Far East Journal of Mathematical Sciences (FJMS)

http://www.pphmj.com
http://dx.doi.org/10.17654/MS102092025
Volume 102, Number 9, 2017, Pages 2025-2052 **ISSN: 0972-0871**

ON EMBEDDING AND EXTENDED SMOOTHNESS OF SPLINE SPACES

Yu. K. Dem'yanovich

Saint-Petersburg State University
Saint-Petersburg
Russia

Abstract

This paper introduces the notion of extended smoothness (which includes usual smoothness), constructs extended smooth splines, defines the necessary and sufficient conditions for uniqueness and the embedding of the spline spaces pointed out. Extended smoothness, introduced here, also considers irregular splines. As an application of the mentioned results, the necessary and sufficient conditions for the embedding of spline spaces are obtained under given condition of maximum smoothness (in the usual sense) for minimal B_φ-splines of mth order. Here two sorts of spline spaces of the first order with irregular generating functions are also analyzed.

1. Introduction

Smoothness of splines is important for the recovery of differentiable functions, for the smoothing of discrete data and so on (see [1, 2, 4, 8, 9, 14, 17]). It is well known (see, for example, [2]) that the maximum smoothness of polynomial splines of degree m (of which support of their basic splines

Received: May 31, 2017; Accepted: July 19, 2017

2010 Mathematics Subject Classification: 65D07.

Keywords and phrases: non-polynomial splines, extended smoothness, embedding, calibration relations, enlargement spline grid.

consists of $m+1$ grid interval) is $m-1$. Maximum smoothness of non-polynomial splines has only been discussed in a few papers (for instance, see [8, 9]).

Another question is the embedding of the function spaces. It is very important for construction of wavelet decomposition (see [3, 5-7, 10-17]). There are well known investigations of embedding for spaces connected with an equidistant grid (see [3, 5-7]). In the case of irregular grid the embedding of polynomial splines was investigated earlier.

There are many difficulties for investigations of the embedding of non-polynomial spline spaces (see [8, 9]). The conditions of embedding have been investigated in some situations (see [14]). Specifically, the embedding of spaces of smooth polynomial splines and the corresponding wavelet decompositions on infinite embedded grids were studied in many works (cf., for example, [10-15] and the references therein). It is important to investigate the embedding of spline spaces in the case of embedding of corresponding spline grids. Such grids can be enlarged by removing grid points one at a time (see [16-19]). Such considerations are based on approximate relations, owing to which it is possible to obtain wavelet decompositions of spline spaces with different level of smoothness and such that their approximate properties are asymptotically optimal with respect to the *N*-width of standard compact sets. The original numerical flow is regarded as a sequence of coefficients of decomposition with respect to the coordinate splines in the space constructed on the original (fine) grid (see [19-21]). This space is projected onto the embedded spline space (on an enlarged grid). As a result, we get a grid obtained by splitting the original numerical information flow into basic flow (formed by coefficients of decomposition relative to the coordinate splines of the embedded space) and wavelet numerical flow, which can be used to restore original numerical flow.

For singular numerical flow (i.e. numerical flow with quick variability), it is important to have irregular grid, non-polynomial approximation and non-smooth spaces.

The aim of this paper is to formulate the notion of extended smoothness (which includes the usual smoothness), to construct extended smooth splines, to define the necessary and sufficient conditions for uniqueness and embedding of the spline spaces pointed out. Extended smoothness, introduced here, let us consider irregular splines. As an application of the mentioned results we obtain necessary and sufficient conditions for embedding of spline spaces, and give such conditions of maximum smoothness (in the usual sense) for minimal B_φ-splines of mth order. We also analyze two sorts of spline spaces of the first order with irregular generating functions.

2. Spaces $(X, \mathbf{A}, \varphi)$-spines of the Order m

We will now discuss sequence **A** of vector columns $\mathbf{a}_i \in \mathbb{R}^{m+1}$; sequence **A** of the vectors is called *a chain* (of the vectors $\mathbf{a}_i$, $i \in \mathbb{Z}$).

There are many enumerations of the vectors in chain **A**, they are distinguished by a constant integer number and by the direction of numbering; for example, if j_0 is a constant integer, then $\mathbf{A} \overset{\text{def}}{=} \{\mathbf{a}_{j'}\}_{j' \in \mathbb{Z}}$ for $j' = -j + j_0$ is another numeration of the same chain.

Chain $\mathbf{A} \overset{\text{def}}{=} \{\mathbf{a}_i\}_{i \in \mathbb{Z}}$ is known as *locally orthogonal* to chain $\mathbf{B} \overset{\text{def}}{=} \{\mathbf{b}_j\}_{j \in \mathbb{Z}}$, if such numeration exists, for which

$$\mathbf{b}_j^T \mathbf{a}_{j-p} = 0, \quad \forall j \in \mathbb{Z}, \quad \forall p \in I_m; \tag{2.1}$$

here $I_m \overset{\text{def}}{=} \{1, 2, ..., m\}$.

Lemma 1. *If chain* **A** *is locally orthogonal to chain* **B**, *then chain* **B** *is locally orthogonal to chain* **A**.

The proof is evident.

Thus, local orthogonality is symmetrical. Therefore in the discussed situations, chains **A** and **B** can be called *locally orthogonal* (to each other).

The chain is called a *singular chain* if there is a zero vector among vectors of the chain; the other chains are called *nonsingular chains*.

Let $A_j \overset{\text{def}}{=} (\mathbf{a}_{j-m}, \mathbf{a}_{j-m+1}, ..., \mathbf{a}_{j-1}, \mathbf{a}_j)$ be a square matrix.

Chain **A** with property $\det A_j \neq 0, \ \forall j \in \mathbb{Z}$ is called a *complete chain*.

It is clear that the complete chain is nonsingular.

Let $\mathcal{A}$ be the class $\{\mathbf{A} \mid \det A_j \neq 0, \ \forall j \in \mathbb{Z}\}$.

Lemma 2. *If chains* $\mathbf{A} \overset{\text{def}}{=} \{\mathbf{a}_i\}_{i \in \mathbb{Z}}$ *and* $\mathbf{B} \overset{\text{def}}{=} \{\mathbf{b}_j\}_{j \in \mathbb{Z}}$ *are locally orthogonal and nonsingular, then chain* **A** *is complete if and only if chain* **B** *is complete*.

Proof. Suppose that chain **B** is complete. We prove that **A** is complete with proof by contradiction. Thus if **A** is not complete, then there exists j such that the representation $\mathbf{a}_j = c_1 \mathbf{a}_{j-1} + c_2 \mathbf{a}_{j-2} + \cdots + c_m \mathbf{a}_{j-m}$ is true; here $c_1, c_2, ..., c_m \in \mathbb{R}^1$. Taking into account (2.1), we obtain

$$\mathbf{b}_j^T \mathbf{a}_j = 0. \tag{2.2}$$

Using the condition of local orthogonality (2.1) once again, we have

$$\mathbf{b}_{j+p}^T \mathbf{a}_j = 0, \quad p \in I_m. \tag{2.3}$$

By relations (2.2) and (2.3), we get $\mathbf{a}_j = 0;$ the last one contradicts completeness of chain **A**. □

Lemma 3. *If chains* $\mathbf{A} \overset{\text{def}}{=} \{\mathbf{a}_i\}_{i \in \mathbb{Z}}$ *and* $\mathbf{B} \overset{\text{def}}{=} \{\mathbf{b}_j\}_{j \in \mathbb{Z}}$ *are complete and the relation* (2.1) *is true, then*

$$\mathbf{b}_j^T \mathbf{a}_j \neq 0, \quad \mathbf{b}_{j+m+1}^T \mathbf{a}_j \neq 0. \tag{2.4}$$

Proof. The first relation in (2.4) can be proved by contradiction: if the number j exists, by which (2.2) is fulfilled, then taking into account the formulae (2.3), we obtain $\mathbf{a}_j = 0;$ the last one contradicts the completeness of chain **A**. The proof of the second assertion can also be obtained by contradiction. □

Lemma 4. *For arbitrary complete chain* **A** *there exists the nonsingular* *chain* **B**, *which is locally orthogonal to* **A**; *the directions of vectors for the last one are defined uniquely*.

Proof. Let $\mathbf{A} = \{\mathbf{a}_j\}_{j\in\mathbb{Z}}$ be a complete chain. Fixing the integer k, we find vector $\mathbf{b}_k$ satisfying to conditions

$$\mathbf{b}_k^T\mathbf{a}_{k-p} = 0, \quad p \in I_m. \tag{2.5}$$

The conditions (2.5) can be represented as a linear system as to unknown vector $\mathbf{b}_k$:

$$(\mathbf{a}_{k-m}, \mathbf{a}_{k-m+1}, ..., \mathbf{a}_{k-1})^T \mathbf{b}_k = 0. \tag{2.6}$$

Using the completeness of chain **A**, we see that vectors $\mathbf{a}_{k-m}, \mathbf{a}_{k-m+1}, ..., \mathbf{a}_{k-1}$ are linear independent. Consequently system (2.6) has the unique (up to constant multiplier) solution $\mathbf{b}_k$, which can be defined, for example, by the identity

$$\mathbf{b}_k^T\mathbf{x} \equiv \det(\mathbf{a}_{k-m}, \mathbf{a}_{k-m+1}, ..., \mathbf{a}_{k-1}, \mathbf{x}).$$

That concludes the proof. □

Corollary 1. *For the arbitrary complete chain* **A**, *there exists chain* **B**, *which is locally orthogonal to* **A**; *the vectors of chain* **B** *are defined up to constant multipliers*.

