Far East Journal of Mathematical Sciences (FJMS)

http://www.pphmj.com
http://dx.doi.org/10.17654/MS102081605
Volume 102, Number 8, 2017, Pages 1605-1611 **ISSN: 0972-0871**

REPRESENTATIONS OF CERTAIN THETA FUNCTION IDENTITIES IN TERMS OF COMBINATORIAL PARTITION IDENTITIES

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Abstract

Villacorta and Chaudhary [10] introduced two new theta function identities. In this sequel, we aim to give representations for these theta function identities in terms of combinatorial partition identities.

1. Introduction and Definitions

For $q, \lambda, \mu \in \mathbb{C}(|q| < 1)$, the basic (or q-) shifted factorial $(\lambda; q)_\mu$ is defined by (see, for example, [1], [7] and [8] dealing with the q-analysis)

$$(\lambda; q)_\mu := \prod_{j=0}^{\infty} \left(\frac{1 - \lambda q^j}{1 - \lambda q^{\mu + j}} \right) \quad (|q| < 1; \lambda, \mu \in \mathbb{C}), \tag{1.1}$$

Received: April 2, 2017; Revised: May 12, 2017; Accepted: June 20, 2017
2010 Mathematics Subject Classification: 33D10, 11A55, 11F27.
Keywords and phrases: Jacobi's triple-product identity, q-product identities, theta function identities, combinatorial partition identities.

so that

$$(a; q)_n := \begin{cases} 1 & (n = 0) \\ \prod_{k=0}^{n-1} (1 - aq^k) & (n \in \mathbb{N}) \end{cases} \tag{1.2}$$

and

$$(\lambda; q)_\infty := \prod_{j=0}^{\infty} (1 - \lambda q^j) \quad (|q| < 1; \lambda \in \mathbb{C}). \tag{1.3}$$

Here and in the following, let $\mathbb{C}$ and $\mathbb{N}$ be the sets of complex numbers and positive integers, respectively. For convenience, we write

$$(a_1, a_2, a_3, ..., a_k; q)_n = (a_1; q)_n (a_2; q)_n (a_3; q)_n \cdots (a_k; q)_n \tag{1.4}$$

and

$$(a_1, a_2, a_3, ..., a_k; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty (a_3; q)_\infty \cdots (a_k; q)_\infty. \tag{1.5}$$

In Chapter 16 of his celebrated Notebooks, Ramanujan defined the general theta function $f(a, b)$ as follows (see [1, p. 31, equation (18.1)]):

$$\begin{aligned} f(a, b) &= 1 + \sum_{n=1}^{\infty} (ab)^{\frac{n(n-1)}{2}} (a^n + b^n) \\ &= \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} = f(b, a) \quad (|ab| < 1), \end{aligned} \tag{1.6}$$

so that, if n is an integer, then we have

$$f(a, b) = a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} f(a(ab)^n, b(ab)^{-n}) = f(b, a). \tag{1.7}$$

Ramanujan also rediscovered Jacobi's famous triple-product identity (see [1, p. 35, Entry 19]):

$$f(a, b) = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty, \tag{1.8}$$

which was, in fact, first proved by Gauss.

Several q-series identities emerging from Jacobi's triple-products identity (8) are worthy of note here (see [1, pp. 36-37, Entry 22]):

$$\phi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2\sum_{n=1}^{\infty} q^{n^2}$$

$$= \{(-q;\, q^2)_\infty\}^2 (q^2;\, q^2)_\infty = \frac{(-q;\, q^2)_\infty (q^2;\, q^2)_\infty}{(q;\, q^2)_\infty (-q^2;\, q^2)_\infty}, \tag{1.9}$$

$$\psi(q) := f(q,\, q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;\, q^2)_\infty}{(q;\, q^2)_\infty} \tag{1.10}$$

and

$$f(-q) := f(-q,\, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}}$$

$$= \sum_{n=0}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} + \sum_{n=1}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}} = (q;\, q)_\infty. \tag{1.11}$$

Equation (11) is known as *Euler's Pentagonal Number Theorem*. The following q-series identity:

$$(-q;\, q)_\infty = \frac{1}{(q;\, q^2)_\infty} = \frac{1}{\chi(-q)} \tag{1.12}$$

provides the *analytic equivalent* of Euler's famous theorem.

The number of partitions of a positive integer n into distinct parts is equal to the number of partitions of n into odd parts.

We now recall the Rogers-Ramanujan continued fraction $R(q)$, which is given by

$$R(q) := q^{\frac{1}{5}} \frac{H(q)}{G(q)} = q^{\frac{1}{5}} \frac{f(-q,\, -q^4)}{f(-q^2,\, -q^3)} = q^{\frac{1}{5}} \frac{(q;\, q^5)_\infty (q^4;\, q^5)_\infty}{(q^2;\, q^5)_\infty (q^3;\, q^5)_\infty}$$

$$= \frac{q^{\frac{1}{5}}}{1+}\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+}\cdots \quad (|q| < 1) \tag{1.13}$$

in terms of the following widely investigated Roger-Ramanujan identities:

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;\, q)_n} = \frac{f(-q^5)}{f(-q,\, -q^4)}$$

$$= \frac{1}{(q;\, q^5)_\infty (q^4;\, q^5)_\infty} = \frac{(q^2;\, q^5)_\infty (q^3;\, q^5)_\infty (q^5;\, q^5)_\infty}{(q;\, q)_\infty} \tag{1.14}$$

and

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;\, q)_n} = \frac{f(-q^5)}{f(-q^2,\, -q^3)}$$

$$= \frac{1}{(q^2;\, q^5)_\infty (q^3;\, q^5)_\infty} = \frac{(q;\, q^5)_\infty (q^4;\, q^5)_\infty (q^5;\, q^5)_\infty}{(q;\, q)_\infty}, \tag{1.15}$$

where the functions $f(a,\, b)$ and $f(-q)$ are defined by (1.6) and (1.11), respectively. For a detailed historical account of (and for various investigating developments stunning from) the Rogers-Ramanujan continued fraction (1.13) and identities (1.14) and (1.15), the reader may be referred to the work [1, p. 77].

