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GLOBAL EXACT AND REGULAR SOLUTIONS OF EINSTEIN EQUATIONS FOR A CLOUD OF DUST ON BIANCHI TYPE I COSMOLOGICAL MODELS

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Abstract

Global dynamic of the Einstein-matter density equations is studied. We obtain, using rigorous mathematical methods, that in one case, the models of universe will be existing just for at most a finite time. In another case, we make change of variables to obtain Bianchi type I models as global exact and regular solutions of the system. It is also proved that the models obtained remain empty or evolve asymptotically towards the empty space-time.

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1. Introduction

In this paper, we study the Einstein-matter density equations for a cloud of dust; the background space-time being the Bianchi type I cosmological models, which are an immediate generalization of the flat Friedman-Lemaitre-Robertson-Walker space-time, also known to be the basic space-time of cosmology. In cosmology, homogeneous phenomena such as the one we consider here are relevant. The whole universe is modeled and particles in the kinetic theory may be particles of ionized gas as nebular galaxies or even cluster of galaxies, burning reactors, solar wind, for which only the evolution in time is really significant. In the case we consider, the evolution is governed by the coupled Einstein-matter density system, the Einstein equations for gravitational field inquiring about gravitational effects, whereas the matter density equation indicating the form of the energy momentum tensor which represents the energetic and material content of the space.

The Einstein theory stipulates that the gravitational field, which depends on the two real valued functions a and b, called *potentials of gravitation*, is determined through the Einstein equations, by the material and energetic content of space-time. In our context, the matter density ρ is also a real valued function of the time variable t.

The coupled Einstein-matter density system turns out to be a non-linear second order differential system to determine a, b and ρ.

The Einstein equations for a cloud of dust have been already considered by several authors. But, in general, they have coupled them with the Boltzmann equation [1, 9-12], establishing only local existence, or global existence using the method of characteristics for the Boltzmann equation. But these methods are unclear. Some of them used techniques similar to those of Hopf and built some complicated function spaces for the Einstein equations to show the global existence, without specifying the form of solutions. Another category of authors have also coupled the Einstein equations with the Maxwell equations [2-7] without obtaining exact solutions of general form. Finally, the question of asymptotic behavior of solutions is not generally studied in those works.

In this paper, the method used and the results obtained are original and different from those used in previous works. First of all, we adopt a rigorous method to clearly establish the system of our equations. Then we study and solve the Cauchy problem and the heavy system of constraints arising from the equations, obtaining that the Hamiltonian constraint is automatically solved if and only if initial data are linked by a particular relation. Our investigations also allow us, to obtain that the matter density ρ is known when the potentials of gravitation a and b are known. Next, we prove that if $\dot{b}(0) = \dot{b}_0 < 0,$ then the Bianchi models of universe exist only for at most a finite time. If $\dot{b}_0 > 0,$ then we obtain; after making some change of variables and using a rigorous analysis that, Bianchi type I cosmological models as global exact and regular solutions to the Einstein equations for a cloud of dust. We also clearly show that those models tend asymptotically towards the empty at future infinity.

The paper organizes as follows:

In Section 2, we establish the equations and solve the problem of constraints.

In Section 3, we study the existence of solutions.

In Section 4, we discuss the behavior of the existing solutions.

Finally, Section 5 gives the conclusion.

2. The Equations

Greek indexes $\alpha, \beta, \gamma, \ldots$ range from 0 to 3, and Latin indexes $i, j, k, \ldots$ from 1 to 3. We adopt the Einstein summation convention:

$$A_\alpha B^\alpha = \sum_\alpha A_\alpha B^\alpha.$$

2.1. The Einstein system of equations

We consider the Bianchi type I space-time $(\mathbb{R}^4, g)$ and denote by $x^\alpha = (x^0, x^i) = (t, x^i)$ the usual coordinates in $\mathbb{R}^4;$ where $x^0 = t$ represents the time and (x^i) the space. $g = (g_{\alpha\beta})$ stands for the unknown metric tensor of hyperbolic signature $(-, +, +, +)$ that can be written as:

$$g = -(dt)^2 + a^2(t)(dx^1)^2 + b^2(t)[(dx^2)^2 + (dx^3)^2] \tag{1}$$

in which $a > 0, b > 0$ are unknown continuously differentiable functions of the single variable t.