Proof. The assertion follows from Lemmas 2 and 4. □

We discuss grid $X \stackrel{\text{def}}{=} \{x_j\}_{j\in\mathbb{Z}}$,

$$X : \ldots < x_{-1} < x_0 < x_1 < \ldots; \quad \alpha \stackrel{\text{def}}{=} \lim_{j\to-\infty} x_j, \quad \beta \stackrel{\text{def}}{=} \lim_{j\to+\infty} x_j$$

in the interval (α, β) with finite values α and β.

Let U_j be a linear space of functions defined on a real interval (x_j, x_{j+1}), $j \in \mathbb{Z}$, let U be the direct production of the spaces U_j,

$$U \stackrel{\text{def}}{=} \bigotimes_{j\in\mathbb{Z}} U_j.$$

By definition, put

$$M \overset{\text{def}}{=} \bigcup_{j\in\mathbb{Z}} (x_j,\ x_{j+1}), \quad S_j \overset{\text{def}}{=} [x_j,\ x_{j+m+1}], \quad J_k \overset{\text{def}}{=} \{k-m, \ldots, k\}.$$

Let $\varphi(t)$ be $m+1$-component vector function (column vector) with components $\varphi_i = \varphi_i(t)$ belonging to U, $i \in \{0, 1, 2, \ldots, m\}$.

If $\mathbf{A} \in \mathcal{A}$, then the approximation relations

$$\sum_{j'\in\mathbb{Z}} \mathbf{a}_{j'}\omega_{j'}(t) \equiv \varphi(t), \quad \forall t \in M, \quad \omega_j(t) = 0 \text{ for } t \notin S_j, \quad \forall j \in \mathbb{Z}, \tag{2.7}$$

uniquely define the functions $\omega_j(t)$, $t \in M$, $j \in \mathbb{Z}$.

If t is a fixed number in an interval $(x_k,\ x_{k+1})$, then relations (2.7) contain no more than $m+1$ nonzero summands:

$$\sum_{j=k-m}^{k} \mathbf{a}_j\omega_j(t) \equiv \varphi(t); \tag{2.8}$$

under the aforementioned t the relations (2.8) are discussed as a system of linear algebraic equations as to unknown $\omega_j(t)$.

Using Cramer's rule and arbitrary fixing $j \in \mathbb{Z}$ in relations (2.8), we successively discuss $k = j,\ j+1,\ j+2, \ldots,\ j+m$; as a result we have

$$\omega_j(t) = \frac{\det(\{\mathbf{a}_{j'}\}_{j'\in J_k,\ j'\neq j} \,\|'^{\,j}\, \varphi(t))}{\det(\{\mathbf{a}_{j'}\}_{j'\in J_k})} \text{ for } t \in (x_k,\ x_{k+1}), \quad k-j \in J_m; \tag{2.9}$$

here the columns of determinants for numerator and denominator are uniformly ordered, and a symbol $\|'^{\,j}\,\varphi(t)$ is denoted that it is necessary to put the column vector $\varphi(t)$ instead of column vector $\mathbf{a}_j$.

Consider a linear space

$$\mathbb{S}_m(X,\ \mathbf{A},\ \varphi) \overset{\text{def}}{=} Cl_p\mathcal{L}\{\omega_j\}_{j\in\mathbb{Z}},$$

where $\mathcal{L}$ denotes the linear hull, and Cl_p is closer in point-wise topology.

The space $\mathbb{S}_m(X, \mathbf{A}, \varphi)$ is called the *space of minimal* $(X, \mathbf{A}, \varphi)$*-splines* of the order m.

Suppose that a set of point-wise functionals in the point ξ is not empty for arbitrary $\xi \in M$; this set is denoted by U_ξ^*.

For each $t \in M$ we discuss a linear functional $f(t)$, which is point-wise one in the point t, that is $f(t) \in U_t^*$.

Union of functionals $\{f(t)\}_{t\in M}$ is a trajectory[1] in the space U^*.

Let u be an element of the space U; if $f(t)u$ is a continuous function on the intervals (x_j, x_{j+1}) and $f(x_j - 0)u = f(x_j + 0)u$, $\forall j \in \mathbb{Z}$, then we write $f(\cdot)u \in C(\alpha, \beta)$.

By definition, put $f(t)\varphi = (f(t)\varphi_0, f(t)\varphi_1, \ldots, f(t)\varphi_m)^T$.

3. Limit Properties of Minimal Splines

Consider a trajectory of point-wise functionals $f(t)$ in the space U^*: $f(t) \in U^*$, $t \in M$, with property $f(\cdot)\varphi_i \in C(\alpha, \beta)$, $\forall i \in \{0, 1, \ldots, m\}$.

Discuss a possibility for continuous prolongation of the functions $f(t)\omega_j$ (for $j \in \mathbb{Z}$, $t \in (x_k, x_{k+1})$, $j, k \in \mathbb{Z}$) on the interval (α, β).

Lemma 5. *Let* $\mathbf{A}$ *be a complete chain; let a number* $k \in \mathbb{Z}$ *and* $t_* \in [x_k, x_{k+1}]$ *be fixed. The relation*

$$\lim_{t\to t_*,\, t\in(x_k, x_{k+1})} f(t)\omega_j = 0$$

is true if and only if the equality

---

[1]If U is the linear space $C^s(M)$, then we can discuss linear point-wise functional $f(t)u \stackrel{\text{def}}{=} u^{(s)}(t)$, $t \in M$, so that we have the trajectory in the space $(C^s(M))^*$.

$$\det(\{\mathbf{a}_{j'}\}_{j'\in J_k,\, j'\neq j} \|'^{j} f(t_*)\varphi) = 0$$

is fulfilled.

Proof. Follows from the formula (2.9). □

Lemma 6. *Let* **A** *be a complete chain. If equalities*

$$\lim_{t\to x_k-0} f(t)\omega_{k-m-1} = 0, \quad \lim_{t\to x_k+0} f(t)\omega_k = 0 \tag{3.1}$$

are right, then the relations

$$\lim_{t\to x_k-0} f(t)\omega_j = \lim_{t\to x_k+0} f(t)\omega_j \ \textit{for}\ j \in \{k-m, k-m+1, ..., k-1\} \tag{3.2}$$

must be fulfilled. If in addition $\mathbf{a}_{k-m-1}$ *and* $\mathbf{a}_k$ *are not collinear, then relations* (3.1) *and* (3.2) *are equivalent.*

Proof. Changing k with $k-1$ in the relation (2.8), we have

$$\sum_{j=k-m-1}^{k-1} \mathbf{a}_j\omega_j(t) \equiv \varphi(t), \quad \forall t \in (x_{k-1}, x_k); \tag{3.3}$$

hence under condition $t \to x_k - 0$ and by the first supposition (3.1), we obtain

$$\sum_{j=k-m}^{k-1} \mathbf{a}_j f(x_k - 0)\omega_j = f(x_k)\varphi. \tag{3.4}$$

Analogously by (2.8) and by the second supposition (3.1) for $t \to x_k + 0$ we have

$$\sum_{j=k-m}^{k-1} \mathbf{a}_j f(x_k + 0)\omega_j = f(x_k)\varphi. \tag{3.5}$$

Because the vectors $\mathbf{a}_{k-m}, \mathbf{a}_{k-m+1}, ..., \mathbf{a}_{k-1}$ are linear independent, then the relations (3.2) follow from (3.4)-(3.5). Thus, the necessity has been proved.

Now taking into account (3.3), we obtain

$$\sum_{j=k-m-1}^{k-1} \mathbf{a}_j f(x_k - 0)\omega_j = f(x_k)\varphi, \tag{3.6}$$

and from (2.8), we have

$$\sum_{j=k-m}^{k} \mathbf{a}_j f(x_k + 0)\omega_j = f(x_k)\varphi. \tag{3.7}$$

Subtracting (3.7) from (3.6) and using relations (3.2), we deduce an equality $\mathbf{a}_{k-m-1} f(x_k - 0)\omega_{k-m-1} = \mathbf{a}_k f(x_k + 0)\omega_k$; taking into account the linear independence of vectors $\mathbf{a}_{k-m-1}$ and $\mathbf{a}_k$, we get the equalities (3.1). □

Theorem 1. *If* **A** *is a complete chain, then for the functions* $f(t)\omega_j$ $(\forall j \in \mathbb{Z})$ *to be continuous on the interval* (α, β) *it is necessary and sufficient to have the zero limit of each of them on the boundary of their support.*

Proof. Sufficiency. Taking into account the continuity of vector function $f(t)\varphi$, we see that it is sufficient to investigate the continuity of the functions $f(t)\omega_j$ in the knots of grid X. If knot x_k belongs to the boundary of set S_j, then the continuity of $f(t)\omega_j$ in the knot x_k follows from the conditions of the discussed theorem. If the knot x_k belongs to the interior of S_j, then the conditions of Lemma 6 are fulfilled, and therefore the relations (3.2) are true. This completes the proof of sufficiency.

