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A REMARK ON LEFT NEAR-COMMUTATIVITY

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Abstract

This note concerns the relations between left near-commutative rings and left duo rings. For the purposes, we study the structure of the skewed trivial extensions in relation with the cases of given monomorphisms to be non-surjective or surjective.

1. Introduction

This study is motivated by the results for right near-commutative rings

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which are obtained by Lee in [8]. Throughout, every ring is associative with identity unless otherwise stated. Let R be a ring. Let $J(R)$, $N_*(R)$, $N^*(R)$, and $N(R)$ denote the Jacobson radical, the prime radical, the upper nilradical (i.e., sum of all nil ideals), and the set of all nilpotent elements in R, respectively. It is well-known that $N^*(R) \subseteq J(R)$ and $N_*(R) \subseteq N^*(R) \subseteq N(R)$. It is also well-known that $N_*(R) = N^*(R) = N(R)$ for any commutative ring R. The n by n full (resp. upper triangular) matrix ring over R is denoted by $Mat_n(R)$ (resp., $T_n(R)$). Let $D_n(R)$ denote the subring $\{M \in T_n(R) |$ the diagonal entries of M are all equal$\}$ of $T_n(R)$. Use e_{ij} for the matrix with (i, j)-entry 1 and 0 elsewhere. $\mathbb{Z}$ $(\mathbb{Z}_n)$ is used to denote the ring of integers (modulo n). $r_R(-)$(resp. $l_R(-)$) is used to denote a right (resp. left) annihilator over R. The group of units in R is denoted by $U(R)$. The center of a ring R is denoted by $C(R)$.

Following Lambek [7], a right R-module M over a ring R is called *symmetric* if $mrs = 0$ implies $msr = 0$ for all $m \in M$ and $r, s \in R$. Thus, a ring R is usually called *symmetric* if $rst = 0$ implies $rts = 0$ for $r, s, t \in R$; while Anderson and Camillo [1] took the term ZC_3 (consequently, ZC_n) for this notion. Lambek proved that a ring R is symmetric if and only if $r_1 r_2 \cdots r_n = 0$, with n any positive integer, implies R is symmetric if and only if $r_{\sigma(1)} r_{\sigma(2)} \cdots r_{\sigma(n)} = 0$ for any permutation σ of the set $\{1, 2, ..., n\}$ and $r_i \in R$ [7, Proposition 1]; while Anderson and Camillo obtained this result independently [1, Theorem 1.1]. This result implies that R is symmetric if and only if R_R is symmetric if and only if ${}_R R$ is symmetric.

Recall that a ring called *Abelian* if every idempotent is central. Symmetric rings are easily shown to be Abelian.

Proposition 1.1 [8, Proposition 2.1]**.** *For a ring R, the following conditions are equivalent*:

(1) *Every right R-module is symmetric*;

(2) *Every cyclic right* R*-module is symmetric*;

(3) $abR = baR$ *for all* $a, b \in R$.

Following Lee [8], a ring R is said to be *right near-commutative* if it satisfies the condition (3) of Proposition 1.1. A ring R is called *left near-commutative* if it satisfies the left version of Proposition 1.1(3), i.e., $Rab = Rba$ for all $a, b \in R$. A ring is called *near-commutative* if it is both right and left near-commutative. Commutative rings are clearly near-commutative. Due to Feller [4], a ring is called *right* (resp. *left*) *duo* if every right (resp. left) ideal is an ideal; a ring is called *duo* if it is both right and left duo. It is obvious that right (resp. left) near-commutative rings are right (resp. left) duo, but the converse is not true in general by [8, Example 2.6].

A ring is usually called *reduced* if it has no nonzero nilpotent elements. Reduced rings are symmetric by a simple computation, but the converse need not hold as can be seen by the many existences of nonreduced commutative rings. It is also obvious that right or left duo rings are Abelian.

We see the relations among commutative rings, right near-commutative rings, and right duo rings. We consider a skewed trivial extension in [12, Definition 1.3] as follows. Let R be a commutative ring with an endomorphism σ and M be an R-module. For $R \oplus M$, the addition and multiplication are given by

$$(r_1, m_1) + (r_2, m_2) = (r_1 + r_2, m_1 + m_2)$$

and

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, \sigma(r_1)m_2 + m_1 r_2),$$

noting $\sigma(r_1)m_2 = m_2\sigma(r_1)$. Then this construction forms a ring. Following the literature, this extension is called the *skew-trivial extension* of R by M, denoted by $R \propto M$.

