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DERIVATIONS OF PKU-ALGEBRAS

Chanwit Prabpayak

Faculty of Science and Technology
Rajamangala University of Technology Phra Nakhon
Thailand

Abstract

We introduce the notions of left and right derivations of PKU-algebras and investigate some derivative properties on PKU-algebras. Also, we introduce the notion of d-invariant on ideals of PKU-algebras and obtain certain related properties.

Introduction

Dudek and Zhang [1] introduced a new notion of ideals in BCC-algebras. They described the connection between ideals and congruences. Jun and Xin [2] applied the notion of derivations in ring and near-ring theories to BCI-algebras, and they introduced a new concept called a regular derivation in BCI-algebras. Later a pseudo KU-algebra was introduced by Leerawat and Prabpayak [3]. A pseudo KU-algebra is known as PKU-algebra, an extension of KU-algebra [5].

In this paper, we introduce the notions of left and right derivations of PKU-algebras. Then we investigate some derivative properties on PKU-algebras. We finally introduce the notion of d-invariant on ideals of PKU-algebras and investigate related properties.

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Preliminaries

By a PKU-algebra, we mean an algebra $(G, \oplus, 0)$ of type $(2, 0)$ with a binary operation $\oplus$ and a constant $0 \in X$ satisfying the following identities: for all $x, y, z \in G$,

(1) $(x \oplus y) \oplus [(y \oplus z) \oplus (x \oplus z)] = 0$,

(2) $0 \oplus x = x$,

(3) if $x \oplus y = 0 = y \oplus x$, then $x = y$.

We define a binary relation $\leq$ by $x \leq y$ if and only if $y \oplus x = 0$. Then $(G, \leq)$ is a partially ordered set. A subset S of a PKU-algebra $(G, \oplus, 0)$ (for brevity, write G) is called a PKU-subalgebra if $x \oplus y \in S$ for all $x, y \in S$. For any PKU-algebra G the following properties hold: for all $x, y, z \in G$,

(1) $x \oplus x = 0$,

(2) $x \oplus [(x \oplus y) \oplus y] = 0$,

(3) $x \oplus (y \oplus z) = y \oplus (x \oplus z)$,

(4) if $x \leq y$, then $y \oplus z \leq x \oplus z$,

(5) if $x \leq y$, then $z \oplus x \leq z \oplus y$.

A non-empty subset A of a PKU-algebra G is called a *PKU-ideal* of G if it satisfies the following conditions:

(a) $0 \in A$,

(b) for all $x, y, z \in X$, $x \oplus (y \oplus z) \in A$ and $y \in A$ implies $x \oplus z \in A$.

For any PKU-ideal A of a PKU-algebra G, the following property holds: for all $x, y \in X$, $xy \in A$ and $x \in A$ implies $y \in A$. For more details, refer to [3].

Derivations of PKU-algebras

For any elements x and y in a PKU-algebra G, we denote $x \wedge y = (x \oplus y) \oplus y$.

Definition 1 [4]**.** Let G be a PKU-algebra, and let $d : G \to G$ be a map from G into G. d is a *left-derivation* (briefly, *l-derivation*) of G if $d(x \oplus y) = (d(x) \oplus y) \wedge (x \oplus d(y))$. d is a *right-derivation* (briefly, *r-derivation*) of G if $d(x \oplus y) = (x \oplus d(y)) \wedge (d(x) \oplus y)$. If d is both l-derivation and r-derivation of G, then d is called a *derivation* of G.

Example. Let $G = \{0, 1, 2\}$ with a binary operation $\oplus$ defined by the following Cayley table:

$\oplus$	0	1	2
0	0	1	2
1	1	0	1
2	0	1	0

Then G is a PKU-algebra (see [3]). Let us define a map $d : G \to G$ by

$$d(x) = \begin{cases} 0, & \text{if } x = 0, 2, \\ 1, & \text{if } x = 1. \end{cases}$$

One can easily check that d is both l-derivation and r-derivation of G. So d is a derivation of G.

Definition 2 [2]**.** A map d of a PKU-algebra G is said to be *regular* if $d(0) = 0$.

In the previous example, d is regular.

Proposition 3. *Let d be a regular map of a PKU-algebra G. Then*

(1) *if d is an l-derivation of G, then $d(x) = x \wedge d(x)$ for all $x \in G$.*

(2) *if d is an r-derivation of G, then $d(x) = d(x) \wedge x$ for all $x \in G$.*

Proof. (1) Let d be an l-derivation of G, and let $x \in G$. Then

$$\begin{aligned} d(x) &= d(0 \oplus x) \\ &= (d(0) \oplus x) \wedge (0 \oplus d(x)) \\ &= (0 \oplus x) \wedge d(x) \\ &= x \wedge d(x). \end{aligned}$$

(2) Let d be an r-derivation of G, and let $x \in G$. Then

$$\begin{aligned} d(x) &= d(0 \oplus x) \\ &= (0 \oplus d(x)) \wedge (d(0) \oplus x) \\ &= d(x) \wedge (0 \oplus x) \\ &= d(x) \wedge x. \end{aligned}$$

□

Theorem 4. *Let d be a regular derivation of a PKU-algebra G. Then*

(1) $0 \leq x \oplus d(x)$ *for all* $x \in G$,

(2) $0 \leq d(x) \oplus x$ *for all* $x \in G$,

(3) $0 \leq d(x) \oplus d(d(x))$ *for all* $x \in G$,

(4) $0 \leq d(d(x)) \oplus d(x)$ *for all* $x \in G$.