The necessity is obvious. □

By definition, put $f_j\varphi \overset{\text{def}}{=} f(x_j)\varphi$.

Theorem 2. *A necessary and sufficient condition for the functions* $f(t)\omega_j$ $(\forall j \in \mathbb{Z})$ *to be continuously prolonged on the interval* (α, β) *is that the relations*

$$\det(\mathbf{a}_{j-m}, \mathbf{a}_{j-m+1}, \ldots, \mathbf{a}_{j-1}, f_j\varphi) = 0, \quad \forall j \in \mathbb{Z} \tag{3.8}$$

are true.

Proof. First of all we show that the equalities (3.1) for $k = j$, $\forall j \in \mathbb{Z}$ are equivalent to the condition (3.8).

Really, by (2.9) the equalities (3.8) are equivalent to the relations

$$f(x_j + 0)\omega_j = 0, \quad \forall j \in \mathbb{Z}; \tag{3.9}$$

changing the index j in (3.8) with $j + m + 1$, we obtain

$$\det(\mathbf{a}_{j+1}, \mathbf{a}_{j+2}, \ldots, \mathbf{a}_{j+m}, f_{j+m+1}\varphi) = 0, \quad \forall j \in \mathbb{Z}. \tag{3.10}$$

Relations (2.9) and (3.10) are equivalent to

$$f(x_{j+m+1} - 0)\omega_j = 0, \quad \forall j \in \mathbb{Z}. \tag{3.11}$$

Thus the equalities (3.8) are equivalent to the relations (3.9) and (3.11). Usage of Theorem 1 completes the proof. □

Theorem 3. *Suppose* $\varphi^{(s)} \in C(\alpha, \beta)$, *where s is a positive integer. A necessary and sufficient condition for the functions* $\omega_j^{(s)}(t)$ $(\forall j \in \mathbb{Z}, t \in M)$ *to be prolonged to continuous functions on the interval* (α, β) *is that the relations*

$$\det(\mathbf{a}_{j-m}, \mathbf{a}_{j-m+1}, \ldots, \mathbf{a}_{j-1}, \varphi^{(s)}(x_j)) = 0, \quad \forall j \in \mathbb{Z}$$

be fulfilled.

Proof. By definition, put

$$U_j \overset{\text{def}}{=} C^{(s)}(x_j, x_{j+1}), \quad f(t)u \overset{\text{def}}{=} u^{(s)}(t), \quad f_j\varphi \overset{\text{def}}{=} \varphi^{(s)}(x_j). \tag{3.12}$$

Now it is sufficient to apply Theorem 2. □

4. On $B_{\varphi, F}$-splines

Discuss a case of relations (2.7) selecting the vector chain $\mathbf{a}_j$ in a special way.

Suppose there is a set $F \overset{\text{def}}{=} \{f^{\langle i \rangle}(t) \mid \forall t \in (\alpha, \beta), i = 0, 1, 2, \ldots, m\}$ of linear point-wise functionals $f^{\langle i \rangle}(t)$, such that functions $f^{\langle i \rangle}(\cdot)\varphi_j \in C(\alpha, \beta)$, $\forall i,\ j \in \{0, 1, 2, ..., m\}$, and

$$|\det(f^{\langle 0 \rangle}(t)\varphi,\ f^{\langle 1 \rangle}(t)\varphi, ...,\ f^{\langle m \rangle}(t)\varphi)| \geq c > 0, \quad \forall t \in (\alpha, \beta). \tag{4.1}$$

By definition, put $f_j^{\langle s \rangle}\varphi \overset{\text{def}}{\equiv} f^{\langle s \rangle}(x_j)\varphi,\ s = 0, 1, 2, ..., m-1.$ We define column vectors $\mathbf{d}_i$ by identity

$$\mathbf{d}_i^T \mathbf{x} \equiv \det(f_i^{\langle 0 \rangle}\varphi,\ f_i^{\langle 1 \rangle}\varphi,\ f_i^{\langle 2 \rangle}\varphi, ...,\ f_i^{\langle m-1 \rangle}\varphi,\ \mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^{m+1}, \quad \forall i \in \mathbb{Z}. \tag{4.2}$$

Taking into account the property (4.1), we can conclude that chain $\{\mathbf{d}_i\}_{i \in \mathbb{Z}}$ is nonsingular.

Let the vectors $\mathbf{a}_j^\star$ be defined by a symbolic determinant

$$\mathbf{a}_j^\star \overset{\text{def}}{\equiv} \det\begin{pmatrix} f_{j+1}^{\langle 0 \rangle}\varphi & f_{j+1}^{\langle 1 \rangle}\varphi & \cdots & f_{j+1}^{\langle m-1 \rangle}\varphi \\ \mathbf{d}_{j+2}^T f_{j+1}^{\langle 0 \rangle}\varphi & \mathbf{d}_{j+2}^T f_{j+1}^{\langle 1 \rangle}\varphi & \cdots & \mathbf{d}_{j+2}^T f_{j+1}^{\langle m-1 \rangle}\varphi \\ \mathbf{d}_{j+3}^T f_{j+1}^{\langle 0 \rangle}\varphi & \mathbf{d}_{j+3}^T f_{j+1}^{\langle 1 \rangle}\varphi & \cdots & \mathbf{d}_{j+3}^T f_{j+1}^{\langle m-1 \rangle}\varphi \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{d}_{j+m}^T f_{j+1}^{\langle 0 \rangle}\varphi & \mathbf{d}_{j+m}^T f_{j+1}^{\langle 1 \rangle}\varphi & \cdots & \mathbf{d}_{j+m}^T f_{j+1}^{\langle m-1 \rangle}\varphi \end{pmatrix}; \tag{4.3}$$

the symbolic determinant should be expanded with the first row.

Lemma 7. *The vector* $\mathbf{a}_j^\star$ *is defined by values* $f_i^{\langle 0 \rangle}\varphi,\ f_i^{\langle 1 \rangle}\varphi, ...,\ f_i^{\langle m-1 \rangle}\varphi,$ *where* $i = j+1,\ j+2, ...,\ j+m.$

Proof. It is clear to see that the vectors $\mathbf{d}_i$ for $j+2 \leq i \leq j+m$ take part in the relations (4.3). According to definition (4.2), each vector $\mathbf{d}_i$ is defined by the vectors $f_i^{\langle 0 \rangle}\varphi,\ f_i^{\langle 1 \rangle}\varphi,\ f_i^{\langle 2 \rangle}\varphi, ...,\ f_i^{\langle m-1 \rangle}\varphi.$ To complete the proof, it is sufficient to use the representation (4.3). This concludes the proof. $\square$

By definition, put $\mathbf{A}^{\star} = \mathbf{A}^{\star}(X, \varphi, F) \stackrel{\text{def}}{=} \{\mathbf{a}_j^{\star}\}_{j\in\mathbb{Z}}$.

Theorem 4. *The chains of vectors* $\{\mathbf{d}_j^T\}_{j\in\mathbb{Z}}$ *and* $\{\mathbf{a}_i^{\star}\}_{i\in\mathbb{Z}}$ *are locally orthogonal*:

$$\mathbf{d}_{j+p}^T\mathbf{a}_j^{\star} = 0, \quad \forall j \in \mathbb{Z}, \quad p \in I_m. \tag{4.4}$$

Proof. Consider relation $\mathbf{d}_{j+p}^T\mathbf{a}_j^{\star}$; according to (4.3) it can be represented in the form

$$\mathbf{d}_{j+p}^T\mathbf{a}_j^{\star} \stackrel{\text{def}}{=} \det\begin{pmatrix} \mathbf{d}_{j+p}^T f_{j+1}^{\langle 0\rangle}\varphi & \mathbf{d}_{j+p}^T f_{j+1}^{\langle 1\rangle}\varphi & \cdots & \mathbf{d}_{j+p}^T f_{j+1}^{\langle m-1\rangle}\varphi \\ \mathbf{d}_{j+2}^T f_{j+1}^{\langle 0\rangle}\varphi & \mathbf{d}_{j+2}^T f_{j+1}^{\langle 1\rangle}\varphi & \cdots & \mathbf{d}_{j+2}^T f_{j+1}^{\langle m-1\rangle}\varphi \\ \mathbf{d}_{j+3}^T f_{j+1}^{\langle 0\rangle}\varphi & \mathbf{d}_{j+3}^T f_{j+1}^{\langle 1\rangle}\varphi & \cdots & \mathbf{d}_{j+3}^T f_{j+1}^{\langle m-1\rangle}\varphi \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{d}_{j+m}^T f_{j+1}^{\langle 0\rangle}\varphi & \mathbf{d}_{j+m}^T f_{j+1}^{\langle 1\rangle}\varphi & \cdots & \mathbf{d}_{j+m}^T f_{j+1}^{\langle m-1\rangle}\varphi \end{pmatrix}.$$

By formula (4.2) for $p = 1$, the first row of this determinant is zero row, and for $p = 2, 3, \ldots, m$, we obtain two identical rows in the discussed determinant; therefore relations (4.4) have been established. This completes the proof. □

Suppose $\mathbf{A}^{\star} \in \mathcal{A}$; now by $\omega_j^{\star}$ denote splines obtained with (2.8), where $\mathbf{a}_j = \mathbf{a}_j^{\star}$.