As in [8], $R \propto R$ is isomorphic to $R[x; \sigma]/(x^2)$ via the corresponding

$$(r, m) \mapsto r + \bar{x}m,$$

where $R[x; \sigma]$ is the skew polynomial ring, with the coefficients written on the right, only subject to $ax = x\sigma(a)$ for $a \in R$ and (x^2) is the ideal of $R[x; \sigma]$ generated by x^2.

If R is a field and σ is not surjective, then $R[x; \sigma]/(x^2)$ is a right near-commutative (hence right duo) ring that is not left duo; but it is near-commutative when σ is bijective, by the following theorem.

Theorem 1.2 [8, Theorem 2.5]**.** (1) *Let K be a field with a monomorphism σ and M be a K-module. If σ is not surjective, then $K \propto M$ is a right near-commutative ring which is not left duo.*

(2) *Let K be a field with a monomorphism σ and M be a K-module. If σ is bijective, then $K \propto M$ is a near-commutative ring.*

(3) *Let D be a commutative domain, with a monomorphism σ, that is not a field, and K be the quotient field of D. Suppose that σ is not surjective. Then $D \propto K$ is a right duo ring that is neither left nor right near-commutative ring, where K is considered as a bimodule over D.*

Proof. In the proof of [8, Theorem 2.5], the proof of (2) is not given, and the case of left near-commutative ring is not considered in (3). We provide here one of (2) and extend one of [8, Theorem 2.5(3)].

(2) Let $E = K \propto M$ and suppose that σ is bijective. Then E is right near-commutative by (1). We will show that E is also left near-commutative. Let $0 \neq x, y \in E$. If $x = (a, b), y = (c, d) \in J(E)$, then $xy = 0$ because $a = c = 0$, and so

$$Exy = Eyx = xyE = yxE = 0.$$

If $x, y \in U(E)$, then

$$Exy = Eyx = xyE = yxE = E.$$

Let

$$x = (f_1, g_1) \in U(E), \ y = (0, g_2) \in J(E).$$

Then

$$xy = (f_1, g_1)(0, g_2) = (0, \sigma(f_1)g_2) = (0, \sigma(f_1) f_1^{-1}g_2 f_1).$$

Since σ is bijective, $f_1^{-1} = \sigma(h)$ for some $h \in K$. So we have

$$\begin{aligned}(0, \sigma(f_1) f_1^{-1}g_2 f_1) &= (0, \sigma(f_1)\sigma(h)g_2 f_1)\\ &= (0, \sigma(f_1h)g_2 f_1)\\ &= (f_1h, 0)(0, g_2 f_1)\\ &= (f_1h, 0)(0, g_2)(f_1, g_1) \in Eyx,\end{aligned}$$

implying $xy \in Eyx$.

Since σ is bijective, $f_1 = \sigma(k)$ for some $k \in K$. So we have

$$\begin{aligned}yx &= (0, g_2)(f_1, g_1) = (0, g_2 f_1) = (0, f_1\sigma(f_1^{-1})\sigma(f_1)g_2)\\ &= (0, \sigma(k)\sigma(f_1^{-1})\sigma(f_1)g_2) = (0, \sigma(kf_1^{-1})\sigma(f_1)g_2)\\ &= (kf_1^{-1}, 0)(f_1, g_1)(0, g_2) \in Exy.\end{aligned}$$

Thus, E is left near-commutative.

(3) Let $E = D \propto K$ and $(f, g), (h, k) \in E$. We first show that E is right duo. Let $f \neq 0$. Then we have

$$\begin{aligned}(h, k)(f, g) &= (hf, \sigma(h)g + kf)\\ &= (fh, gh + \sigma(f)\sigma(f)^{-1}(\sigma(h)g + kf - gh))\\ &= (f, g)(h, \sigma(f)^{-1}(\sigma(h)g + kf - gh)) \in (f, g)E.\end{aligned}$$

If $f = 0$ and $g \neq 0$, then

$$(h, k)(0, g) = (0, \sigma(h)g) = (0, g\sigma(h)) = (0, g)(\sigma(h), 0) \in (0, g)E.$$

Thus, E is right duo.

Next we claim that E is neither left nor right near-commutative, assuming that σ is not surjective. Then $s \notin \sigma(D)$ for some $s \in D$.

