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VARIOUS CENTROIDS AND SOME CHARACTERIZATIONS OF CATENARIES

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Abstract

For every interval $[a, b]$, we denote by $(\bar{x}_A, \bar{y}_A)$ and $(\bar{x}_L, \bar{y}_L)$ the geometric centroid of the area under a catenary curve $y = k\cosh((x-c)/k)$ defined on this interval and the centroid of the curve itself, respectively. Then it is well-known that $\bar{x}_L = \bar{x}_A$ and $\bar{y}_L = 2\bar{y}_A$.

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In this paper, we fix an end point, say 0, and we show that $\bar{y}_L/\bar{x}_L = 2\bar{y}_A/\bar{x}_A$ for every interval with an end point 0 characterizes the family of catenaries among nonconstant positive C^2 functions.

1. Introduction

A well-known property of the catenary curve $y = k\cosh((x-c)/k)$, $k > 0$ is that the ratio of the area under the curve to the arc length of the curve is independent of the interval over which these quantities are concurrently measured. For a positive C^1 function $f(x)$ defined on an interval I and an interval $[a, b] \subset I$, we consider the area $A(a, b)$ over the interval $[a, b]$ and the arc length $L(a, b)$ of the graph of $f(x)$. Then the catenary curve $y = k\cosh((x-c)/k)$, $k > 0$ satisfies for every interval $[a, b] \subset I$, $A(a, b) = kL(a, b)$. This property characterizes the family of catenaries $y = k\cosh((x-c)/k)$ among nonconstant C^2 functions [14]. Thus, we have the following:

Proposition 1.1. *For a nonconstant positive C^2 function $f(x)$ defined on an interval I, the following are equivalent*:

(1) *There exists a positive constant k such that for every interval $[a, b] \subset I$, $A(a, b) = kL(a, b)$.*

(2) *The function $f(x)$ satisfies $f(x) = k\sqrt{1 + y'(x)^2}$, where k is a positive constant.*

(3) *For some $k > 0$ and $c \in \mathbb{R}$,*

$$f(x) = k\cosh\left(\frac{x-c}{k}\right).$$

Two higher dimensional generalizations of Proposition 1.1 were established in [2]. For a positive C^1 function $f(x)$ defined on an interval I

and an interval $[a, b] \subset I$, we denote by $(\bar{x}_A, \bar{y}_A) = (\bar{x}_A(a, b), \bar{y}_A(a, b))$ and $(\bar{x}_L, \bar{y}_L) = (\bar{x}_L(a, b), \bar{y}_L(a, b))$ the geometric centroid of the area under the graph of $f(x)$ defined on this interval and the centroid of the graph itself, respectively. Then, for a catenary curve, we have the following [14]:

Proposition 1.2. *A catenary curve* $y = k\cosh((x - c)/k)$ *satisfies the following*:

(1) *For every interval* $[a, b] \subset I$, $\bar{x}_L(a, b) = \bar{x}_A(a, b)$.

(2) *For every interval* $[a, b] \subset I$, $\bar{y}_L(a, b) = 2\bar{y}_A(a, b)$.

In this paper, we consider intervals with a fixed end point, say 0. For a nonzero real number *x*, we denote by I_x the interval defined by

$$I_x = \begin{cases} [0, x], & \text{if } x > 0, \\ [x, 0], & \text{if } x < 0. \end{cases} \tag{1.1}$$

We also denote by $A(x)$, $L(x)$, $(\bar{x}_A(x), \bar{y}_A(x))$ and $(\bar{x}_L(x), \bar{y}_L(x))$ the area under the graph of $f(x)$, the arc length of the graph of $f(x)$, the geometric centroid of the area under the graph of $f(x)$ and the centroid of the graph itself over the interval I_x, respectively.

Recently, the following characterization was established [5, 11]. See also the recent paper [1].

Proposition 1.3. *For a nonconstant positive* C^2 *function* $f(x)$ *defined on an interval I containing* 0, *the following are equivalent*:

(1) *For every nonzero real number* $x \in I$, $\bar{x}_L(x) = \bar{x}_A(x)$.

(2) *For every nonzero real number* $x \in I$, $\bar{y}_L(x) = 2\bar{y}_A(x)$.