The Christoffel symbols of g are given by:

$$\Gamma^{\lambda}_{\alpha\beta} = \frac{1}{2} g^{\lambda\mu}[\partial_\alpha g_{\mu\beta} + \partial_\beta g_{\alpha\mu} + \partial_\mu g_{\alpha\beta}]. \tag{2}$$

One considers the case of a cloud of dust whose stress-matter tensor is given by:

$$T_{\alpha\beta} = \rho u_\alpha u_\beta \tag{3}$$

in which $\rho \geq 0$ is an unknown function of the single variable t representing the matter density, $u = (u^\alpha)$ is a unit time-like vector, tangent to the time axis at any point, which means that $u_\alpha u^\alpha = -1$, $u^i = u_i = 0$, $i = 1, 2, 3$. The particles are then supposed to be spatially on rest.

Following [8], the Einstein-matter density equations for the metric tensor g read:

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = 8\pi T_{\alpha\beta}, \tag{4}$$

where

$R_{\alpha\beta}$ is the Ricci tensor, contracted of the curvature tensor, and whose expression is given by:

$$R_{\alpha\beta} = (\partial_\lambda \Gamma^{\lambda}_{\beta\mu} - \partial_\beta \Gamma^{\lambda}_{\lambda\mu}) + (\Gamma^{\lambda}_{\lambda\nu}\Gamma^{\nu}_{\beta\mu} - \Gamma^{\lambda}_{\beta\nu}\Gamma^{\nu}_{\lambda\mu}), \tag{5}$$

$R = g^{\alpha\beta} R_{\alpha\beta}$ is the scalar curvature, contracted of the Ricci tensor;

$T_{\alpha\beta}$ is the energy-momentum tensor defined in equation (3):

2.2. Expression of the Einstein equations in *a* and *b*

Using relations (1), (2) and (5), the components $R_{\alpha\beta}$ of the Ricci tensor and the scalar curvature R are computed to give:

$$\begin{cases} R_{00} = -\dfrac{\ddot{a}}{a} - 2\dfrac{\ddot{b}}{b},\ R_{11} = a\ddot{a} + 2a\dot{a}\dfrac{\dot{b}}{b},\ R_{22} = R_{33} = b\ddot{b} + (\dot{b})^2 + b\dot{b}\dfrac{\dot{a}}{a}, \\ R = 2\left(\dfrac{\ddot{a}}{a} + 2\dfrac{\ddot{b}}{b} + 2\dfrac{\dot{a}}{a}\dfrac{\dot{b}}{b} + \left(\dfrac{\dot{b}}{b}\right)^2\right). \end{cases} \tag{6}$$

Now, using the expression (3) and the fact that $u_\alpha u^\alpha = -1$, $u^i = u_i = 0$, $i = 1, 2, 3;$ one gets:

$$T_{00} = \rho, \quad T_{11} = T_{22} = T_{33} = 0, \quad T_{0i} = 0, \quad i \neq 0. \tag{7}$$

Putting together the relations (6) and (7), since $a > 0$, $b > 0$, the Einstein-matter density equations (4) transform in:

$$2\frac{\dot{a}\dot{b}}{ab} + \left(\frac{\dot{b}}{b}\right)^2 = 8\pi\rho, \tag{8}$$

$$2\frac{\ddot{b}}{b} + \left(\frac{\dot{b}}{b}\right)^2 = 0, \tag{9}$$

$$\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\dot{a}\dot{b}}{ab} = 0. \tag{10}$$

2.3. The Cauchy problem and the constraints

We suppose that $a(0) = a_0$, $\dot{a}(0) = \dot{a}_0$, $b(0) = b_0$, $\dot{b}(0) = \dot{b}_0$, $\rho(0) = \rho_0 > 0$, are given and we take $t \in [0, +\infty[$.

We also consider the quantities H^α defined as follows:

$$H^\alpha = S^{\alpha 0} - 8\pi T^{\alpha 0} \tag{11}$$

in which $S_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}$ is the Einstein tensor appearing in (4).

Invoking the fact that $\nabla_\alpha S^{\alpha\beta} = 0$ and $\nabla_\alpha T^{\alpha\beta} = 0,$ where ∇_α is the usual covariant derivative in g, we obtain the relation:

$$\nabla_\alpha H^\alpha = 0. \tag{12}$$

Using the expression (2) of $\Gamma^\lambda_{\alpha\beta},$ equation (12) gives, since $H^\alpha = H^\alpha(t)$:

$$\partial_0 H^0 + \left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right)H^0 = 0. \tag{13}$$

Now integrating (13) over $[0, t],$ we obtain:

$$H^0(t) = \frac{a_0 b_0^2}{ab^2(t)} H^0(0). \tag{14}$$

The relation (14) shows that $H^0(t) = 0 \Leftrightarrow H^0(0) = 0.$

But $H^0 = 0 \Leftrightarrow S_{00} = 8\pi T_{00}.$ So $S_{00}(0) = 8\pi T_{00}(0) \Leftrightarrow S_{00}(t) = 8\pi T_{00}(t)$ $\Leftrightarrow$ equation (8) holds.