Set $x = (0, \sigma(s)^{-1})$ and $y = (s, 0) \in E$. Then

$$xy = (0, s\sigma(s)^{-1}) \text{ and } yx = (s, 0)(0, \sigma(s)^{-1}) = (0, 1).$$

So we have that

$$xy = (0, s\sigma(s)^{-1}) \neq (f, g)(0, 1) = (0, \sigma(f)) \in Eyx$$

and

$$xy = (0, s\sigma(s)^{-1}) \neq (0, 1)(f, g) = (0, f) \in yxE$$

for any $(f, g) \in E$, because $s\sigma(s)^{-1} \notin D$ from the fact that $s \notin \sigma(D)$. Thus, E is neither left nor right near-commutative.

Consider the proof of (3) of Theorem 1.2. For the case of $f \neq 0$, we also have

$$\begin{aligned}(f, g)(h, k) &= (fh, \sigma(f)k + gh)\\ &= (fh, \sigma(h)g + f^{-1}(\sigma(f)k + gh - \sigma(h)g)f)\\ &= (h, f^{-1}(\sigma(f)k + gh - \sigma(h)g))(f, g) \in E(f, g).\end{aligned}$$

Let $f = 0$ and $g \neq 0$. In this case, suppose that σ is bijective. Then we also have

$$(0, g)(h, k) = (0, gh) = (0, \sigma(\sigma^{-1}(h))g) = (\sigma^{-1}(h), 0)(0, g) \in E(0, g),$$

noting $h \in \sigma(D)$. Thus, S is also left duo.

One can see the applications of Theorem 1.2 in [8, Example 2.6].

2. Left Near-commutative Rings

In this section, we study the structure of left near-commutative rings in relation with several ring theoretic properties which have roles in noncommutative ring theory. Recall that a ring R is said to be *left near-commutative* if $Rab = Rba$ for all $a, b \in R$. Left near-commutative rings are clearly left duo, but not conversely by Example 2.3 to follow. Note that left near-commutative rings are Abelian since left duo rings are Abelian. In this section, we study the basic properties of left near-commutative rings introduced by Lee [8]. Our arguments are motivated by the results for right near-commutative rings.

Following Lambek [7], a left R-module M over a ring R is called *symmetric* if $rsm = 0$ implies $srm = 0$ for all $m \in M$ and $r, s \in R$.

Proposition 2.1. *For a ring R, the following conditions are equivalent*:

(1) *Every left R-module is symmetric*;

(2) *Every cyclic left R-module is symmetric*;

(3) $Rab = Rba$ *for all* $a, b \in R$.

Proof. We apply the proof of Proposition 1.1. (2) $\Rightarrow$ (3) Consider the cyclic left R-module R/Rab. By condition, $ab(1 + Rab) = 0$ implies $ba(1 + Rab) = 0$, entailing $Rba \subseteq Rab$. From $ba(1 + Rba) = 0$, we get $ab(1 + Rba) = 0$, implying $Rab \subseteq Rba$.

(3) $\Rightarrow$ (1) Let M be a left R-module, and suppose $abm = 0$ for $m \in M$ and $a, b \in R$. Then $Rab \subseteq l_R(m)$, but $Rab = Rba$ by the condition; hence $Rba \subseteq rl_R(m)$, implying $bam = 0$.

(1) $\Rightarrow$ (2) is obvious. □

We consider the left version of the skewed trivial extension, as in [5, Example 1.2(2)]. Let R be a commutative ring with an endomorphism σ and M be an R-module. Applying [12, Definition 1.3], for $R \oplus M$, the addition and multiplication are given by

$$(r_1, m_1) + (r_2, m_2) = (r_1 + r_2, m_1 + m_2)$$

and

$$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1\sigma(r_2)),$$

noting $\sigma(r_2)m_1 = m_1\sigma(r_2)$. Then this construction also forms a ring. This extension is also called the *skew-trivial extension* of R by M, denoted by $R \propto M$.

As in [8], $R \propto R$ is isomorphic to $R[x; \sigma]/(x^2)$ via the corresponding

$$(r, m) \mapsto r + m\bar{x},$$

where $R[x; \sigma]$ is the skew polynomial ring, with the coefficients written on the left, only subject to $xa = \sigma(a)x$ for $a \in R$ and (x^2) is the ideal of $R[x; \sigma]$ generated by x^2.

If R is a field and σ is not surjective, then $R[x; \sigma]/(x^2)$ is a left near-commutative (hence left duo) ring that is not right duo; but it is near-commutative when σ is bijective, by the following theorem.