Proof. (1) Since d is an l-derivation of G, we have $d(x) = x \wedge d(x)$ by Proposition 3. So we have

$$d(x) = (x \oplus d(x)) \oplus d(x).$$

Since d is an r-derivation of G, $d(x) = d(x) \wedge x$ by Proposition 3. So we have

$$d(x) = (d(x) \oplus x) \oplus x.$$

The above equations yield

$$(x \oplus d(x)) \oplus d(x) = (d(x) \oplus x) \oplus x,$$

$$d(x) \oplus [(x \oplus d(x)) \oplus d(x)] = d(x) \oplus [(d(x) \oplus x) \oplus x],$$

$$d(x) \oplus [(x \oplus d(x)) \oplus d(x)] = 0,$$

$$((x \oplus d(x))) \oplus (d(x) \oplus d(x)) = 0,$$

$$((x \oplus d(x)))0 = 0.$$

Therefore $0 \leq x \oplus d(x)$.

(2) From (1) we have

$$(d(x) \oplus x) \oplus x = (x \oplus d(x)) \oplus d(x),$$

$$x \oplus [(d(x) \oplus x) \oplus x] = x \oplus [(x \oplus d(x)) \oplus d(x)],$$

$$x \oplus [(d(x) \oplus x) \oplus x] = 0,$$

$$(d(x) \oplus x) \oplus (x \oplus x) = 0,$$

$$(d(x) \oplus x) \oplus 0 = 0.$$

Hence $0 \leq d(x) \oplus x$.

(3) By Proposition 3, we obtain

$$d(d(x)) = (d(x) \oplus d(d(x))) \oplus d(d(x))$$

and

$$d(d(x)) = (d(d(x)) \oplus d(x)) \oplus d(x),$$

$$(d(x) \oplus d(d(x))) \oplus d(d(x)) = (d(d(x)) \oplus d(x)) \oplus d(x),$$

$$d(x(x)) \oplus [(d(x) \oplus d(d(x))) \oplus d(d(x))]$$
$$= d(d(x)) \oplus [(d(d(x)) \oplus d(x)) \oplus d(x)],$$

$$d(x(x)) \oplus [(d(x) \oplus d(d(x))) \oplus d(d(x))] = 0,$$

$$(d(x) \oplus d(d(x))) \oplus (d(x(x)) \oplus d(x(x))) = 0,$$

$$(d(x) \oplus d(d(x))) \oplus 0 = 0.$$

Thus $0 \leq d(x) \oplus d(d(x))$.

(4) From (3) we have

$$(d(d(x)) \oplus d(x)) \oplus d(x) = (d(x) \oplus d(d(x))) \oplus d(d(x)),$$

$$d(x) \oplus [(d(d(x)) \oplus d(x)) \oplus d(x)] = d(x) \oplus [(d(x) \oplus d(d(x))) \oplus d(d(x))],$$

$$d(x) \oplus [(d(d(x)) \oplus d(x)) \oplus d(x)] = 0,$$

$$(d(d(x)) \oplus d(x)) \oplus (d(x) \oplus d(x)) = 0,$$

$$(d(d(x)) \oplus d(x)) \oplus 0.$$

Hence $0 \leq d(d(x)) \oplus d(x)$. □

Definition 5. Let A be a PKU-ideal of a PKU-algebra G. Let d be a regular derivation of G. We denote the set $d(A) = \{d(x) \mid x \in A\}$. Then A is said to be *d-invariant* if $d(A) \subset A$.

Example. From the previous example, let $A = \{0, 2\} \subset G$. It is easy to check that A is a PKU-ideal of G. A regular derivation d of G is defined by

$$d = \begin{cases} 0, & \text{if } x = 0, 2, \\ 1, & \text{if } x = 1. \end{cases}$$

Then $d(A) = \{0\}$ which is contained in A. Thus A is d-invariant.

Theorem 6. *Let d be a regular derivation of a PKU-ideal G. Then a PKU-ideal A of G is d-invariant if* $x \oplus d(x) = 0$ *for all* $x \in A$.

Proof. Let A be a PKU-ideal of G. Suppose that $x \in d(A)$. Then there exists an element $y \in A$ such that $x = d(y)$. Then $y \oplus x = y \oplus d(y)$. By our assumption, we obtain $y \oplus x = 0 \in A$. Since A is a PKU-ideal of G, $x \in A$. Hence $d(A) \subset A$. Therefore, A is d-invariant. □

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