The following continued fraction was reported in [7] from an earlier work cited therein:

$$(q^2;\, q^2)_\infty (-q;\, q)_\infty$$

$$= \frac{(q^2;\, q^2)_\infty}{(q;\, q^2)_\infty}$$

$$= \frac{1}{1-}\frac{q}{1+}\frac{q(1-q)}{1-}\frac{q^3}{1+}\frac{q^2(1-q^2)}{1-}\frac{q^5}{1+}\frac{q^3(1-q^3)}{1-\cdots}. \tag{1.16}$$

Finally, we turn to the recent investigation by Andrews et al. [2], involving combinatorial partition identities associated with the following general family:

$$R(s,\, t,\, l,\, u,\, v,\, w) := \sum_{n=0}^{\infty} q^{s\binom{n}{2}+tn} r(l,\, u,\, v,\, w;\, n), \tag{1.17}$$

where

$$r(l,\, u,\, v,\, w;\, n) := \sum_{j=0}^{\left[\frac{n}{u}\right]} (-1)^j \frac{q^{uv\binom{j}{2}+(w-ul)j}}{(q;\, q)_{n-uj}(q^{uv};\, q^{uv})_j}. \tag{1.18}$$

In particular, we recall the following combinatorial partition identities [2, p. 106, Th. 3]:

$$R(2,\, 1,\, 1,\, 1,\, 2,\, 2) = (-q;\, q^2)_\infty, \tag{1.19}$$

$$R(2,\, 2,\, 1,\, 1,\, 2,\, 2) = (-q^2;\, q^2)_\infty, \tag{1.20}$$

$$R(m,\, m,\, 1,\, 1,\, 1,\, 2) = \frac{(q^{2m};\, q^{2m})_\infty}{(q^m;\, q^{2m})_\infty}. \tag{1.21}$$

The outline of this paper is as follows. In Section 2, we record a set of known results which are required in this paper. In Section 3, we state and prove our main results associated with the family $R(s,\, t,\, l,\, u,\, v,\, w)$ defined by (1.17), which depict representations of certain theta function identities in terms of combinatorial partition identities.

2. A Set of Preliminary Results

The following identities will be required in proving our main results in Section 3 (see [10]):

$$f^2(-q^3) = \psi(q^2)\phi(q^9) - q^2\psi(q^{18})\phi(q) \tag{2.1}$$

and

$$f(-q^3)(\psi(q) - q\psi(q^9)) = f(-q^6)\phi(-q^9), \tag{2.2}$$

where all the symbols having their usual meanings.

3. The Main Results

We state our main results.

Theorem 1. *Each of the following identities holds true*:

$$\begin{aligned} f^2(-q^3) = {} & R(2, 2, 1, 1, 1, 2)(-q^9; q^{18})_\infty^2 (q^{18}; q^{18})_\infty \\ & - R(18, 18, 1, 1, 1, 2)\{R(2, 1, 1, 1, 2, 2)\}^2 q^2 (q^2; q^2)_\infty, \end{aligned} \tag{3.1}$$

$$\frac{f(-q^3)}{f(-q^6)} = \frac{(q^9; q^{18})_\infty^2 (q^{18}; q^{18})_\infty}{R(1, 1, 1, 1, 2, 2) - q \cdot R(9, 9, 1, 1, 1, 2)} \tag{3.2}$$

and

$$\begin{aligned} & \frac{R(2, 2, 1, 1, 1, 2)(-q^9; q^{18})_\infty^2 (q^{18}; q^{18})_\infty}{f^2(-q^6)} \\ & - \frac{R(18, 18, 1, 1, 1, 2)\{R(2, 1, 1, 1, 2, 2)\}^2 q^2 (q^2; q^2)_\infty}{f^2(-q^6)} \\ & = \frac{(q^9; q^{18})_\infty^4 (q^{18}; q^{18})_\infty^2}{\{R(1, 1, 1, 1, 2, 2) - q \cdot R(9, 9, 1, 1, 1, 2)\}^2}. \end{aligned} \tag{3.3}$$

Proof. With the help of (1.9) and (1.10), we can compute the values for $\phi(q)$, $\phi(q)$, $\phi(-q^9)$, $\psi(q)$, $\psi(q^2)$, $\psi(q^9)$, $\psi(q^{18})$, respectively. Further, using these results into (2.1) and (2.2), and applying results (1.19) and (1.21), by little algebra, we get (3.1) and (3.2). Again, squaring on both the sides of (3.2) and using (3.1), we obtain (3.3).

Acknowledgement

The authors thank the anonymous referees for their valuable suggestions which led to the improvement of the manuscript.

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