Theorem 2.2. (1) *Let K be a field with a monomorphism σ and M be a K-module. If σ is not surjective, then $K \propto M$ is a left near-commutative ring which is not right duo.*

(2) *Let K be a field with a monomorphism σ and M be a K-module. If σ is bijective, then $K \propto M$ is a near-commutative ring.*

(3) *Let D be a commutative domain, with a monomorphism σ, that is not a field, and K be the quotient field of D. If σ is not surjective, then $D \propto K$ is a left duo ring that is neither left nor right near-commutative ring, where K is considered as a bimodule over D.*

Proof. We apply the proofs of [8, Theorem 2.5] and Theorem 1.2.

(1) Let $E = K \propto M$. First note that $J(E) = 0 \oplus M$ and $E/J(E) \cong K$, and so

$$U(E) = \{(r, m) \in E \mid r \neq 0\}.$$

Assume that σ is not surjective. Then $s \notin \sigma(K)$ for some $s \in K$. Consider $0 \neq (0, m) \in E$. Then

$$(s, 0)(0, m) = (0, sm) \neq (0, \sigma(t)m) = (0, m)(t, n)$$

for any $(t, n) \in E$ since $s \notin \sigma(K)$. This means

$$(s, 0)(0, m) \notin (0, m)E,$$

and hence E is not a right duo ring.

Next we claim that E is left near-commutative. Let $0 \neq x, y \in E$. If $x = (a, b),\ y = (c, d) \in J(E)$, then $xy = 0$ because $a = c = 0$, and so

$$Exy = Eyx = xyE = yxE = 0.$$

If $x, y \in U(E)$, then

$$Exy = Eyx = xyE = yxE = E.$$

Let

$$x = (f_1, g_1) \in U(E), \quad y = (0, g_2) \in J(E).$$

Then

$$xy = (f_1, g_1)(0, g_2) = (0, f_1g_2) = (0, f_1\sigma(f_1^{-1})g_2\sigma(f_1))$$

$$= (f_1\sigma(f_1^{-1}), 0)(0, g_2\sigma(f_1)) = (f_1\sigma(f_1^{-1}), 0)(0, g_2)(f_1, g_1) \in Eyx$$

and

$$yx = (0, g_2)(f_1, g_1) = (0, g_2\sigma(f_1))$$

$$= (0, \sigma(f_1)f_1^{-1}f_1g_2) = (\sigma(f_1)f_1^{-1}, 0)(f_1, g_1)(0, g_2) \in Exy.$$

Thus, E is left near-commutative.

(2) Let $E = K \propto M$ and suppose that σ is bijective. Then E is left near-commutative by (1). We will show that E is also right near-commutative.

Let $0 \neq x,\ y \in E.$ If $x = (a,\ b),\ y = (c,\ d) \in J(E),$ then $xy = 0$ because $a = c = 0,$ and so

$$Exy = Eyx = xyE = yxE = 0.$$

If $x,\ y \in U(E),$ then

$$Exy = Eyx = xyE = yxE = E.$$

Let

$$x = (f_1,\ g_1) \in U(E), \quad y = (0,\ g_2) \in J(E).$$

Then

$$xy = (f_1,\ g_1)(0,\ g_2) = (0,\ f_1 g_2) = (0,\ g_2 \sigma(f_1)\sigma(f_1^{-1}) f_1).$$

Since σ is bijective, $f_1 = \sigma(v)$ for some $v \in K.$ So we have

$$\begin{aligned}(0,\ g_2\sigma(f_1)\sigma(f_1^{-1}) f_1) &= (0,\ g_2\sigma(f_1)\sigma(f_1^{-1})\sigma(v)) \\ &= (0,\ g_2\sigma(f_1)\sigma(f_1^{-1}v)) \\ &= (0,\ g_2)(f_1,\ g_1)(f_1^{-1}v,\ 0) \in yxE,\end{aligned}$$

implying $xy \in yxE.$

Since σ is bijective, $f_1^{-1} = \sigma(w)$ for some $w \in K.$ So we have

$$\begin{aligned}yx = (0,\ g_2)(f_1,\ g_1) &= (0,\ g_2\sigma(f_1)) \\ &= (0,\ g_2 f_1 f_1^{-1}\sigma(f_1)) \\ &= (0,\ g_2 f_1\sigma(w)\sigma(f_1)) = (0,\ g_2 f_1\sigma(wf_1)) \\ &= (f_1,\ g_1)(0,\ g_2)(wf_1,\ 0) \in xyE.\end{aligned}$$

Thus, E is right near-commutative.

