Far East Journal of Mathematical Sciences (FJMS)
http://www.pphmj.com
http://dx.doi.org/10.17654/MS102071523
Volume 102, Number 7, 2017, Pages 1523-1532 **ISSN: 0972-0871**

WAVE KERNEL FOR SCHRÖDINGER OPERATOR WITH THE MORSE POTENTIAL AND APPLICATIONS

Y. O. Mohamed Abdelhaye[1,2], **Badahi Ould Mohamed**[1] and **M. V. Ould Moustapha**[1,2]

[1]College of Arts and Sciences at Al Qurayat
Al-Jouf University
Kingdom of Saudi Arabia

[2]Faculté des Sciences et Techniques
Université de Nouakchott Al-Aasriya
Nouakchott, Mauritania

Abstract

In this article, we give the solution of the wave equation with the Morse potential in closed explicit form. As applications, we obtain the solutions of the wave equation with constant potential, the telegraph equation and the wave equation on the Lobacheviskii plane in closed explicit forms.

1. Introduction

Consider the following Cauchy problem for the wave equation associated to the Schrödinger operator with the Morse potential:

Received: March 22, 2017; Revised: May 3, 2017; Accepted: July 12, 2017
2010 Mathematics Subject Classification: 97Z55, 87K75, 67A10, 80A40, 90B31, 49T45.
Keywords and phrases: Morse potential, wave equation, hyperbolic plane, the Bessel equation, the Cauchy problem, the Schrödinger operator.

$$\begin{cases} \left(\dfrac{\partial^2}{\partial X^2} - k^2 e^{2X} \right) U(t, X) = \dfrac{\partial^2}{\partial t^2} U(t, X), (t, X) \in \mathbb{R}_+^* \times \mathbb{R}, \\ U(0, X) = 0, U_t(0, X) = U_1(X), U_1 \in C_0^\infty(\mathbb{R}), \end{cases} \qquad (WM)_k$$

where for $k \in \mathbb{R}$,

$$\Lambda_k = \frac{\partial^2}{\partial X^2} - k^2 e^{2X} \tag{1.1}$$

is the Schrödinger operator with Morse diatomic molecular potential on the real line $\mathbb{R}$. The purpose of this paper is to give the solution of the Cauchy problem $(WM)_k$ in closed explicit form and by using this explicit solution we give three applications that is to give in closed explicit form the solutions of the Cauchy problems for the wave equation with constant potential, for the telegraph equation and for the wave equation on Lobacheviskii plane.

The Schrödinger operator with the Morse diatomic molecular potential $-\Lambda^k$ is self-adjoint positive definite and has an absolute continuous spectrum. The importance of the Morse potential for both theory and application in mathematics and physics may be found in literature (Ikeda and Matsumoto [5]). For example, the purely vibrational levels of diatomic molecules with angular momentum $l = 0$ have been described by the Morse potential for long time (Tasseli [9]). Another application is in string theory: the wave equation with the Morse potential is the equation of motion for a re-scaled tachyon field and is nothing but Wheeler-de Wit equation satisfied by the macroscopic loop (Li [8]). To use numerical method for a simple modeling of some diatomic molecules like HCl, H2 and O2 in MATLAB (Fidiani [3]), see information and new results on the Morse potential (Hassanabadi and Zare [4]) and (Znojil [10]).

2. Wave Equation with the Morse Potential

In this section, we give the solution of the Cauchy problem $(WM)_k$ for the wave equation with the Morse potential. We begin with the following lemma:

Lemma 2.1. *For* $k, t, X, X \in \mathbb{R}$, *set*:

$$Z = |k|\sqrt{2e^{X+X'}(\cosh t - \cosh(X - X'))}, \tag{2.1}$$

then

$$\frac{\partial Z}{\partial X} = \frac{1}{2}Z - k^2 e^{X+X'}\sinh(X - X')Z^{-1}, \tag{2.2}$$

$$\frac{\partial^2 Z}{\partial X^2} = \frac{1}{4}Z - k^2 e^{2X}Z^{-1} - k^4 e^{2X+2X'}\sinh^2(X - X')Z^{-3}, \tag{2.3}$$

$$\frac{\partial Z}{\partial t} = k^2 e^{X+X'}\sinh tZ^{-1}, \tag{2.4}$$

$$\frac{\partial^2 Z}{\partial t^2} = k^2 e^{X+X'}\cosh tZ^{-1} - k^4 e^{2X+2X'}\sinh^2 tZ^{-3}. \tag{2.5}$$

The proof of this lemma is straightforward calculation and in consequence is left to the reader.