(3) *For some* $k > 0$ *and* $c \in \mathbb{R}$,

$$f(x) = k\cosh\left(\frac{x - c}{k}\right).$$

In Section 3, we prove the following characterization theorem for catenary curves:

Theorem 1.4. *For a nonconstant positive* C^2 *function* $y = f(x)$ *defined on an open interval I containing* $0 \in \mathbb{R}$, *the following are equivalent*:

(1) *For every nonzero real number* $x \in I$,

$$\frac{\bar{y}_L(x)}{\bar{x}_L(x)} = 2\,\frac{\bar{y}_A(x)}{\bar{x}_A(x)}.$$

(2) *For some* $k > 0$ *and* $c \in \mathbb{R}$,

$$f(x) = k\cosh\left(\frac{x-c}{k}\right).$$

Remark 1.5. Suppose that a continuous positive function $y = f(x)$ defined on an open interval I containing $0 \in \mathbb{R}$ satisfies the following for every nonzero real number $x \in I$:

$$\frac{\bar{y}_L(x)}{\bar{x}_L(x)} = \lambda\,\frac{\bar{y}_A(x)}{\bar{x}_A(x)},$$

where λ is a constant. Then, for the function $w(x) = \sqrt{1 + f'(x)^2}$, we have

$$\frac{\lambda}{2}\,\frac{\int_0^x tw(t)dt}{\int_0^x tf(t)dt} = \frac{\int_0^x f(t)w(t)dt}{\int_0^x f(t)^2 dt}. \tag{1.2}$$

By putting $x \to 0$, we get $\lambda = 2$.

In order to find the centroid of polygons, see [4]. For the perimeter centroid of a polygon, we refer [3]. In [12], mathematical definitions of centroid of planar bounded domains were given. For various centroids of higher dimensional simplexes, see [13]. The relationships between various centroids of a quadrangle were given in [7, 10].

Archimedes proved the area properties of parabolic sections and then formulated the centroid of parabolic sections [15]. Some characterizations of parabolas using these properties were given in [6, 8, 9].

2. Some Lemmas

In this section, we prove a lemma which is useful in the proof of Theorem 1.4 stated in Section 1. We consider a positive C^2 function $f(x)$ and a positive C^1 function $w(x)$ which are defined on an interval I containing $0 \in \mathbb{R}$. We denote by $k(x)$ and $\phi(x)$ the functions defined as follows:

$$k(x) = \frac{f(x)}{w(x)} \tag{2.1}$$

and

$$\phi(x) = \int_0^x w(t)f(t)dt - l(x)\int_0^x tw(t)dt, \tag{2.2}$$

where $l(x) = f(x)/x$ defined on the open set $I_0 = I \setminus \{0\}$.

Lemma 2.1. *We consider a positive C^2 function $f(x)$ and a positive C^1 function $w(x)$ which are defined on an interval I containing $0 \in \mathbb{R}$. Suppose that $f(x)$ and $w(x)$ satisfy the following*:

$$\frac{\int_0^x tf(t)dt}{\int_0^x tw(t)dt} = \frac{\int_0^x f(t)^2 dt}{\int_0^x w(t)f(t)dt}, \quad x \in I, \quad x \neq 0. \tag{2.3}$$

Then, on the open $I_2 = \{x \in I_0 \mid k'(x) \neq 0, \phi(x) \neq 0\}$, we have $l'(x) = 0$.

Proof. Suppose that $f(x)$ and $w(x)$ satisfy (2.3). Then, for all $x \in I$, we get

$$\int_0^x tf(t)dt\int_0^x w(t)f(t)dt = \int_0^x tw(t)dt\int_0^x f(t)^2 dt. \tag{2.4}$$

By differentiating (2.4) with respect to the variable *x*, we obtain

$$xf(x)\int_0^x w(t)f(t)dt + w(x)f(x)\int_0^x tf(t)dt$$

$$= xw(x)\int_0^x f(t)^2 dt + f(x)^2\int_0^x tw(t)dt. \tag{2.5}$$

The function $k(x)$ given in (2.1) is a C^1 function on the interval *I*. It follows from (2.5) that for the function $l(x) = f(x)/x$ on the open set $I_0 = I\backslash\{0\}$, we have

$$k(x)\int_0^x w(t)f(t)dt + l(x)\int_0^x tf(t)dt$$

$$= \int_0^x f(t)^2 dt + k(x)l(x)\int_0^x tw(t)dt. \tag{2.6}$$

On the open set I_0, differentiating (2.6) with respect to *x* gives

$$k'(x)\int_0^x w(t)f(t)dt + l'(x)\int_0^x tf(t)dt = (k(x)l(x))'\int_0^x tw(t)dt. \tag{2.7}$$

