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SOME PROPERTIES OF THE INTEGRAL BOX PROBLEM

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Abstract

The box problem is a problem concerning a rectangular sheet of paper which is cut from each corner by an identical square and finding the size of the excised square so that the resulting paper folds into an open box of maximum volume. In [2], Hotchkiss obtained the necessary and sufficient conditions for the existence of the integral solution and the rational solution to the box problem. Our aim in this paper is to study various properties concerning the integral volume of the box. Under the existence of the integral solution, some conditions about the integral volume are obtained and the minimum volume is known. Moreover, there are finitely many box problems that have the same solution and the same integral volume.

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1. Introduction

Let a and b be side-lengths of a rectangular sheet of paper which is cut from each corner by the identical square of a side-length s. In general, the open box obtained by folding the resulting paper has the maximum volume when s is a positive real number satisfying

$$\frac{d}{ds}(a-2s)(b-2s)s = 0 \text{ and } \frac{d^2}{ds^2}(a-2s)(b-2s)s < 0,$$

and hence, using an elementary calculation, one can show that

$$s = \frac{(a+b)-\sqrt{a^2-ab+b^2}}{6}.$$

This event is known as the box problem and s is the solution to the problem. If we are interested in finding the rational solution, then the problem is called the rational box problem. Similarly, if the solution is an integer, then the problem is called the *integral box problem*. Besides the solution above, in 2002, Hotchkiss [2] used some geometric argument to obtain the necessary and sufficient conditions for the existence of the solution to the rational box problem and the integral box problem in the case that a and b are rational numbers as follows.

Theorem 1.1 (Rational box problem) [2]**.** *The rational solution to the box problem, obtained by the paper of rational side-lengths a and b, exists if and only if* $a = r(1-m^2)$ *and* $b = r(2m-m^2)$, *where r and m are rational numbers with* $0 < m < 1$.

Theorem 1.2 (Integral box problem) [2]**.** *The integral solution to the box problem, obtained by the paper of rational side-lengths a and b, exists if and only if* $a = r(1-m^2)$, $b = r(2m-m^2)$, *and* $r(m-m^2)$ *is an even integer, where r and m are rational numbers with* $0 < m < 1$.

Note that by the theorems above if the solution to the box problem exists, then one can show that $s = \frac{r(m-m^2)}{2}$. That is why $r(m-m^2)$ must be divided by 2 for the case of the existence of the integral solution. Moreover, the volume of the box is

$$V = (a - 2s)(b - 2s)s = \frac{r^3(m - m^2)^2}{2} = 2rs^2.$$

For examples, the pair $(a, b) = (3, 8)$ yields the rational solution $s = \frac{2}{3}$ since a and b satisfy Theorem 1.1 when $r = \frac{25}{3}$ and $m = \frac{4}{5}$, and the pair $(a, b) = (5, 8)$ yields the integral solution $s = 1$ since a and b satisfy Theorem 1.2 when $r = 9$ and $m = \frac{2}{3}$.

Now consider the box problem with distinct integral side-lengths. That is, a and b are distinct positive integers. Without of loss of generality, we assume that $a < b$. We say that a pair (a, b) is minimal for the box problem if (a, b) gives rise to a solution to the problem and for each pair (c, d) with $c < d$ giving rise to a solution to such a problem, $a < c$ or, $a = c$ and $b < d$. In 2009, Chuang [1] showed that the pairs (3, 8) and (5, 8) are minimal for the rational box problem and the integral box problem, respectively.

In this paper, we focus on the integral box problem obtained by the paper of rational side-lengths. Some various properties are obtained, including the properties about the volume of the box.

2. Integral Box Problem

Throughout this section, we assume that the integral box problem, obtained by the paper of rational side-lengths a and b, has the solution. By Theorem 1.2, there exist $r, m \in \mathbb{Q}^+$ with $0 < m < 1$ such that

$$a = r(1 - m^2) \in \mathbb{Q}^+, \; b = r(2m - m^2) \in \mathbb{Q}^+, \text{ and } s = \frac{r(m - m^2)}{2} \in \mathbb{Z}^+.$$

Here s is the solution to the problem which is the side-length of the excised square. Moreover, the volume of the box is

$$V = (a - 2s)(b - 2s)s = 2rs^2.$$

Theorem 2.1. *With the above notations,* $r \in \mathbb{Z}^+$ *if and only if* $a, b \in \mathbb{Z}^+$. *In other words, r is a positive integer if and only if the paper dimensions are integers.*

Proof. Assume that $m = \frac{u}{t}$ for some $u, t \in \mathbb{Z}^+$, $\gcd(u, t) = 1$ and $u < t$. Note that $\gcd(u, t) = 1$ implies $\gcd(u, t^2) = \gcd(t - u, t^2) = 1$.