Theorem 5. *If* $\mathbf{A}^{\star} \in \mathcal{A}$, *then the formulae* $f^{\langle s\rangle}(\cdot)\omega_j^{\star} \in C(\alpha, \beta)$, $\forall j \in \mathbb{Z}$, $\forall s = 0, 1, \ldots, m-1$ *are right*.

Proof. The relations (4.4) can be rewritten in the form $\mathbf{d}_j \perp \mathbf{a}_{j-i}^{\star}$, $i \in I_m$; the last one is equivalent to

$$\mathbf{d}_j \perp \mathcal{L}(\mathbf{a}_{j-m}^{\star}, \mathbf{a}_{j-m+1}^{\star}, \ldots, \mathbf{a}_{j-2}^{\star}, \mathbf{a}_{j-1}^{\star}). \tag{4.5}$$

On the other hand, according to formula (4.2), we have

$$\mathbf{d}_j \perp \mathcal{L}(f_j^{\langle 0\rangle}\varphi,\ f_j^{\langle 1\rangle}\varphi,\ f_j^{\langle 2\rangle}\varphi, \ldots,\ f_j^{\langle m-1\rangle}\varphi). \tag{4.6}$$

By (4.5) and (4.6), it follows

$$\mathcal{L}(\mathbf{a}^{\star}_{j-m}, \mathbf{a}^{\star}_{j-m+1}, \ldots, \mathbf{a}^{\star}_{j-2}, \mathbf{a}^{\star}_{j-1}) = \mathcal{L}(f_j^{\langle 0\rangle}\varphi,\ f_j^{\langle 1\rangle}\varphi,\ f_j^{\langle 2\rangle}\varphi, \ldots,\ f_j^{\langle m-1\rangle}\varphi),$$

whence

$$f_k^{\langle s\rangle}\varphi \in \mathcal{L}(\mathbf{a}^{\star}_{j-m}, \mathbf{a}^{\star}_{j-m+1}, \ldots, \mathbf{a}^{\star}_{j-2}, \mathbf{a}^{\star}_{j-1})$$

for $s = 0, 1, 2, \ldots, m-1;$ thus

$$\det(\mathbf{a}^{\star}_{j-m}, \mathbf{a}^{\star}_{j-m+1}, \ldots, \mathbf{a}^{\star}_{j-2}, \mathbf{a}^{\star}_{j-1},\ f_k^{\langle s\rangle}\varphi) = 0.$$

Now use Theorem 2. □

Consider the linear space $\mathbb{S}^{\star}_m(X, \varphi, F) \stackrel{\text{def}}{=} Cl_p\mathcal{L}\{\omega^{\star}_j\}_{j\in\mathbb{Z}}$.

Space $\mathbb{S}^{\star}_m(X, \varphi, F)$ is called the *space* of $B_{\varphi, F}$ -splines.

We discuss the set of spaces

$$\mathfrak{S}_m(X, \varphi) \stackrel{\text{def}}{=} \{\mathbb{S}_m(X, \mathbf{A}, \varphi) \mid \forall \mathbf{A} \in \mathcal{A}\}.$$

Next we use the subspace C_F of space U:

$$C_F \stackrel{\text{def}}{=} \{u \mid u \in U,\ f^{\langle s\rangle}(\cdot)u \in C(\alpha, \beta),\ \forall s = 0, 1, 2, \ldots, m-1\}.$$

Theorem 6. *In the set* $\mathfrak{S}_m(X, \varphi)$ *there exists the unique space, which belongs to* C_F; *the mentioned space is* $\mathbb{S}^{\star}_m(X, \varphi)$.

Proof. The existence of the space, which belongs to C_F, follows from Theorem 5. Now we will prove its uniqueness.

According to Theorem 3, for the functions ω_j to be in space C_F it is necessary and sufficient to have the true relations (3.8) for $f \stackrel{\text{def}}{=} f^{\langle s\rangle}$, $s =$

$0, 1, 2, ..., m-1$. Suppose chain $\{\mathbf{a}_j\}_{j\in\mathbb{Z}}$ satisfies the condition

$$\det(\mathbf{a}_{j-m}, \mathbf{a}_{j-m+1}, ..., \mathbf{a}_{j-2}, \mathbf{a}_{j-1}, f_j^{\langle s\rangle}\varphi) = 0, \quad \forall j \in \mathbb{Z}, \tag{4.7}$$

$$s = 0, 1, 2, ..., m-1.$$

We are going to show that vectors $\mathbf{a}_j$ differ from vectors $\mathbf{a}_j^\star$ with a nonzero multiplier.

Let chain $\mathbf{b}$ be locally orthogonal to chain $\{\mathbf{a}_j\}_{j\in\mathbb{Z}}$:

$$\mathbf{b}_j \perp \mathcal{L}\{\mathbf{a}_{j-m}, \mathbf{a}_{j-m+1}, ..., \mathbf{a}_{j-2}, \mathbf{a}_{j-1}\};$$

according to Lemma 4 such a chain exists and is defined uniquely up to nonzero constant multipliers. By (4.7), we have

$$f_j^{\langle s\rangle}\varphi \in \mathcal{L}\{\mathbf{a}_{j-m}, \mathbf{a}_{j-m+1}, ..., \mathbf{a}_{j-2}, \mathbf{a}_{j-1}\}, \quad s = 0, 1, 2, ..., m-1,$$

whence $\mathbf{b}_j \perp f_j^{\langle s\rangle}\varphi$, $s = 0, 1, 2, ..., m-1$; it follows

$$\mathbf{b}_j \perp \mathcal{L}\{f_j^{\langle 0\rangle}\varphi, f_j^{\langle 1\rangle}\varphi, ..., f_j^{\langle m-1\rangle}\varphi\}.$$

By the condition (4.1), the vectors $f_j^{\langle 0\rangle}\varphi, f_j^{\langle 1\rangle}\varphi, ..., f_j^{\langle m-1\rangle}\varphi$ are linear independent, and therefore (taking into account that all discussed vectors are $m+1$-dimensional) the relations $\mathbf{b}_j = c_j\mathbf{d}_j$ are true (see definition (4.2) of vectors $\mathbf{d}_j$); here c_j are nonzero constants. The chains $\{\mathbf{a}_j\}_{j\in\mathbb{Z}}$ and $\{\mathbf{a}_j^\star\}_{j\in\mathbb{Z}}$ are locally biorthogonal to chains $\{\mathbf{b}_j\}_{j\in\mathbb{Z}}$ and $\{\mathbf{d}_j\}_{j\in\mathbb{Z}}$ accordingly and they differ only in nonzero constants: $\mathbf{a}_j = \mathbf{c}_j^\star\mathbf{a}_j^\star$. It follows $\omega_j = \omega_j^\star / c_j^\star$ (see Lemma 4). This completes the proof. □

Let us multiply the relation

$$\sum_{i=k-m}^{k} \mathbf{a}_i^\star\omega_i^\star(t) = \varphi(t), \quad \forall t \in (x_k, x_{k+1}) \tag{4.8}$$

on the left by the vector rows $\mathbf{d}_j^T$ for $j = k - m, k - m + 1, ..., k - 1, k.$ Taking into account the local orthogonality of the chains $\{\mathbf{d}_j^T\}_{j\in\mathbb{Z}}$ and $\{\mathbf{a}_i^\star\}_{i\in\mathbb{Z}}$, we obtain $m + 1$ scalar equations, which can be written in the form

$$\begin{pmatrix} \mathbf{d}_{k-m}^T\mathbf{a}_{k-m}^\star & \mathbf{d}_{k-m}^T\mathbf{a}_{k-m+1}^\star & \cdots & \mathbf{d}_{k-m}^T\mathbf{a}_{k-1}^\star & \mathbf{d}_{k-m}^T\mathbf{a}_{k}^\star \\ 0 & \mathbf{d}_{k-m+1}^T\mathbf{a}_{k-m+1}^\star & \cdots & \mathbf{d}_{k-m+1}^T\mathbf{a}_{k-1}^\star & \mathbf{d}_{k-m+1}^T\mathbf{a}_{k}^\star \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \mathbf{d}_{k-1}^T\mathbf{a}_{k-1}^\star & \mathbf{d}_{k-1}^T\mathbf{a}_{k}^\star \\ 0 & 0 & \cdots & 0 & \mathbf{d}_{k}^T\mathbf{a}_{k}^\star \end{pmatrix}$$

$$\times \begin{pmatrix} \omega_{k-m}^\star(t) \\ \omega_{k-m+1}^\star(t) \\ \cdots \\ \omega_{k-1}^\star(t) \\ \omega_{k}^\star(t) \end{pmatrix} = \begin{pmatrix} \mathbf{d}_{k-m}^T\varphi(t) \\ \mathbf{d}_{k-m+1}^T\varphi(t) \\ \cdots \\ \mathbf{d}_{k-1}^T\varphi(t) \\ \mathbf{d}_{k}^T\varphi(t) \end{pmatrix}. \tag{4.9}$$