In conclusion, equation (8) is automatically satisfied if and only if

$$2\frac{\dot{a}_0\dot{b}_0}{a_0 b_0} + \left(\frac{\dot{b}_0}{b_0}\right)^2 = 8\pi\rho_0 \tag{15}$$

is satisfied. Equation (8) will be called the *Hamiltonian constraint*.

To solve the Cauchy problem (8)-(9)-(10) with the initial data $a(0) = a_0,$ $\dot{a}(0) = \dot{a}_0,$ $b(0) = b_0,$ $\dot{b}(0) = \dot{b}_0,$ $\rho(0) = \rho_0 > 0$ satisfying (15), we just need to solve the system of equations (9)-(10).

2.4. Expression of the matter density ρ

Differentiating the relation $u_\alpha u^\alpha = -1$ with respect to t, we obtain:

$$u_\alpha \nabla_\alpha u^\beta = 0. \tag{16}$$

Now, the conservation law of $T_{\alpha\beta} = \rho u_\alpha u_\beta$ allows to write:

$$\nabla_\alpha (\rho u^\alpha u^\beta) = 0. \tag{17}$$

Developing and multiplying the relation (17) by u^β and using (16), we obtain the following equation:

$$\nabla_\alpha (\rho u^\alpha) = 0. \tag{18}$$

Solving Equation (18) as in (12), we obtain:

$$\rho(t) = \frac{a_0 b_0^2}{ab^2(t)} \rho_0. \tag{19}$$

The relation (19) shows that $\rho_0 = 0 \Leftrightarrow \rho(t) = 0$ and (3) gives $T_{\alpha\beta}(t) = 0$.

This means that the space-time stays empty during its evolution without creation of the matter.

3. Global Existence of Solutions

Equations (9)-(10) are the Einstein evolution equations.

Equation (9) also writes:

$$\frac{\ddot{b}}{\dot{b}} = -\frac{1}{2}\frac{\dot{b}}{b}. \tag{20}$$

Integrating equation (20), we obtain:

$$\dot{b}\sqrt{b} = \dot{b}_0 \sqrt{b_0}. \tag{21}$$

Integrating equation (21) over $[0, t]$, we get:

$$b^{\frac{3}{2}}(t) = \left(\frac{3}{2}\dot{b}_0 b_0^{\frac{1}{2}}\right) t + b_0^{\frac{3}{2}}. \tag{22}$$

Now, if $\dot{b}_0 < 0,$ since $b > 0,$ the left hand side of relation (22) is nonnegative. So is for the right hand side of the same relation.

Thus

$$\left(\frac{3}{2}\dot{b}_0 b_0^{\frac{1}{2}}\right)t + b_0^{\frac{3}{2}} \geq 0$$

or equivalently,

$$0 \leq t \leq t_0 = \frac{-\frac{2}{3}b_0}{\dot{b}_0}. \tag{23}$$

If $\dot{b}_0 > 0,$ since $t \in [0, +\infty[,$ relation (22) shows that b is defined all over the interval $[0, +\infty[$ and consequently the solution of equation (9) is global.

Posing $u = \frac{\dot{a}}{a},$ $v = \frac{\dot{b}}{b}$ the system of equations (9)-(10) transforms into the following:

$$\dot{u} = -u^2 - uv + \frac{1}{2}v^2, \tag{24}$$

$$\dot{v} = -\frac{3}{2}v^2. \tag{25}$$

Already notice that equation (25) has a unique solution such that $v(0) = v_0,$ where v_0 is given.

In fact, if v_1 and v_2 are two solutions of equation (25) such that $v_1(0) = v_2(0) = v_0,$ then setting

$$w = v_1 - v_2,$$

we get

$$\dot{w} + A(t)w = 0, \tag{26}$$

where

$$A(t) = \frac{3}{2}(v_1 + v_2)(t).$$

But

$$\dot{w} + A(t)w = 0 \Leftrightarrow e^{\int_0^t A(s)\,ds}(\dot{w} + Aw) = 0 \Leftrightarrow \left(\widehat{we^{\int_0^t A(s)\,ds}}\right)^{\cdot} = 0.$$

Integrating the above relation over $[0, t]$ leads to the following:

$$w(t)e^{\int_0^t A(s)\,ds} - w(0) = 0.$$

Using the fact that $w(0) = 0$, we obtain the equality

$$v_1 = v_2.$$

Notice that if

$$\dot{b}_0 = 0, \tag{27}$$

then the only solution of the Einstein equation (9) is

$$b(t) = b_0. \tag{28}$$

Equation (24) is a Riccati equation whose unknown is u. Its integration is possible if we know a particular solution u_0. In this case, we pose

$$u = u_0 + z.$$

z will, then solve a Bernoulli equation.

