\left(x_{n}, x^{*}\right)=$
$\lim _{n \rightarrow \infty} d\left(x^{*}, x_{n}\right)=0$. From (3.6) and (3.7) and $n \rightarrow \infty$, we get

$$
d\left(x^{*}, T x^{*}\right)=0
$$

and also

$$
d\left(T x^{*}, x^{*}\right)=0
$$

which is only possible if $x^{*} \in T x^{*}$.
Now we have to show that for every $y \in X$ and $y \in T x^{*}, y=x^{*}$.
Suppose that there exists $y \in X$ such that $y \in T x^{*}$.
Since $x^{*} \in T x^{*}$ and $y \in T x^{*}$, using Lemma 2.7 and taking $\varepsilon=d\left(y, x^{*}\right)$, there exists $z \in T y$ such that

$$
d\left(x^{*}, z\right) \leq H\left(T x^{*}, T y\right)+d\left(y, x^{*}\right) \leq \frac{d(y, T y) d\left(T x^{*}, x^{*}\right)}{1+p\left(d(y, T y)+d\left(T x^{*}, x^{*}\right)\right)}+d\left(y, x^{*}\right)
$$

and

$$
d(z, y) \leq H\left(T y, T x^{*}\right)+d\left(y, x^{*}\right) \leq \frac{d\left(x^{*}, T x^{*}\right) d(T t y, y)}{1+p\left(d\left(x^{*}, T x^{*}\right)+d(T y, y)\right)}+d\left(y, x^{*}\right) .
$$

Since $x^{*} \in T x^{*}$, we have $d\left(T x^{*}, x^{*}\right)=d\left(x^{*}, T x^{*}\right)=0$ which implies that

$$
d\left(x^{*}, z\right) \leq d\left(y, x^{*}\right)
$$

and

$$
d(z, y) \leq d\left(y, x^{*}\right) .
$$

Now, by using (2.1), we have

$$
d\left(x^{*}, y\right) \leq \alpha d\left(y, x^{*}\right)+\frac{1}{2} b\left(d\left(x^{*}, z\right)+d(z, y)\right)
$$

which can be written as:

$$
d\left(x^{*}, y\right) \leq \alpha d\left(y, x^{*}\right)+\frac{1}{2} b\left(d\left(y, x^{*}\right)+d\left(y, x^{*}\right)\right)=\alpha d\left(y, x^{*}\right)+b d\left(y, x^{*}\right)
$$

yielding

$$
\begin{equation*}
d\left(x^{*}, y\right) \leq(\alpha+b) d\left(y, x^{*}\right) . \tag{3.8}
\end{equation*}
$$

Now from $x^{*} \in T x^{*}$ and $y \in T x^{*}$, by using Lemma 2.8 and taking $\varepsilon=$ $d\left(x^{*}, y\right)$, there exists $w \in T y$ such that
$d\left(w, x^{*}\right) \leq H\left(T y, T x^{*}\right)+d\left(x^{*}, y\right) \leq \frac{d\left(x^{*}, T x^{*}\right) d(T y, y)}{1+p\left(d\left(x^{*}, T x^{*}\right)+d(T y, y)\right)}+d\left(x^{*}, y\right)$
and

$$
d(y, w) \leq H\left(T x^{*}, T y\right)+d\left(x^{*}, y\right) \leq \frac{d(T y, y) d\left(T x^{*}, x^{*}\right)}{1+p\left(d(T y, y)+d\left(T x^{*}, x^{*}\right)\right)}+d\left(x^{*}, y\right)
$$

Since $x^{*} \in T x^{*}$, we have $d\left(T x^{*}, y\right)=d\left(x^{*}, T x^{*}\right)=0$ which gives

$$
d\left(w, x^{*}\right) \leq d\left(x^{*}, y\right)
$$

and

$$
d(y, w) \leq d\left(x^{*}, y\right) .
$$

Now using (2.1), thus

$$
\begin{aligned}
d\left(y, x^{*}\right) & \leq \alpha d\left(x^{*}, y\right)+\frac{1}{2} b\left(d(y, w)+d\left(w, x^{*}\right)\right) \\
& \leq \alpha d\left(x^{*}, y\right)+\frac{1}{2} b\left(d\left(x^{*}, y\right)+d\left(x^{*}, y\right)\right)
\end{aligned}
$$

yields

$$
\begin{equation*}
d\left(y, x^{*}\right) \leq(\alpha+b) d\left(x^{*}, y\right) . \tag{3.9}
\end{equation*}
$$

From (3.8) and (3.9), we get

$$
d\left(x^{*}, y\right)=d\left(y, x^{*}\right)=0
$$

which implies that $y=x^{*}$.

Since $x^{*} \in T x^{*}$ and $y$ is arbitrary element in $X$ and $y \in T x^{*}$, we obtain $T x^{*}=\left\{x^{*}\right\}$ which shows that $T$ has a unique fixed point in $X$.

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## References

[1] I. A. Bakhtin, The contraction mapping principle in almost metric space, functional analysis, Ul’yanovsk. Gos. Ped. Inst., Ul’yanovsk 30 (1989), 26-37 (in Russian).
[2] S. Czerwik, Contraction mappings in b-metric spaces, Acta Math. Inform. Univ. Ostraviensis 1 (1995), 5-11.
[3] S. B. Nadler, Multivalued contraction mappings, Pacific J. Math. 30 (1969), 475-488.
[4] J. M. Joseph and E. Ramganesh, Fixed point theorem on multivalued mappings, Int. J. Anal. Appl. 1 (2013), 127-132.
[5] J. M. Joseph, D. D. Roselin and M. Marudai, Fixed Point Theorems on Multivalued Mappings in $b$-metric Spaces, Springer Plus 5 (2016), 217.
[6] C. Chifu and G. Petrusel, Fixed points for multivalued contractions in $b$-metric spaces with applications to fractals, Taiwanese J. Math. 18 (2014), 1365-1375.
[7] R. Miculescu and A. Mihail, New fixed point theorems for set-valued contractions in $b$-metric spaces, J. Fixed Point Appl. 19(3) (2017), 2153-2163.
[8] H. Aydi, M. F. Bota, E. Karapinar and S. Mitrovi'c, A fixed point theorem for set-valued quasi-contractions in $b$-metric spaces, Fixed Point Theory Appl. 2012 (2012), 88.
[9] M. Boriceanu, Fixed point theory for multivalued generalized contraction on a set with two $b$-metrics, Stud. Univ. Babes-Bolyai Math. LIV(3) (2009), 1-14.
[10] B. Nurwahyu, Fixed point theorems for generalized contraction mappings in quasi $a b$-metric space, Far East J. Math. Sci. (FJMS) 101(8) (2017), 1813-1832.