Together with (2.1), this shows that

$$k'(x)\phi(x) = \psi(x), \tag{2.8}$$

where

$$\phi(x) = \int_0^x w(t)f(t)dt - l(x)\int_0^x tw(t)dt,$$

$$\psi(x) = k(x)l'(x)\int_0^x tw(t)dt - l'(x)\int_0^x tf(t)dt. \tag{2.9}$$

It follows from (2.8) that on the open set I_1 defined by $I_1 = \{x \in I_0 \,|\, \phi(x) \neq 0\}$, $k'(x)$ is given by $k'(x) = \psi(x)/\phi(x)$, and hence on the open set I_1, $k'(x)$ is a C^1 function. That is, $k(x)$ is a C^2 function on the open set I_1.

First, suppose that the open set I_2 defined by $I_2 = \{x \in I_1 \,|\, k'(x) \neq 0\}$ is nonempty. Then, on the open set I_2, from (2.7), we get

$$\int_0^x w(t) f(t) dt + \eta(x) \int_0^x t f(t) dt = \delta(x) \int_0^x t w(t) dt, \tag{2.10}$$

where we put

$$\eta(x) = \frac{l'(x)}{k'(x)}, \quad \delta(x) = \frac{(k(x) l(x))'}{k'(x)}. \tag{2.11}$$

Note that $\eta(x)$ and $\delta(x)$ are C^1 functions on I_2. By differentiating (2.10) with respect to x, we obtain

$$\eta'(x) \int_0^x t f(t) dt = \delta'(x) \int_0^x t w(t) dt, \quad x \in I_2. \tag{2.12}$$

Next, suppose that the open set I_3 defined by $I_3 = \{x \in I_2 \,|\, \eta'(x) \neq 0\}$ is nonempty. Then, on the open set I_3, from (2.12), we get

$$\int_0^x t f(t) dt = h(x) \int_0^x t w(t) dt, \tag{2.13}$$

where

$$h(x) = \frac{\delta'(x)}{\eta'(x)}. \tag{2.14}$$

It follows from (2.13) that the function $h(x)$ is a C^2 function on the open set I_3. Differentiating (2.13) with respect to x shows that

$$x f(x) - x h(x) w(x) = h'(x) \int_0^x t w(t) dt. \tag{2.15}$$

Finally, suppose that the open set I_4 defined by $I_4 = \{x \in I_3 \,|\, h'(x) \neq 0\}$ is nonempty. Then, on the open set I_4, from (2.15), we get

$$(xj(x)w(x))' = xw(x), \tag{2.16}$$

where we put

$$j(x) = \frac{k(x) - h(x)}{h'(x)}. \tag{2.17}$$

It follows from (2.15) that the function $j(x)$ is a C^1 function on the open set I_4.

On the other hand, together with (2.13), (2.3) implies

$$\int_0^x f(t)^2 dt = h(x)\int_0^x w(t) f(t) dt, \quad x \in I_3, \tag{2.18}$$

where $h(x)$ is given in (2.14). Differentiating (2.18) with respect to x gives

$$f(x)^2 - h(x)w(x) f(x) = h'(x)\int_0^x w(t) f(t) dt. \tag{2.19}$$

Hence, as in the discussions above, on the open set I_4, from (2.19), we obtain

$$(j(x)w(x) f(x))' = w(x) f(x), \tag{2.20}$$

where $j(x)$ is defined in (2.17) and we use (2.1).