Now since $v = \frac{\dot{b}}{b}$ is a continuous function of the variable t and since the right hand side of equation (24) is C^∞ with respect to u, and thus locally Lipschitz, the existence of a local solution u of Equation (24), and accordingly of the solution a of equation (10) is then guaranteed by the standard theory on first order ordinary differential systems.

If we set

$$U = \dot{u} + \frac{v}{2}$$

and suppose that

$$\dot{b}_0 > 0, \quad U_0 > 0,$$

then equations (24) and (25) imply that:

$$\dot{U} = -U^2. \tag{29}$$

Equation (29) has a trivial solution $W = 0$. But one knows that if the solutions U and $W = 0$ of equation (29) are equal at a point t_0, then they are equal all over the domain of their existence. So, one will also have:

$$U(0) = 0$$

contradicting the fact that

$$U(0) = U_0 > 0.$$

In conclusion, the function U never vanishes.

Consequently, equation (29) is equivalent to the following equation:

$$-\frac{\dot{U}}{U^2} = +1,$$

which solves to give the solution:

$$U = \frac{U_0}{1 + U_0 t} = u + \frac{v}{2}. \tag{30}$$

One also has using the relation (30):

$$u = \frac{U_0}{1 + U_0 t} - \frac{v}{2}. \tag{31}$$

Invoking the fact that $v = \frac{\dot{b}}{b}$ and using the relation (21) in which $\dot{b}\sqrt{b} = \dot{b}_0\sqrt{b_0}$, one gets:

$$\frac{\dot{b}}{b} = \frac{\dot{b}_0}{\frac{3}{2}\dot{b}_0 t + b_0} = v. \tag{32}$$

Reporting the relation (32) in equation (31), and reminding that $u = \frac{\dot{a}}{a}$, leads to the following:

$$\frac{\dot{a}}{a} = \frac{U_0}{1 + U_0 t} - \frac{\dot{b}_0}{3\dot{b}_0 t + 2b_0}. \tag{33}$$

Integrating the equality (33) over $[0, t]$, we reach the following solution:

$$a(t) = a_0 \sqrt[3]{2b_0}\left(\frac{1 + U_0 t}{\sqrt[3]{3\dot{b}_0 t + 2b_0}}\right), \quad t \geq 0. \tag{34}$$

Relations (22) and (34) are global exact solutions of the system (9)-(10) in a and b.

Consequently, the relation (19) also provides a global solution for the matter density ρ.

4. Discussion

If $\rho(0) = \rho_0 = 0$, then the relation (19) shows that the space-time remains empty during its evolution without creation of the matter.

In case $\rho(0) = \rho_0 \neq 0$:

If $\dot{b}_0 < 0$, then equation (23) shows that the models of universe exist only for at most a finite time.

If $\dot{b}_0 > 0,$ then relations (22) and (34) show that Bianchi type I cosmological models are general global exact regular solutions to the Einstein equations for a cloud of matter.

At late times (that is when $t \to +\infty),$ relations (22) and (34) also show that:

$$ab^2(t) \approx \frac{a_0 b_0 \dot{b}_0 U_0}{2} t^2. \tag{35}$$

Putting together relations (7) and (35), we obtain that:

$$T_{\alpha\beta}(t) \to 0, \quad t \to +\infty. \tag{36}$$

This means that, Bianchi type I cosmological models with dust will asymptotically evolve towards the empty space-time at late times.

5. Conclusion

The physical significance of the work we did in the present paper, is the study of the global dynamics of the Einstein equations for a cloud of dust. We have seen that if $\rho_0 = 0,$ then the space-time remain empty during its evolution without creation of the matter. If $\rho_0 \neq 0,$ then if the derivative of the potential of gravitation *b* with respect to the time at the early time is negative, the Bianchi type I models of universe solve the Einstein-Matter density equations only at most for a finite time. But if this derivative is non negative, then Bianchi type I cosmological models are Global exact and regular solutions for the equations. The discussion reveals also that the models obtained will asymptotically evolve towards the empty space-time at late times.

In our future investigations, we intend to study the asymptotic behavior and the geodesic completeness of global solutions for the Einstein equations with Yang-Mills charge on Bianchi models. These investigations seem to have a great interest. In fact, the system obtained leads to a heavy problem of physical constraints which we find in several natural phenomena.

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