Proposition 2.2. *The general solution of the wave equation in* $(WM)_k$ *is given by*:

$$W_k(t, X, X') = aJ_0(|k|\sqrt{2e^{X+X'}(\cosh t - \cosh(X - X'))})$$

$$+ bY_0(|k|\sqrt{2e^{X+X'}(\cosh t - \cosh(X - X'))}),$$

where $a, b \in \mathbb{C}$ *and* J_0, Y_0 *are Bessel functions of the first and second kind*, *respectively*.

Proof. Let

$$W(t, X, X') = \Phi(Z), \tag{2.6}$$

where Z is as in (2.1). Then we have

$$D := \frac{\partial^2 W(t, X, X')}{\partial X^2} - \frac{\partial^2 W(t, X, X')}{\partial t^2}$$

$$= \left[\left(\frac{\partial Z}{\partial X}\right)^2 - \left(\frac{\partial Z}{\partial t}\right)^2\right]\Phi''(Z) + \left[\frac{\partial^2 Z}{\partial X^2} - \frac{\partial^2 Z}{\partial t^2}\right]\Phi'(Z). \tag{2.7}$$

Using the formulas (2.2)-(2.5), we obtain

$$D = \left[\frac{1}{4}Z^2 - k^2 e^{X+X'}\sinh(X - X')\right.$$

$$\left. + k^4 e^{2X+2X'}[\sinh^2(X - X') - \sinh^2 t]Z^{-2}\right]\Phi''(Z)$$

$$+ \left[\frac{1}{4}Z - k^2 e^{2X} Z^{-1} - k^2 e^{X+X'}\cosh t Z^{-1}\right.$$

$$\left. + k^4 e^{2X+2X'}[\sinh^2 t - \sinh^2(X - X')]Z^{-3}\right]\Phi'(Z). \tag{2.8}$$

From the Bessel equation (Lebedev [7, p. 98])

$$Z^2\Phi''(Z) + Z\Phi'(Z) + (Z^2 - \nu^2)\Phi(Z) = 0$$

for $\nu = 0$ and $Z \neq 0,$ we have

$$\Phi''(Z) = -Z^{-1}\Phi'(Z) - \Phi(Z).$$

Replacing in (2.8), we obtain

$$D = \left[\frac{1}{4}Z^2 - k^2 e^{X+X'}(\cosh(X - X') + \cosh t)Z^{-1}\right.$$

$$\left. - k^4 e^{2X+2X'}(\sinh^2(X - X') - \sinh^2 t) \times Z^{-3}\right]\Phi'(Z)$$

$$+\left[\frac{1}{2}k^2e^{X+X'}(\cosh(X-X')+\cosh t)\right.$$

$$\left.-k^4e^{2X+2X'}(\sinh^2(X-X')-\sinh^2 t)Z^{-2}\right]\Phi(Z)+k^2e^{2X}\Phi(Z),$$

that is, we have

$$D = A(t,\ X,\ X')[2Z^{-1}\Phi'(Z)+\Phi(Z)]+k^2e^{2X}\Phi(Z) \tag{2.9}$$

with

$$A(t,\ X,\ X') = -\frac{1}{2}k^2e^{X+X'}(\cosh(X-X')+\cosh t)$$

$$-k^4e^{2X+2X'}(\sinh^2(X-X')-\sinh^2 t)Z^{-2}.$$

Taking into account (2.1) and the formula $\sinh^2 y-\sinh^2 z=\cosh^2 y-\cosh^2 z$, we obtain $A(t,\ X,\ X')=0$, and the proof of the proposition is finished.

Theorem 2.3. *The Cauchy problem* $(WM)_k$ *for the wave equation with the Morse potential has the unique solution given by*:

$$U(t,\ X)$$

$$=\int_{|X-X'|<t} J_0\left(|k|\sqrt{2e^{X+X'}(\cosh t-\cosh(X-X'))}\right)f(X')dX'. \tag{2.10}$$

Proof. By Proposition 2.2 and the fact that the uniqueness of the solution of the problem $(WM)_k$ is a consequence of the classical theory of hyperbolic operator, the proof of the theorem will be finished by showing limit conditions. For this, set $\sinh\left(\frac{X-X'}{2}\right)=z\sinh\frac{t}{2}$ to get

$$U(t,\ X) = \int_0^1 J_0\left(2|\, k \,|\sinh\frac{t}{2}\sqrt{e^{X+X'}(1-z^2)}\right)$$

$$\times\left[f\left(X + 2\arg\sinh\left(z\sinh\frac{t}{2}\right)\right) + f\left(X - 2\arg\sinh\left(z\sinh\frac{t}{2}\right)\right)\right]$$

$$\times\frac{\sinh\frac{t}{2}}{\sqrt{(1+z^2\sinh^2\frac{t}{2})}}\, dz. \tag{2.11}$$