Analogously, multiplying equation (4.8) on the left by vector rows $\mathbf{d}_j^T$ for $j = k + 1, k + 2, ..., k + m, k + m + 1$ and using local orthogonality of chains $\{\mathbf{d}_j^T\}_{j\in\mathbb{Z}}$ and $\{\mathbf{a}_i^\star\}_{i\in\mathbb{Z}}$ with each other, we have another system of relations

$$\begin{pmatrix} \mathbf{d}_{k+1}^T\mathbf{a}_{k-m}^\star & 0 & \cdots & 0 & 0 \\ \mathbf{d}_{k+2}^T\mathbf{a}_{k-m}^\star & \mathbf{d}_{k+2}^T\mathbf{a}_{k-m+1}^\star & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{d}_{k+m}^T\mathbf{a}_{k-m}^\star & \mathbf{d}_{k+m}^T\mathbf{a}_{k-m+1}^\star & \cdots & \mathbf{d}_{k+m}^T\mathbf{a}_{k-1}^\star & 0 \\ \mathbf{d}_{k+m+1}^T\mathbf{a}_{k-m}^\star & \mathbf{d}_{k+m+1}^T\mathbf{a}_{k-m+1}^\star & \cdots & \mathbf{d}_{k+m+1}^T\mathbf{a}_{k-1}^\star & \mathbf{d}_{k+m+1}^T\mathbf{a}_{k}^\star \end{pmatrix}$$

$$\times \begin{pmatrix} \omega^{\star}_{k-m}(t) \\ \omega^{\star}_{k-m+1}(t) \\ \cdots \\ \omega^{\star}_{k-1}(t) \\ \omega^{\star}_{k}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{d}^T_{k+1}\varphi(t) \\ \mathbf{d}^T_{k+2}\varphi(t) \\ \cdots \\ \mathbf{d}^T_{k+m}\varphi(t) \\ \mathbf{d}^T_{k+m+1}\varphi(t) \end{pmatrix}. \tag{4.10}$$

The relations (4.9) and (4.10) are discussed as systems of linear algebraic equations with respect to unknowns $\omega^{*}_{j}(t), \ \ j \in J_k$. Triangular matrices of these equations are nonsingular because the diagonal elements are nonzero (according to Lemma 3).

By definition, put $I^1_p \overset{\text{def}}{\equiv} \{-p, -p+1, ..., m\}$, $I^2_p \overset{\text{def}}{\equiv} \{-m+1, -m+2, ..., m+1-p\}$, where $p = 0, 1, 2, ..., m.$

Lemma 8. *The functions* $\omega^{\star}_{k-p}(t), \ \ p = 0, 1, 2, ..., m,$ *on interval* $t \in (x_k, x_{k+1})$ *are only defined by values of the vector function* $\varphi(t)$ *on the mentioned interval, complemented by values of components of vectors* $f^{\langle s\rangle}(t)\varphi,\ s \in \{0, 1, ..., m-1\},$ *in the knots* $t = x_{k+i},\ \forall i \in I^1_{p.}$

Proof. The vector $\mathbf{d}_j$ is defined by vectors $f^{\langle s\rangle}_j\varphi,\ s \in \{0, 1, 2, ..., m-1\}$ (see formulae (4.2)), and according to Lemma 7 the vector $\mathbf{a}^{\star}_j$ is defined by vectors $f^{\langle s\rangle}_i\varphi,\ \ s \in \{0, 1, 2, ..., m-1\}, i \in \{j+1,\ j+2, ...,\ j+m\}.$

Taking into account the system (4.9), we see that the function $\omega^{\star}_k(t)$ is defined by the values of the vector function $\varphi(t)$ for $t \in (x_k, x_{k+1})$ and by vectors $\mathbf{d}_k, \mathbf{a}^{\star}_k$. The last ones are defined by the vectors $f^{\langle s\rangle}_i\varphi,$ where $s \in \{0, 1, 2, ..., m-1\}, i \in \{k, k+1, k+2, ..., k+m\}.$

Analogously we define dependence of function $\omega^{\star}_{k-1}(t)$ on the values of vector function $\varphi(t)$ for $t \in (x_k, x_{k+1})$ and on the vectors $\mathbf{d}_k, \mathbf{d}_{k-1}, \mathbf{a}_k^\star, \mathbf{a}_{k-1}^\star$; the last ones are defined by vectors $f_i^{\langle s \rangle}\varphi$, where $s \in \{0, 1, 2, ..., m-1\}$, $i \in \{k-1, k, ..., k+m\}$.

Continuing such reasons, at last we find that the function $\omega_{k-m}^\star(t)$ is defined by the values of $\varphi(t)$ for (x_k, x_{k+1}) and by vectors $\mathbf{d}_k, \mathbf{d}_{k-1}, ...,$ $\mathbf{d}_{k-m}, \mathbf{a}_k^\star, \mathbf{a}_{k-1}^\star, ..., \mathbf{a}_{k-m}^\star$; the vectors defined by the other vectors: they are $f_i^{\langle s \rangle}\varphi$, where $s \in \{0, 1, 2, ..., m-1\}$, $i \in \{k-m, k-m+1, ..., k+m\}$. □

Lemma 9. *The functions* $\omega_{k-p}^\star(t)$, $p = 0, 1, 2, ..., m$ *on interval* $t \in (x_k, x_{k+1})$ *are defined by the values of vector function* $\varphi(t)$ *on discussed interval and complemented by values of component of vectors* $f^{\langle s \rangle}(t)\varphi$, $s \in \{0, 1, ..., m-1\}$ *in knots* x_{k+i}, $\forall i \in I_p^2$.

Proof. According to formulae (4.2), vectors $\mathbf{d}_j$ depend on the values of vectors $f_j^{\langle s \rangle}\varphi$, $s \in \{0, 1, 2, ..., m-1\}$, and by Lemma 7 the vector $\mathbf{a}_j^\star$ is defined by vectors $f_i^{\langle s \rangle}\varphi$, $s \in \{0, 1, 2, ..., m-1\}$, $i \in \{j+1, j+2, ..., j+m\}$.

Using the system (4.10) for the definition of the function $\omega_{k-m}^\star(t)$, we see that the function is defined by the vector function $\varphi(t)$ on the interval (x_k, x_{k+1}) and by vectors $\mathbf{d}_{k+1}$, $\mathbf{a}_{k-m}^\star$. The last ones are defined by the vectors $f_i^{\langle s \rangle}\varphi$, where $s \in \{0, 1, 2, ..., m-1\}$, $i \in \{k-m+1, k-m+2, ..., k+1\}$.

By the system (4.10), we deduce that the function $\omega_{k-m+1}^\star(t)$ is defined by the values of $\varphi(t)$ for $t \in (x_k, x_{k+1})$ and by the vectors $\mathbf{d}_{k+1}, \mathbf{d}_{k+2}$, $\mathbf{a}_{k-m}^\star, \mathbf{a}_{k-m+1}^\star$; conversely the last ones defined by vectors $f_i^{\langle s \rangle}\varphi$, where $s \in \{0, 1, 2, ..., m-1\}$, $i \in \{k-m+1, k-m+2, ..., k+2\}$.

Analogously we define the dependence of the functions $\omega^{\star}_{k-p}(t)$, $p = m-2,\ m-3,\ \ldots,\ 1,\ 0,$ on the vectors $f_i^{\langle s\rangle}\varphi$.

At the end (for $p = 0$) we see that the function $\omega^{\star}_{k}(t)$ is defined by the values of $\varphi(t)$ for $t \in (x_k,\ x_{k+1})$ and by the vectors $\mathbf{d}_{k+1},\ \mathbf{d}_{k+2},\ \ldots,\ \mathbf{d}_{k+m+1}$, $\mathbf{a}^{\star}_{k-m},\ \mathbf{a}^{\star}_{k-m+1},\ \ldots,\ \mathbf{a}^{\star}_{k}$; the vectors are defined by $f_i^{\langle s\rangle}\varphi$, where $s \in \{0,\ 1,\ 2,\ \ldots,\ m-1\}$, $i \in \{k-m+1,\ k-m+2,\ \ldots,\ k+m+1\}$.

This completes the proof. □

We discuss the next condition:

(A) The functionals $f_i^{\langle s\rangle}$, $s \in \{0,\ 1,\ 2,\ \ldots,\ m-1\}$, $i \in \mathbb{Z}$, are linear independent.

In what follows we suppose that the condition (A) is valid.

Lemma 10. *The functions* $\omega^{\star}_{k-p}(t)$, $p = 0,\ 1,\ 2,\ \ldots,\ m$, *for* $t \in (x_k,\ x_{k+1})$ *are defined by values of the vector function* $\varphi(t)$ *on the interval* $(x_k,\ x_{k+1})$ *and by values of* $f_j^{\langle s\rangle}\varphi$, $\forall j$, $j-k \in I_p^1 \cap I_p^2$.