($\Rightarrow$) Suppose that $r \in \mathbb{Z}^+$. Since $\frac{ru(t - u)}{t^2} = rm(1 - m) = 2s \in \mathbb{Z}^+$, we obtain that $t^2 \mid ru(t - u)$, and so $t^2 \mid r$ because of $\gcd(u(t - u), t^2) = 1$. We assume that $r = t^2 l$ for some $l \in \mathbb{Z}^+$. Therefore,

$$a = r(1 - m^2) = \frac{r(t^2 - u^2)}{t^2} = l(t^2 - u^2) \in \mathbb{Z}^+,$$

and

$$b = r(2m - m^2) = \frac{r(2ut - u^2)}{t^2} = l(2ut - u^2) \in \mathbb{Z}^+.$$

Moreover, $V = s(a - 2s)(b - 2s) \in \mathbb{Z}^+$ because of $a, b, s \in \mathbb{Z}^+$.

($\Leftarrow$) Suppose that $a, b, V \in \mathbb{Z}^+$ and $r = \frac{p}{q}$ for some $p, q \in \mathbb{Z}^+$ and $\gcd(p, q) = 1$.

We will show that $q = 1$. Since $\frac{p(t^2 - u^2)}{qt^2} = r(1 - m^2) = a \in \mathbb{Z}^+$, we obtain that $q \mid p(t^2 - u^2)$, and so $q \mid (t^2 - u^2)$ because of $\gcd(p, q) = 1$. Similarly,

$$\frac{p(2ut - u^2)}{qt^2} = r(2m - m^2) = b \in \mathbb{Z}^+,$$

and

$$\frac{p(ut - u^2)}{qt^2} = rm(1 - m) = 2s \in \mathbb{Z}^+,$$

imply $q \mid (2ut - u^2)$ and $q \mid (ut - u^2)$. Hence, $q \mid \gcd(2ut - u^2, ut - u^2)$. However,

$$\begin{aligned} \gcd(2ut - u^2, ut - u^2) &= \gcd(2ut - u^2 - (ut - u^2), ut - u^2) \\ &= \gcd(ut, ut - u^2) \\ &= \gcd(ut, u^2) \\ &= u \cdot \gcd(t, u) \\ &= u. \end{aligned}$$

Then $q \mid u$, and so $q \mid u^2$. Since $q \mid (t^2 - u^2)$, $q \mid t^2$. Therefore, $q \mid \gcd(u, t^2) = 1$, i.e., $q = 1$. Finally, we can conclude that $r = \frac{p}{q} = p \in \mathbb{Z}^+$. □

Corollary 2.2. *With the above notations,* $r \in \mathbb{Z}^+$ *if and only if* $a, b, V \in \mathbb{Z}^+$. *In other words, r is a positive integer if and only if the paper dimensions and the volume are all integers.*

Proof. Use the fact that if $a, b, \in \mathbb{Z}^+$, then $V = (a - 2s)(b - 2s)s \in \mathbb{Z}^+$. □

Next we are interested in the integral volumes of the boxes in the integral box problem which has the solution. The following theorem shows that the minimum of all possible volumes is 16 and, it occurs when the box is obtained by the square paper of side-length 6 and the excised square of side-length 1.

Theorem 2.3. *If the volumes in the integral box problem are integers, then the minimum of all possible volumes is* 16.

Proof. Suppose that $V \in \mathbb{Z}^+$. The proof is divided into two cases.

Case 1. $s = 1$. Assume that $m = \frac{u}{t}$ and $r = \frac{p}{q}$ for some $u, t, p, q \in \mathbb{Z}^+$, $\gcd(u, t) = \gcd(p, q) = 1$ and $u < t$. Note that $\gcd(u, t) = 1$ implies $\gcd(u, t^2) = \gcd(t - u, t^2) = 1$.

Moreover, $\frac{pu(t-u)}{2qt^2} = \frac{rm(1-m)}{2} = s = 1$, i.e., $2qt^2 = pu(t-u)$. Since $\frac{2p}{q} = 2r = 2rs^2 = V \in \mathbb{Z}^+$, we have $q \mid 2p$, and so $q \mid 2$ because of $\gcd(p, q) = 1$. That is, $q = 1$ or $q = 2$.

Case 1.1. $q = 1$. We have $2t^2 = pu(t-u)$. Then $t^2 \mid p$, i.e., there exists $l \in \mathbb{Z}^+$ such that $p = t^2 l$. Then $2 = lu(t-u)$. Therefore, we obtain the following possible cases.

(1) If $l = 1$, $u = 1$, then $t - u = 2$, $t = 3$, and so $V = \frac{2p}{q} = 2t^2 l = 18$.

(2) If $l = 1$, $u = 2$, then $t - u = 1$, $t = 3$, and so $V = 2t^2 l = 18$.

(3) If $l = 2$, $u = 1$, then $t - u = 1$, $t = 2$, and so $V = 2t^2 l = 16$.

Case 1.2. $q = 2$. We have $4t^2 = pu(t-u)$. Similarly, there exists $l \in \mathbb{Z}^+$ such that $p = t^2 l$. Then $4 = lu(t-u)$. Note that $\gcd(u, t-u) = 1$. Now we obtain the following possible cases.

(1) If $l = 1$, $u = 1$, then $t - u = 4$, $t = 5$, and so $V = \frac{2p}{q} = t^2 l = 25$.