On the open set I_4, it follows from (2.16) and (2.20) that for the function $l(x) = f(x)/x$,

$$j(x)w(x)(f'(x) - l(x)) = 0, \tag{2.21}$$

and hence we have from $w(x) > 0$,

$$j(x)(f'(x) - l(x)) = 0. \tag{2.22}$$

Since $f'(x) - l(x) = xl'(x)$, it follows from (2.22) that

$$j(x)l'(x) = 0. \tag{2.23}$$

Now, we claim that the derivative $l'(x)$ of the function $l(x)$ vanishes on I_4. Otherwise, on a nonempty open set I_5 contained in I_4, the function $j(x)$ vanishes. Then it follows from (2.20) that on I_5, $w(x)f(x) = 0$, which is a contradiction. This contradiction completes the proof of the claim. Since $l'(x)$ vanishes on I_4, (2.11) implies that $\eta(x) = 0$ on I_4, which contradicts to the hypothesis on I_3. This contradiction shows that I_4 must be empty, that is, $h'(x)$ vanishes on the open set I_3.

Since $h'(x) = 0$ on the open set I_3, it follows from (2.1) and (2.15) that $k = h$, and hence $k'(x)$ also vanishes on I_3. This contradiction shows that the open set I_3 must be empty, that is, $\eta'(x)$ vanishes on the open set I_2.

Together with (2.12), the vanishing of $\eta'(x)$ on the open set I_2 shows that on any fixed connected component I_2^0 of the open set I_2, $\eta(x) = a$ and $\delta(x) = b$ for some constants a and b. Hence, (2.11) yields that on the connected component I_2^0 for some constants c and d,

$$l(x) = ak(x) + c, \quad k(x)l(x) = bk(x) + d. \tag{2.24}$$

This implies that $k(x)$ is a root of the quadratic polynomial $q(t) = at^2 + (c - b)t - d$. If the quadratic polynomial $q(t)$ is nontrivial, then the ratio $k(x)$ must be constant on the connected component $I_2^{(0)}$. This contradiction shows that $q(t)$ is trivial. That is, we have $a = 0$, $b = c$ and $d = 0$, and hence from (2.24), we see that on any fixed connected component I_2^0 of the open set I_2 the function $l(x)$ is constant. This shows that on the open set I_2, $l'(x)$ vanishes. This completes the proof of Lemma 2.1. □

3. Proof of Theorem 1.4

In this section, with the help of Lemma 2.1 in Section 2, we prove Theorem 1.4 stated in Section 1.

Suppose that a nonconstant positive C^2 function $f(x)$ defined on an interval I containing $0 \in \mathbb{R}$ satisfies for every nonzero real number $x \in I$,

$$\frac{\bar{y}_L(x)}{\bar{x}_L(x)} = 2\frac{\bar{y}_A(x)}{\bar{x}_A(x)}. \tag{3.1}$$

Then for all $x \in I_0 = I \backslash \{0\}$, we have

$$\frac{\int_0^x f(t)w(t)dt}{\int_0^x tw(t)dt} = \frac{\int_0^x f(t)^2 dt}{\int_0^x tf(t)dt}, \tag{3.2}$$

where we put

$$w(x) = \sqrt{1 + f'(x)^2}. \tag{3.3}$$

For the ratio $k(x) = f(x)/w(x)$, the function $\phi(x)$ is given by

$$\phi(x) = \int_0^x w(t)f(t)dt - l(x)\int_0^x tw(t)dt, \tag{3.4}$$

where $l(x) = f(x)/x$ for $x \in I_0$, we consider the following open set:

$$I_2 = \{x \in I_0 \,|\, k'(x) \neq 0,\ \phi(x) \neq 0\}. \tag{3.5}$$

Suppose that the open set I_2 is nonempty. Then it follows from Lemma 2.1 that on the open set I_2, we have $l'(x) = 0$. Hence, on any fixed connected component I_2^0 of the open set I_2, we have for some constant a,

$$l(x) = a, \quad f(x) = ax, \quad w(x) = \sqrt{1 + a^2}. \tag{3.6}$$

Furthermore, together with (2.11), this implies that $\eta(x) = 0$ and $\delta(x) = a$. Hence, we get from (2.10) that

$$\int_0^x w(t) f(t) dt = a \int_0^x tw(t) dt. \tag{3.7}$$

This shows that on the connected component I_2^0, $\phi(x) = 0$, which contradicts to the definition of I_2. This contradiction shows that the open set I_2 must be empty. That is, on the open set $J = \{x \in I \mid k'(x) \neq 0\}$, the function $\phi(x)$ vanishes.