Proof. The proof follows from Lemma 8 and Lemma 9 because the values of functionals $f_i^{\langle s\rangle}u$, $s \in \{0,\ 1,\ 2,\ \ldots,\ m-1\}$, $i \in \mathbb{Z}$, for $u \in U$ can be unrestricted. □

Theorem 7. *The function* $\omega^{\star}_{j}(t)$ *is defined by the values of vector function* $\varphi(t)$, *and the values of vectors* $f_i^{\langle s\rangle}\varphi$, $s \in \{0,\ 1,\ 2,\ \ldots,\ m-1\}$, *are also used, namely*:

(A) *if function* $\omega^{\star}_{j}(t)$ *is calculated for* $t \in (x_j,\ x_{j+1})$, *then vectors* $f_{j+i}^{\langle s\rangle}\varphi$, $i = 0,\ 1,\ 2,\ \ldots,\ m$, *are required*,

(B) *if* $t \in (x_{j+k},\ x_{j+k+1}),\ \ k = 1, 2, \ldots, m-1,$ *then* $f_{j+i}^{\langle s \rangle}\varphi,\ \ i = 0, 1, 2, \ldots, m, m+1,$ *are required*,

(C) *finally, if* $t \in (x_{j+m},\ x_{j+m+1}),$ *then we are required to calculate* $f_{j+i}^{\langle s \rangle}\varphi,\ i = 1, 2, \ldots, m, m+1.$

Proof. If we successively put $k = j,\ j+1, \ldots,\ j+m-1,\ j+m$ and use Lemma 10, then we obtain the desired result. □

Theorem 8. *For the definition of function* $\omega_j^{\star}(t)$ *it is sufficient to know the values of the vector function* $\varphi(t)$ *on the interval* $(x_j - \varepsilon,\ x_{j+m+1} + \varepsilon)$ *for arbitrary* $\varepsilon > 0.$

Proof. According to Theorem 7, all values of function $\omega_j^{\star}(t)$ are defined by the values of vector function $\varphi(t)$ in the points of interval $[x_j,\ x_{j+m+1}]$ and by the values of the functions $f^{\langle s \rangle}(t)\varphi$ in the points $x_{j+i},$ where $i \in \{0, 1, 2, \ldots, m+1\},\ \ s \in \{0, 1, \ldots, m-1\}.$ Taking into account the definition of point functionals $f^{\langle s \rangle}(t),$ we conclude that ε-neighborhood of the interval $(x_j,\ x_{j+m+1})$ is sufficient for the calculation of mentioned functionals. □

5. Calibration Relations

Here we discuss an enlargement of grid X by removal of knot x_{k+1}: we put

$$\tilde{x}_j = x_j \text{ for } j \le k, \quad \tilde{x}_j = x_{j+1} \text{ for } j \ge k+1.$$

Thus $\tilde{X} \stackrel{\text{def}}{=} \{\tilde{x}_j\}_{j \in \mathbb{Z}},$

$$\tilde{X} : \ldots < \tilde{x}_{-1} < \tilde{x}_0 < \tilde{x}_1 < \ldots .$$

By definition, put $\tilde{f}_j^{\langle s\rangle}\varphi \stackrel{\text{def}}{=} f^{\langle s\rangle}(\tilde{x}_j)\varphi,\ s \in \{0, 1, 2, ..., m-1\}$. Let us denote

$$\tilde{\mathbf{d}}_j^T\mathbf{x} \equiv \det(\tilde{f}_j^{\langle 0\rangle}\varphi,\ \tilde{f}_j^{\langle 1\rangle}\varphi, ...,\ \tilde{f}_j^{\langle m-1\rangle}\varphi,\ \mathbf{x}), \quad \forall\mathbf{x} \in \mathbb{R}^{m+1}. \tag{5.1}$$

Now we introduce the vectors with symbolic determinant

$$\tilde{\mathbf{a}}_j^{\star} \stackrel{\text{def}}{=} \det\begin{pmatrix} \tilde{f}_j^{\langle 0\rangle}\varphi & \tilde{f}_j^{\langle 1\rangle}\varphi & \tilde{f}_j^{\langle 2\rangle}\varphi & \cdots & \tilde{f}_j^{\langle m-1\rangle}\varphi \\ \tilde{\mathbf{d}}_{j+2}^T\tilde{f}_j^{\langle 0\rangle}\varphi & \tilde{\mathbf{d}}_{j+2}^T\tilde{f}_j^{\langle 1\rangle}\varphi & \tilde{\mathbf{d}}_{j+2}^T\tilde{f}_j^{\langle 2\rangle}\varphi & \cdots & \tilde{\mathbf{d}}_{j+2}^T\tilde{f}_j^{\langle m-1\rangle}\varphi \\ \tilde{\mathbf{d}}_{j+3}^T\tilde{f}_j^{\langle 0\rangle}\varphi & \tilde{\mathbf{d}}_{j+3}^T\tilde{f}_j^{\langle 1\rangle}\varphi & \tilde{\mathbf{d}}_{j+3}^T\tilde{f}_j^{\langle 2\rangle}\varphi & \cdots & \tilde{\mathbf{d}}_{j+3}^T\tilde{f}_j^{\langle m-1\rangle}\varphi \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \tilde{\mathbf{d}}_{j+m}^T\tilde{f}_j^{\langle 0\rangle}\varphi & \tilde{\mathbf{d}}_{j+m}^T\tilde{f}_j^{\langle 1\rangle}\varphi & \tilde{\mathbf{d}}_{j+m}^T\tilde{f}_j^{\langle 2\rangle}\varphi & \cdots & \tilde{\mathbf{d}}_{j+m}^T\tilde{f}_j^{\langle m-1\rangle}\varphi \end{pmatrix}. \tag{5.2}$$

Suppose $\tilde{\mathbf{A}}^{\star} \stackrel{\text{def}}{=} \{\tilde{\mathbf{a}}_j^{\star}\}_{j\in\mathbb{Z}} \in \mathcal{A}$. We define splines $\tilde{\omega}_j^{\star}(t)$ by the approximation relations

$$\sum_{j\in\mathbb{Z}}\tilde{\mathbf{a}}_j^{\star}\tilde{\omega}_j^{\star}(t) \equiv \varphi(t), \quad \forall t \in \tilde{M}, \quad \omega_j^{\star}(t) = 0 \text{ for } t \notin \tilde{S}_j, \quad \forall j \in \mathbb{Z}, \tag{5.3}$$

where $\tilde{M} \stackrel{\text{def}}{=} \bigcup_{j\in\mathbb{Z}}(\tilde{x}_j, \tilde{x}_{j+1}),\ \tilde{S}_j \stackrel{\text{def}}{=} [\tilde{x}_j, \tilde{x}_{j+m+1}]$.

Consider the space of $B_{\varphi, F}$ -splines according to the new grid

$$\mathbb{S}_m^{\star}(\tilde{X}, \varphi) \stackrel{\text{def}}{=} Cl_p\mathcal{L}\{\tilde{\omega}_j^{\star}\}_{j\in\mathbb{Z}}.$$

Theorem 9. *Under the discussed conditions, the space of* $B_{\varphi, F}$ *-splines constructed for grid X contains the space of* $B_{\varphi, F}$ *-splines for the* $\tilde{X}$:

$$\mathbb{S}_m^{\star}(\tilde{X}, \varphi) \subset \mathbb{S}_m^{\star}(X, \varphi). \tag{5.4}$$

Proof. By the formulae (5.1)-(5.3), it is easy to obtain relations

$$\tilde{\omega}_j^\star(t) \equiv \omega_j^\star(t), \quad \tilde{\mathbf{a}}_j^\star = \mathbf{a}_j^\star \text{ for } j \le k - m - 1, \tag{5.5}$$

$$\tilde{\omega}_j^\star(t) \equiv \omega_{j+1}^\star(t), \quad \tilde{\mathbf{a}}_j^\star = \mathbf{a}_{j+1}^\star \text{ for } j \ge k + 1. \tag{5.6}$$

From (2.7) and (5.3), it follows

$$\sum_{j' \in \mathbb{Z}} \tilde{\mathbf{a}}_{j'}^\star \tilde{\omega}_{j'}^\star(t) \equiv \sum_{j \in \mathbb{Z}} \mathbf{a}_j^\star \omega_j^\star(t), \quad \forall t \in (\alpha, \beta),$$

whence by annihilation of identical summands, defined by relations (5.5)-(5.6), we obtain

$$\begin{aligned} &\tilde{\mathbf{a}}_{k-m}^\star \tilde{\omega}_{k-m}^\star(t) + \tilde{\mathbf{a}}_{k-m+1}^\star \tilde{\omega}_{k-m+1}^\star(t) + \ldots + \tilde{\mathbf{a}}_{k-1}^\star \tilde{\omega}_{k-1}^\star(t) + \tilde{\mathbf{a}}_k^\star \tilde{\omega}_k^\star(t) \\ \equiv\ &\mathbf{a}_{k-m}^\star \omega_{k-m}^\star(t) + \mathbf{a}_{k-m+1}^\star \omega_{k-m+1}^\star(t) + \ldots + \mathbf{a}_{k-1}^\star \omega_{k-1}^\star(t) \\ &+ \mathbf{a}_k^\star \omega_k^\star(t) + \mathbf{a}_{k+1}^\star \omega_{k+1}^\star(t), \quad \forall t \in (\alpha, \beta). \end{aligned} \tag{5.7}$$