(2) If $l = 1$, $u = 4$, then $t - u = 1$, $t = 5$, and so $V = t^2 l = 25$.

(3) If $l = 2,\ u = 1,$ then $t - u = 2,\ t = 3,$ and so $V = t^2 l = 18.$

(4) If $l = 2,\ u = 2,$ then $t - u = 1,\ t = 3$ and so $V = t^2 l = 18.$

(5) If $l = 4,\ u = 1,$ then $t - u = 1,\ t = 2,$ and so $V = t^2 l = 16.$

Case 2. $s \geq 2.$ Consider $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = x(1 - x)$ for all $x \in (0, 1).$ We have for each $x \in (0, 1),$

$$f(x) > 0,\ \frac{1}{f(x)} > 0,\ f'(x) = 1 - 2x \text{ and } f''(x) = -2.$$

Then $f(x)$ has the maximum value at $x = \frac{1}{2}$ and $f\left(\frac{1}{2}\right) = \frac{1}{4}.$ Therefore

$$x(1 - x) \leq \frac{1}{4},\ \text{i.e.,}\ \frac{1}{x(1 - x)} \geq 4$$

for all $x \in (0, 1).$ Since $0 < m < 1$ and $s \geq 2,$

$$\frac{1}{m(1 - m)} \geq 4, \text{ and so } r = \frac{2s}{m(1 - m)} \geq 16.$$

Hence, $V = 2rs^2 \geq 128.$

By the two cases above, we conclude that the minimum of the volume is 16. □

Now consider $r = \frac{2^{2k+3}}{2^{2k} - 1} \in \mathbb{Q}^+$ and $m = \frac{2^k - 1}{2^{k+1}} \in \mathbb{Q}^+,$ where $k \in \mathbb{Z}^+.$ It is obvious that $0 < m < 1.$ Furthermore, we can show that

$$a = \frac{2(3 \cdot 2^k - 1)}{2^k - 1},\ b = \frac{2(3 \cdot 2^k + 1)}{2^k + 1},\ \text{and } s = 1.$$

This implies that there are infinitely many integral box problems that have the same solution. If we further fix the volume of the boxes to be some positive integer, then we come up with the question that “Are there still infinitely many integral box problems?" The answer is "No" as seen in the following theorem.

Theorem 2.4. *If the solution to the integral box problems with the paper dimensions a and b, and the integral volume of the boxes are given, then there are finitely many pairs of* (a, b).

Proof. Suppose that $V \in \mathbb{Z}^+$ and s are fixed. Assume that $m = \frac{u}{t}$ and $r = \frac{p}{q}$ for some $u, t, p, q \in \mathbb{Z}^+$, $\gcd(u, t) = \gcd(p, q) = 1$ and $u < t$. Note that $\frac{2ps^2}{q} = 2rs^2 = V \in \mathbb{Z}^+$ implies $1 \le q \le 2s^2$. Then there are finitely many possible values of q because s is given. Now consider each value of q. Since $\frac{pu(t-u)}{qt^2} = rm(1-m) = 2s \in \mathbb{Z}^+$ and $\gcd(u(t-u), t^2) = 1$, we obtain that $t^2 \mid p$. We assume that $p = t^2 l$ for some $l \in \mathbb{Z}^+$. Then $2s = \frac{lu(t-u)}{q}$, and so $lu(t-u) = 2sq \ge 2$. By the fundamental theorem of arithmetic, $2sq = p_1^{a_1} p_2^{a_2} p_3^{a_3} \dots p_n^{a_n}$, where p_i is a prime number and $a_i \in \mathbb{Z}^+$ for all $i = 1, \dots, n$. Let A and B denote the sets

$$\{(l, u, t) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{Z}^+ \mid 2sq = lu(t-u) \text{ and } u < t\},$$

and

$$\{(x, y, z) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{Z}^+ \mid 2sq = xyz\},$$

respectively. We will claim that $|A| = |B|$. Define $\varphi : B \to A$ by $\varphi(x, y, z) = (x, y, y+z)$ for all $(x, y, z) \in B$. For each $(x, y, z) \in B$, $xy(y+z-y) = xyz = 2sq$ and $y + z > y$ which imply that $(x, y, y+z) \in A$. Then φ is well-defined. It is easy to check that φ is injective. For each $(l, u, t) \in A$, $2sq = lu(t-u)$ and $t - u \in \mathbb{Z}^+$ which imply that $(l, u, t-u) \in B$, and also

$\varphi(l, u, t-u) = (l, u, t)$. Then φ is surjective. We have the claim. Since

$$B \subseteq C = \{(i, j, k) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{Z}^+ \mid i, j, k \text{ are factors of } 2sq\},$$

and C is finite, so is A. For each $(l, u, t) \in A$,

$$(a, b) = \left(\frac{l(t^2 - u^2)}{q}, \frac{l(2ut - u^2)}{q}\right).$$

Therefore, for each q there are finitely many pairs of (a, b). Since the number of possible values of q is finite, the theorem is proved. □

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