(3) Let $E = D \propto K$ and $0 \neq (f, g), (h, k) \in E.$ We first show that E is left duo. Let $f \neq 0.$ Then we have

$$
\begin{aligned}
(f, g)(h, k) &= (fh, fk + g\sigma(h)) \\
&= (hf, hg + \sigma(f)\sigma(f)^{-1}(fk + g\sigma(h) - hg)) \\
&= (h, \sigma(f)^{-1}(fk + g\sigma(h) - hg))(f, g) \in E(f, g).
\end{aligned}
$$

Let $f = 0$ and $g \neq 0.$ Then we have

$$(0, g)(h, k) = (0, g\sigma(h)) = (0, \sigma(h)g) = (\sigma(h), 0)(0, g) \in E(0, g).$$

Thus, E is left duo.

Next we show that E is not left near-commutative. Assume that σ is not surjective. Then $s \notin \sigma(D)$ for some $s \in D.$

Set $x = (0, \sigma(s)^{-1})$ and $y = (s, 0) \in E.$ Then

$$xy = (0, \sigma(s)^{-1}\sigma(s)) = (0, 1)$$

and

$$yx = (s, 0)(0, \sigma(s)^{-1}) = (0, s\sigma(s)^{-1}).$$

But

$$yx = (0, s\sigma(s)^{-1}) \neq (0, \sigma(f)) = (0, 1)(f, g) \in xyE$$

and

$$yx = (0, s\sigma(s)^{-1}) \neq (f, g)(0, 1) = (0, f) \in Exy$$

for any $(f, g) \in E$ because $s\sigma(s)^{-1} \notin D$ from the fact that $s \notin \sigma(D).$

Thus, E is neither left nor right near-commutative. $\square$

Consider the proof of (3) of Theorem 2.2. For the case of $f \neq 0,$ we also have

$$(h, k)(f, g) = (hf, hg + k\sigma(f))$$

$$= (hf, g\sigma(h) + ff^{-1}(hg + k\sigma(f) - g\sigma(h)))$$

$$= (f, g)(h, f^{-1}(hg + k\sigma(f) - g\sigma(h))) \in (f, g)E.$$

Let $f = 0$ and $g \neq 0.$ In this case, suppose that σ is bijective. Then we also have

$$(h, k)(0, g) = (0, hg) = (0, g\sigma(\sigma^{-1}(h))) = (0, g)(\sigma^{-1}(h), 0) \in (0, g)E,$$

noting $h \in \sigma(D).$ Thus, E is also right duo.

In the following we see elaborations of Theorem 2.2.

Example 2.3. (1) There exists a left near-commutative ring that is not right duo. The construction is due to Chatters and Xue [2]. Let F be a field, $F[x]$ be the polynomial ring with an indeterminate x over F, and $F(x)$ be the quotient field of $F[x]$.

Let E be the ring $F(x) \times F(x)$ with operations $(f_1(x), g_1(x)) + (f_2(x), g_2(x)) = (f_1(x) + f_2(x), g_1(x) + g_2(x))$ and

$$(f_1(x), g_1(x))(f_2(x), g_2(x)) = (f_1(x) f_2(x), f_1(x) g_2(x) + g_1(x) f_2(x^2))$$

for $f_i(x), g_i(x) \in F(x).$ Then E is isomorphic to the skew-trivial extension $K \propto M$ with $K = F(x) = M$ and

$$\sigma\left(\frac{f(x)}{g(x)}\right) = \frac{f(x^2)}{g(x^2)}.$$

So E is left near-commutative but not right duo by Theorem 2.2(1).

(2) There is a duo ring that is neither left nor right near-commutative. This construction is due to [8, Example 3], modifying the ring and monomorphism in (1). Let E be the subring $F[x] \times F[x]$ of $F(x) \times F(x)$ with operations

$$(f_1(x),\ g_1(x)) + (f_2(x),\ g_2(x)) = (f_1(x) + f_2(x),\ g_1(x) + g_2(x))$$

and

$$(f_1(x),\ g_1(x))(f_2(x),\ g_2(x)) = (f_1(x)f_2(x),\ f_1(x)g_2(x) + g_1(x)f_2(x^2))$$

for $f_1(x),\ f_2(x) \in F[x]$ and $g_1(x),\ g_2(x) \in F(x)$. Then E is isomorphic to the skew-trivial extension $D \propto K$ with $K = F(x)$, $D = F[x]$, and $\sigma(f(x)) = f(x^2)$. So E is a left duo ring that is neither left nor right near-commutative by Theorem 2.2(3), letting $s = x$ we get $s \notin \sigma(D)$ and $s\sigma(s)^{-1} = xx^{-2} = x^{-1} \notin D$.

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