Considering the relation (5.7) as a system of linear algebraic equations with respect to unknown functions $\tilde{\omega}_{k-m}^\star(t), \tilde{\omega}_{k-m+1}^\star(t), ..., \tilde{\omega}_{k-2}^\star(t), \tilde{\omega}_{k-1}^\star(t),$ $\tilde{\omega}_k^\star(t),$ and taking into account that matrix of the system is nonsingular (because the chain $\tilde{\mathbf{A}}^\star$ is complete), we express aforementioned functions through $\omega_{k-m}^\star(t), \omega_{k-m+1}^\star(t), ..., \omega_{k-2}^\star(t), \omega_{k-1}^\star(t), \omega_k^\star(t), \omega_{k+1}^\star(t).$

By relations (5.5)-(5.7) we see that basic functions $\tilde{\omega}_j^\star(t)$ of the space $\mathbb{S}_m^\star(\tilde{X}, \varphi)$ can be expressed with basic functions $\omega_j^\star(t)$ of the space $\mathbb{S}_m^\star(X, \varphi).$ The inclusion (5.4) has been proved. □

The relations, which express coordinate splines on an enlargement grid through the coordinate splines on the original grid are called *calibration relations*.

Let $X' = X_0 \subset X_1 \subset X_2 \subset \ldots \subset X_n = X$ be sequence embedded grids, let $\mathbf{A}_i^\star$ be a chain (of vectors) which is relevant to grid X_i. Suppose that chain $\mathbf{A}_i^\star$ is constructed by scheme (5.2) and $\mathbf{A}_i^\star \in \mathcal{A}$. By (4.1) system $\{ f^{(s)}(t)\varphi \,|\, s = 0, 1, 2, ..., m-1;\ t \in X_i\}$ is linear independent system of vectors, $i = 0, 1, 2, ..., n$; therefore

$$\mathbb{S}_m^\star(X', \varphi) \subset \mathbb{S}_m^\star(X, \varphi) \subset C_F.$$

6. Applications

6.1. Minimal splines with maximum smoothness

By definition, put $U_i \overset{\text{def}}{=} C^m(x_i, x_{i+1})$, $f^{\langle i\rangle}(t)u \overset{\text{def}}{=} u^{(i)}(t)$, $t \in M$. In the discussed case the suppositions $f^{\langle i\rangle}(\cdot)\varphi_j \in C(\alpha, \beta)$, $i, j \in \{0, 1, 2, ..., m\}$, are equivalent to implication $\varphi \in C^m(\alpha, \beta)$, and the condition (4.1) may be represented in the form

$$|\det(\varphi, \varphi', \varphi'', ..., \varphi^{(m)})(t)| \geq c > 0, \quad \forall t \in (\alpha, \beta).$$

The vectors $\mathbf{a}_j^*$ are defined by relations (4.3), where $\mathbf{d}_i$ satisfies identity

$$\mathbf{d}_i^T \mathbf{x} \equiv \det(\varphi_i, \varphi_i', \varphi_i'', ..., \varphi_i^{(m-1)}, \mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^{m+1}, \quad \forall i \in \mathbb{Z};$$

here $\varphi_i \overset{\text{def}}{=} \varphi(x_i)$, $\varphi_i^{(s)} \overset{\text{def}}{=} \varphi^{(s)}(x_i)$, $s = 1, 2, ..., m$.

Analogously we get the chain $\{\tilde{\mathbf{a}}_j^\star\}_{j\in\mathbb{Z}}$.

If the derivatives $\varphi^{(i)}(t)$ are uniformly bounded on the interval (α, β) for $i \in \{0, 1, ..., m\}$, and parameter $h \overset{\text{def}}{=} \sup_{j\in\mathbb{Z}}(\tilde{x}_{j-1} - \tilde{x}_j)$ is sufficiently small, then the chains $\{\mathbf{a}_j^\star\}_{j\in\mathbb{Z}}$ and $\{\tilde{\mathbf{a}}_j^\star\}_{j\in\mathbb{Z}}$ are complete (it is possible to prove by the Taylor formula).

As a result we obtain B_φ-splines (see [15]). We denote the spaces of B_φ-splines for grids X and $\tilde{X}$ by $\grave{\mathbb{S}}_m^\star(X, \varphi)$ and $\grave{\mathbb{S}}_m^\star(\tilde{X}, \varphi)$, respectively.

Considering $C_F \stackrel{\text{def}}{=} C^{m-1}(\alpha, \beta)$ and applying Theorem 1 and Theorem 9, we obtain the next assertions.

Theorem 10. *In set* $\mathfrak{S}_m(X, \varphi)$ *there exists a unique space, which belongs to* $C^{m-1}(\alpha, \beta)$; *the mentioned space is* $\grave{\mathbb{S}}^{\star}_m(X, \varphi)$.

Theorem 11. *Under the discussed conditions, the space of* B_φ*-splines constructed for grid X contains the space of* B_φ*-splines for the* $\widetilde{X}$:

$$\grave{\mathbb{S}}^{\star}_m(\widetilde{X}, \varphi) \subset \grave{\mathbb{S}}^{\star}_m(X, \varphi).$$

6.2. Minimal splines with discontinuous derivative of generating function

Discuss the case of $m = 1$, $\varphi(t) \stackrel{\text{def}}{=} (1, \psi(t))^T$, where $\psi \in C(\alpha, \beta)$. Suppose that $t^* \in (\alpha, \beta)$, and there exists continuous derivative $\psi'(t)$ in intervals (α, t^*) and (t^*, β) with property $|\psi'(t)| \ge c_1 > 0$. We also suppose that finite limits $a = \psi'(t^* - 0)$, $b = \psi'(t^* + 0)$ exist, and $a \ne b$, $ab \ne 0$. Consider function

$$\chi_{a,b}(t) \stackrel{\text{def}}{=} \begin{cases} 1/a \text{ for } t \in (\alpha, t^*), \\ 1/b \text{ for } t \in (t^*, \beta), \end{cases}$$

and introduce the functionals $\hat{f}^{\langle 0 \rangle}(t)$ and $\hat{f}^{\langle 1 \rangle}(t)$ by formulae

$$\hat{f}^{\langle 0 \rangle}(t)u \stackrel{\text{def}}{=} u(t), \qquad \hat{f}^{\langle 1 \rangle}(t)u \stackrel{\text{def}}{=} \chi_{a,b}(t)u'(t).$$

It is clear to see that $\hat{f}^{\langle 0 \rangle}(t)\varphi$ and $\hat{f}^{\langle 1 \rangle}(t)\varphi$ are continuous vector-functions on the interval (α, β),

$$|\det(\hat{f}^{\langle 0 \rangle}(t)\varphi, \hat{f}^{\langle 1 \rangle}(t)\varphi)| = |\chi_{a,b}(t)\psi'(t)| \ge c > 0,$$

where c is a positive constant.

According to the formulae (4.2) and (5.2), we have

$$\mathbf{a}_j^\star = \hat{f}_{j+1}^{\langle 0\rangle}\varphi = (\hat{f}_{j+1}^{\langle 0\rangle}\varphi_0,\ \hat{f}_{j+1}^{\langle 0\rangle}\varphi_1)^T,$$

so that in the discussed case we have $\mathbf{a}_j^\star = (1,\ \psi(x_{j+1}))^T$; analogously $\tilde{\mathbf{a}}_j^\star = (1,\ \psi(\tilde{x}_{j+1}))^T$. Hence

$$\det(\mathbf{a}_{j-1}^\star,\ \mathbf{a}_j^\star) = \psi(x_{j+1}) - \psi(x_j), \qquad \det(\tilde{\mathbf{a}}_{j-1}^\star,\ \tilde{\mathbf{a}}_j^\star) = \psi(\tilde{x}_{j+1}) - \psi(\tilde{x}_j).$$

If the function $\psi(t)$ is strongly monotone on the interval $(\alpha,\ \beta)$, then the chains of vectors $\{\mathbf{a}_j^\star\}_{j\in\mathbb{Z}}$ and $\{\tilde{\mathbf{a}}_j^\star\}_{j\in\mathbb{Z}}$ are complete. In this case, by relations (2.8) and (5.3) we have splines $\omega_j^\star$ and $\tilde{\omega}_j^\star$; let $\hat{\mathbb{S}}_1^\star(X,\ \varphi)$ and $\hat{\mathbb{S}}_1^\star(\tilde{X},\ \varphi)$ be the spaces of the mentioned splines, respectively.

Next, we use the space

$$\hat{C}_F \overset{\text{def}}{=} \{u\,|\,u \in U,\ f^{\langle 0\rangle}(\cdot)u \in C(\alpha,\ \beta)\}.$$

By Theorem 6 and Theorem 9, we make the following assertions.