It is not hard to see the limit conditions from (2.11) and the formula (Lebedev [7, p. 134])

$$J_\nu(x) \approx \frac{x^\nu}{2^\nu\Gamma(1+\nu)}, \qquad x \to 0. \tag{2.12}$$

3. The Wave Equation with Constant Potential and the Telegraph Equation

In this section, we give two applications of Theorem 2.3, the explicit solutions to the wave equation with constant potential on $\mathbb{R}$ and the telegraph equation

Corollary 3.1. *The Cauchy problem for the wave equation with constant potential*

$$\begin{cases}\left(\dfrac{\partial^2}{\partial X^2} - k^2\right)U(t,\ X) = \dfrac{\partial^2}{\partial t^2}U(t,\ X),\ (t,\ X) \in \mathbb{R}_+^* \times \mathbb{R}, \\ U(0,\ X) = 0,\ U_t(0,\ X) = U_1(X),\ U_1 \in C^\infty(\mathbb{R})\end{cases} \qquad (WC)_k$$

has the unique solution given by

$$U(t,\ X) = \int_{|\, X-X'\,|<t} J_0(|\, k\,|\sqrt{(t^2-(X-X')^2)})f(X')dX', \tag{3.1}$$

where J_0 *is the Bessel function of the first kind and of order* 0.

Proof. Replacing k by $\frac{k}{\lambda}$, X by λX and t by λt and letting $\lambda \to 0$ in the formula (2.10), we get the solution (3.1) of the Cauchy problem for the wave equation with constant potential:

Note that the telegraph equation satisfied by the voltage or the current v as a function of the time t and the position X along the cable from initial point;

$$\begin{cases} \dfrac{\partial^2}{\partial X^2} v(t, X) = \left(\dfrac{\partial^2}{\partial t^2} + (\alpha + \beta)\dfrac{\partial}{\partial t} + \alpha\beta \right) v(t, X), \ (t, X) \in \mathbb{R}_+^* \times \mathbb{R}, \\ v(0, X) = 0, \ v_t(0, X) = v_1(X), \ v_1 \in C^\infty(\mathbb{R}) \end{cases} \qquad (W\mathbb{R})_k$$

can be reduced to the wave equation with constant potential, where $k = \frac{(\alpha - \beta)^2}{4}$; by introducing $U = e^{((\alpha+\beta)/2)t} v$ (Courant and Hilbert [1, pp. 192-193; 695]).

4. Wave Equation on the Lobacheviskii Plane

Let $\mathbb{H}^2 = \{z = x + iy \in \mathbb{C}, \ y > 0\}$ be the hyperbolic Poincaré half plane, endowed with the usual hyperbolic metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2} \tag{4.1}$$

with the hyperbolic surface form $d\mu(z)$:

$$d\mu(z) = \frac{dxdy}{y^2}, \tag{4.2}$$

the hyperbolic distance $d(z, z')$ is given by

$$\cosh d(z, z') = \frac{(x - x')^2 + y^2 + y'^2}{2yy'} \tag{4.3}$$

with the Laplace-Beltrami operator

$$\Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right). \tag{4.4}$$

In this section, we use Theorem 2.3 to give the solution of the Cauchy problem for the wave equation on Lobacheviskii plane associated to modified Laplacian

$$L = \Delta + \frac{1}{4}. \tag{4.5}$$

Corollary 4.1 (See Intissar and Ould Moustapha [6]). *The Cauchy problem for the wave equation on the Lobacheviskii plane*

$$\begin{cases} Lu(t, x, y) = \dfrac{\partial^2}{\partial t^2} u(t, x, y), (t, x, y) \in \mathbb{R}_+^* \times \mathbb{R}, \\ u(0, x, y) = 0, u_t(0, x, y) = u_1(x, y), u_1 \in C^\infty(\mathbb{R}) \end{cases} \tag{$(W\mathbb{H}^2)_k$}$$

has the unique solution given by

$$u(t, z) = \frac{1}{\sqrt{2\pi}} \int_{d(z, z')<t} (\cosh t - \cosh d(z, z'))^{-\frac{1}{2}} f(z') d\mu(z'), \tag{4.6}$$

where $d(z, z')$ *is the geodesic distance on* $\mathbb{H}^2$ *given by* (4.3).