We consider two cases as follows:

Case 1. *J* is empty. Then, on the whole interval *I*, the ratio $k(x)$ is constant. Hence, for some constant $k \in \mathbb{R}$, we have $f(x) = kw(x)$. Therefore, Proposition 1.1 implies that for some $c \in \mathbb{R}$,

$$f(x) = k \cosh\left(\frac{x - c}{k}\right). \tag{3.8}$$

Case 2. *J* is nonempty. Then, it follows from the discussions above, we see that on *J*, $\phi(x)$ vanishes. Differentiating $\phi(x)$ in (3.4) shows that on the open set *J*, $l'(x)$ vanishes. Hence, on any fixed connected component J^0 of *J*, $f'(x) = a$ for some nonzero constant *a*. In particular, on *J*, $f''(x)$ vanishes.

Suppose that the complement J^c of the open set *J* has nonempty interior *K*. Then, on the nonempty interior *K*, $k'(x)$ vanishes. Hence, on any fixed connected component K^0 of the nonempty interior *K*, Proposition 1.1 shows that the function $f(x)$ is a constant or a function given by (3.8) for some constants $k > 0$ and *c*. This yields that on *K*, $f(x)$ satisfies either $f'(x) = 0$ or $f''(x) \geq 1/k > 0$, which contradicts to the properties of *f* on *J* because $f(x)$ is a C^2 function. This contradiction shows that the complement J^c of the open set J has empty interior. That is, the open set J is an open dense set contained in the domain I.

Since $l'(x)$ vanishes on J and J is dense in the domain I, we see that on the whole interval I, the derivative $l'(x)$ vanishes. Hence, for some constant $k \in \mathbb{R}$, we have $f(x) = kx$. But, such function cannot be positive on the open interval I containing $0 \in \mathbb{R}$. This contradiction shows that J must be nonempty.

Combining the above two cases completes the proof of (1) $\Rightarrow$ (2).

Conversely, it follows from Proposition 1.2 that (2) $\Rightarrow$ (1). This completes the proof of Theorem 1.4.

References

[1] V. Coll and J. Dodd, A characteristic averaging property of the catenary, Amer. Math. Monthly 123 (2016), 683-688.

[2] V. Coll and M. Harrison, Two generalizations of a property of the catenary, Amer. Math. Monthly 121 (2014), 109-119.

[3] Mark J. Kaiser, The perimeter centroid of a convex polygon, Appl. Math. Lett. 6 (1993), 17-19.

[4] B. Khorshidi, A new method for finding the center of gravity of polygons, J. Geom. 96 (2009), 81-91.

[5] D.-S. Kim, S.-O. Bang and D. W. Yoon, Various centroids and some characterizations of catenary curves, submitted.

[6] D.-S. Kim and D. S. Kim, Centroid of triangles associated with a curve, Bull. Korean Math. Soc. 52 (2015), 571-579.

[7] D.-S. Kim, W. Kim, K. S. Lee and D. W. Yoon, Various centroids of polygons and some characterizations of rhombi, Commun. Korean Math. Soc. 32(1) (2017), 135-145.

[8] D.-S. Kim and Y. H. Kim, On the Archimedean characterization of parabolas, Bull. Korean Math. Soc. 50 (2013), 2103-2114.

[9] D.-S. Kim, Y. H. Kim and S. Park, Center of gravity and a characterization of parabolas, Kyungpook Math. J. 55 (2015), 473-484.

[10] D.-S. Kim, K. S. Lee, K. B. Lee, Y. I. Lee, S. Son, J. K. Yang and D. W. Yoon, Centroids and some characterizations of parallelograms, Commun. Korean Math. Soc. 31(3) (2016), 637-645.

[11] D.-S. Kim, H. T. Moon and D. W. Yoon, Centroids and some characterizations of catenary curves, Commun. Korean Math. Soc. 32(3) (2017), 709-714.

[12] Steven G. Krantz, A matter of gravity, Amer. Math. Monthly 110 (2003), 465-481.

[13] Steven G. Krantz, John E. McCarthy and Harold R. Parks, Geometric characterizations of centroids of simplices, J. Math. Anal. Appl. 316 (2006), 87-109.

[14] E. Parker, A property characterizing the catenary, Math. Mag. 83 (2010), 63-64.

[15] S. Stein, Archimedes: What did he do besides cry Eureka?, Mathematical Association of America, Washington, DC, 1999.