Theorem 12. *In the set* $\mathfrak{S}_1(X,\ \varphi)$ *there exists a unique space, which belongs to* $\hat{C}_F$; *the mentioned space is* $\hat{\mathbb{S}}_1^\star(X,\ \varphi)$.

Theorem 13. *Under the discussed conditions, the space of splines constructed for grid X contains the space of splines for the* $\tilde{X}$:

$$\hat{\mathbb{S}}_m^\star(\tilde{X},\ \varphi) \subset \hat{\mathbb{S}}_m^\star(X,\ \varphi).$$

6.3. Minimal splines with discontinuous generating function

Consider differential operator $Lw \overset{\text{def}}{=} w'' + \frac{1}{t}w' + w,$ and discuss Bessel's differential equation

$$Lw(t) = w''(t) + \frac{1}{t}w'(t) + w(t) = 0 \ \text{ for } \ t \in (1,\ 0) \cup (0,\ -1).$$

Let $J_0(t)$ and $Y_0(t)$ be the linear independent solutions of the equation with the next asymptotic behavior

$$J_0(t) \approx 1, \quad Y_0(t) \approx \frac{2}{\pi}\left(\ln\frac{t}{2} + \gamma\right) \text{ for } t \to 0,$$

where γ is Euler's constant $(\gamma = 0.577215...)$.

Suppose that $m = 1$, $\varphi(t) \stackrel{\text{def}}{=} (\varphi_0(t), \varphi_1(t))^T$, where

$$\varphi_0(t) = J_0(t) + 1, \qquad \varphi_1(t) = Y_0(t) + t^2.$$

Consider functionals

$$\breve{f}^{\langle 0\rangle}(t)u \stackrel{\text{def}}{=} Lu(t), \quad \breve{f}^{\langle 1\rangle}(t)u \stackrel{\text{def}}{=} L^2u(t) \text{ for } t \in (1, 0) \cup (0, -1).$$

Because

$$L\varphi_0(t) = 1, \quad L\varphi_1(t) = t^2 + 4, \quad L^2\varphi_0(t) = 1, \quad L^2\varphi_1(t) = t^2 + 8,$$

we have

$$\det(\breve{f}^{\langle 0\rangle}(t)\varphi, \breve{f}^{\langle 1\rangle}(t)\varphi) = \begin{vmatrix} L\varphi_0(t) & L^2\varphi_0(t) \\ L\varphi_1(t) & L^2\varphi_1(t) \end{vmatrix} = 4.$$

Thus, the vector functions $\breve{f}^{\langle 0\rangle}(t)\varphi$, $\breve{f}^{\langle 1\rangle}(t)\varphi$ are continuous on the interval $(-1, 1)$, and determinant $\det(\breve{f}^{\langle 0\rangle}(t)\varphi, \breve{f}^{\langle 1\rangle}(t)\varphi)$ is not zero.

In the discussed case, we obtain

$$\mathbf{a}_j^\star = (L\varphi_0(x_{j+1}), L\varphi_1(x_{j+1}))^T = (1, x_{j+1}^2)^T,$$

$$\tilde{\mathbf{a}}_j^\star = (L\varphi_0(\tilde{x}_{j+1}), L\varphi_1(\tilde{x}_{j+1}))^T = (1, \tilde{x}_{j+1}^2)^T,$$

therefore

$$\det(\mathbf{a}_{j-1}, \mathbf{a}_j) = x_{j+1}^2 - x_j^2, \qquad \det(\tilde{\mathbf{a}}_{j-1}, \tilde{\mathbf{a}}_j) = \tilde{x}_{j+1}^2 - \tilde{x}_j^2.$$

If $0 \in \tilde{X}$, then the chains of vectors $\{\mathbf{a}_j^\star\}_{j\in\mathbb{Z}}$ and $\{\tilde{\mathbf{a}}_j^\star\}_{j\in\mathbb{Z}}$ are complete. The relations (2.7) and (5.3) give us the splines $\omega_j^\star$ and $\tilde{\omega}_j^\star$; in this case the spaces of the discussed splines are denoted by $\breve{\mathbb{S}}_1^\star(X, \varphi)$ and by $\breve{\mathbb{S}}_1^\star(\tilde{X}, \varphi)$, respectively.

Next, we use the space

$$\breve{C}_F \stackrel{\text{def}}{=} \{u \,|\, u \in U,\ \breve{f}^{\langle 0\rangle}(\cdot)u \in C(\alpha, \beta)\}.$$

Using Theorem 6 and Theorem 9, we can deduce the next statements.

Theorem 14. *In set* $\mathfrak{S}_1(X, \varphi)$ *there exists a unique space, which belongs to* $\breve{C}_F$: *the mentioned space is* $\breve{\mathbb{S}}_1^\star(X, \varphi)$.

Theorem 15. *Under the discussed conditions the space of splines constructed for grid X contains the space of splines for* $\tilde{X}$:

$$\breve{\mathbb{S}}_1^\star(\tilde{X}, \varphi) \subset \breve{\mathbb{S}}_1^\star(X, \varphi).$$

Acknowledgements

This research was partly supported by Grant RFFI No. 15-01-008847.

References

[1] J. J. Goel, Construction of basis functions for numerical utilization of Ritz's method, Numer. Math. 12 (1968), 435-447.

[2] L. L. Schumaker, On super splines and finite elements, SIAM J. Numer. Anal. 26 (1989), 997-1005.

[3] S. Mallat, Multiresolution approximation and wavelets. Trans. Amer. Math. Soc. 315 (1989), 69-88.

[4] S. G. Mikhlin, Error Analysis in Numerical Processes, Wiley, Chichester, New York, 1991.

[5] Y. Meyer, Wavelets and Operators, Cambridge University Press, Cambridge, 1992.

[6] I. Daubeshies, Ten lectures on wavelets, CBMS-NSR Series in Appl. Math., SIAM, 1992.

[7] C. K. Chui, An Introduction to Wavelets, Academic Press, 1992.

[8] M. D. Buhmann, Multiquadric prewavelets on nonequally spaced knots in one dimension, Math. Comput. 64(212) (1995), 1611-1625.

[9] W. Lawton, S. L. Lee and Z. Chen, Characterization of compactly supported refinable splines, Adv. Comput. Math. 3(1-2) (1995), 137-145.

[10] G. Gripenberg, A necessary and sufficient condition for the existence of father wavelet, Studia Mathematica 114(3) (1995), 207-226.

[11] I. Ya. Novikov and S. B. Stechkin, Basic wavelet theory, Usp. Mat. Nauk 53(6) (1998), 53-128 (in Russian); English transl.: Russ. Math. Surv. 53(6) (1998), 1159-1231.

[12] Yu. K. Dem'yanovich, Wavelet expansions in spline spaces on an irregular grid, Dokl. Akad. Nauk, Ross. Akad. Nauk 382(3) (2002), 313-316 (in Russian); English transl.: Dokl. Math. 65(1) (2002), 47-50.

[13] I. Ya. Novikov, V. Yu. Protasov and M. A. Scopina, Wavelet theory, AMS, Vol. 239, 2011.

[14] Yu. K. Dem'yanovich, Embedded spaces of trigonometric splines and their wavelet expansion, Math Notes 78(5) (2005), 615-630.

[15] Yu. K. Dem'yanovich, Wavelet decompositions on nonuniform grid, Proceedings of the St. Petersburg Mathematical Society XIII, N. N. Uraltseva, ed., American Mathematical Society Translations, Series 2, Vol. 222, 2007, pp. 23-43, American Mathematical Society, RI, USA, 2008.

[16] Yu. K. Dem'yanovich, Minimal splines and wavelets, Vestn. St.-Petersbg. Univ., Ser. I (2) (2008), 8-22 (in Russian); English transl.: Vestn. St.-Petersbg. Univ., Math. 41(2) (2008), 88-101.

[17] Yu. K. Dem'yanovich, Minimal splines of Lagrange type, Probl. Mat. Anal. 50 (2010), 21-64 (in Russian); English transl.: J. Math. Sci., New York 170(4) (2010), 444-495.

[18] Yu. K. Dem'yanovich and O. M. Kosogorov, Calibration relations for non-polynomial splines, J. Math Sci. 164(3) (2010), 364-382.

[19] Yu. K. Dem'yanovich and I. D. Miroshnichenko, Calibration relations to splines of the fourth order, J. Math. Sci. 178(6) (2011), 576-588.

[20] Yu. K. Dem'yanovich and I. G. Burova, On adaptive processing of discrete flow, WSEAS Trans. Math. 14 (2015), 226-236.

[21] Yu. K. Dem'yanovich, Adaptive properties of Hermite splines, The 1st IFAC Conference on Modelling, Identification and Control of Nonlinear Systems, Final Program and Book of Abstracts, MICNON 2015, Saint-Petersburg, June, 24-26.

[22] E. Janke, F. Emde and F. Lösch, Tafeln Höherer Funktionen, Teubner Verlaggesellschaft, Stuttgart, 1960.