Proof. The Fourier transform with respect to the variable x transforms the Cauchy problems $(W\mathbb{H}^2)_k$ to the following Cauchy problem:

$$\begin{cases} \left[y^2 \dfrac{\partial^2}{\partial y^2} - k^2 y^2 + \dfrac{1}{4}\right] \hat{u}(t, k, y) = \dfrac{\partial^2 \hat{u}(t, k, y)}{\partial t^2}, (t, k, y) \in \mathbb{R}_+^* \times \mathbb{R}, \\ \hat{u}(0, k, y) = 0, \dfrac{\partial \hat{u}(0, k, y)}{\partial t}, \ \hat{f}(k, y), \ f \in C_0^\infty(\mathbb{H}^2). \end{cases} \tag{$(W\mathbb{R})_{\alpha\beta}$}$$

Set

$$\hat{u}(t, k, y) = y^{\frac{1}{2}} v(t, k, y), \quad y = e^X, \quad v(t, k, y) = U(t, k, X). \tag{4.7}$$

In the above problem, we get

$$\begin{cases}\left(\dfrac{\partial^2}{\partial X^2}-k^2e^{2X}\right)U(t, k, X)=\dfrac{\partial^2}{\partial t^2}U(t, k, X), (t, k, X)\in\mathbb{R}_+^*\times\mathbb{R}, \\ U(0, k, X)=0, U_t(0, k, X)=e^{-\frac{X}{2}}\hat{f}(k, e^X), \; f\in C^\infty(\mathbb{R}).\end{cases} \qquad (WM)_k$$

Using Theorem 2.3, we get

$$U(t, k, X)$$

$$= \int_{|X-X'|<t} J_0(|k|\sqrt{2e^{X+X'}(\cosh t-\cosh(X-X'))})e^{-\frac{X'}{2}}\hat{f}(k, e^{X'})dX'.$$

From (4.7), we obtain

$$u(t, z) = \frac{\sqrt{yy'}}{2\pi}\int_{-\infty}^{\infty}\int_{\left|\ln\frac{y}{y'}\right|<t}\int_{-\infty}^{\infty}e^{-ik(x-x')}$$

$$\times J_0(|k|\sqrt{2yy'\cosh t-y^2-y'^2})dkf(x', y')\frac{dx'dy'}{y'^2},$$

$$u(t, z) = \frac{\sqrt{yy'}}{\pi}\int_{-\infty}^{\infty}\int_{\left|\ln\frac{y}{y'}\right|<t}\int_{0}^{\infty}\cos k(x-x')$$

$$\times J_0(k\sqrt{2yy'\cosh t-y^2-y'^2})dkf(x', y')\frac{dx'dy'}{y'^2}$$

and hence by the formula (Ditkine and Prudnikov [2, 7.165, p. 182])

$$\int_0^\infty \cos utJ_0(at)dt = \begin{cases}(a^2-u^2)^{-\frac{1}{2}}, & \text{if } u<a, \\ 0, & u>a\end{cases}$$

and the formulas (4.2) and (4.3), we get the proof of the corollary.

Acknowledgement

The authors thank the anonymous referees for their valuable suggestions and comments.

References

[1] R. Courant and D. Hilbert, Methods of Mathematical Physics, Volume II, Interscience Publisher, 1962.

[2] V. Ditkine and A. Proudnikov, Transformations intégrales et calcul opérationnel, Éditions, Mir, Moscow, 1978.

[3] E. Fidiani, Modeling of diatomic molecule using the Morse potential and the Verlet algorithm, AIP Conferences Proceedings 1719 (2016), 030001.
DOI: 10.1063/1.4943696.

[4] H. Hassanabadi and S. Zare, Investigation of quasi-Morse potential in position-dependent mass formalism, Eur. Phys. J. Plus 132 (2017), 49.
DOI: 10.1140/epjp/i2017-11319-x.

[5] N. Ikeda and H. Matsumoto, Brownian motion one the hyperbolic plane and Selberg trace formula, J. Funct. Anal. 163 (1999), 63-110.

[6] A. Intissar and M. V. Ould Moustapha, Solution explicite de l'équation des ondes dans l'espace symétrique de type non compact de rang 1, C. R. Acad. Sci. Paris Sér. I Math. 321 (1995), 77-80.

[7] N. N. Lebedev; Special Functions and their Applications, Dover Publications, Inc., New York, 1972.

[8] Miao Li, Some remarks on tachyon action in 2d string theory; arXiv:hep:th/9212061vl, 9 Dec. 1992.

[9] H. Taseli, Exact solutions for vibrational levels of Morse potential, J. Phys. A 31 (1998), 779-788.

[10] M. Znojil, Morse potential, symmetric Morse potential and bracketed bound-states energies, Mod. Phys. Letter A 31 (2016), 34.
DOI: http://dx.doi.org/10.1142/S0217732316500887.