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A TOUR OF EXTREMES: THE GEOMETRY OF A FAMILY OF POLYNOMIALS WITH THREE ROOTS

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Abstract

We study the critical points of complex-valued polynomials of the form $p(z) = (z-1)^k (z - r_1)(z - r_2)$ with $|r_1| = |r_2| = 1$ and $k \in \mathbb{N}$. The Gauss-Lucas theorem guarantees that the critical points of such a polynomial will lie within the unit disk. This paper further explores the location and structure of these critical points. For example, there is a 'desert', the open disk $\left\{z \in \mathbb{C} : \left|z - \frac{2}{k+2}\right| < \frac{k}{k+2}\right\}$, in which the critical points cannot occur. Furthermore, a critical point of such a polynomial almost always determines the polynomial uniquely.

Several recent papers [1, 2, 4] have studied the geometry of polynomials with three roots. Frayer et al. [1] studied the critical points of the family of polynomials

$$\{p : \mathbb{C} \to \mathbb{C} \mid p(z) = (z-1)(z-r_1)(z-r_2), |r_1| = |r_2| = 1\}.$$

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The Gauss-Lucas theorem guarantees that the critical points of such a polynomial will lie in the unit disk. The results of [1] include that a critical point almost always determines p uniquely, and there is a *desert*, $\left\{z \in \mathbb{C} : \left| z - \frac{2}{3} \right| < \frac{1}{3}\right\}$, in which the critical points cannot occur.

For $k, m, n \in \mathbb{N}$, a natural extension of [1] is to study

$$P(k, m, n) = \{p : \mathbb{C} \to \mathbb{C} \mid p(z) = (z-1)^k (z - r_1)^m (z - r_2)^n, \ |r_1| = |r_2| = 1\}.$$

The family $P(1, k, k)$ is characterized in [2]. Similar to [1], a critical point almost always determines $p \in P(1, k, k)$ uniquely, and the unit disk contains a desert in which critical points cannot occur.

Marden's text, *The Geometry of Polynomials* [3], provides an interesting physical interpretation. The critical points of a polynomial are the equilibrium points of a force field. The field is generated by particles placed at the roots of the polynomial, the particles having masses equal to the multiplicity of the roots and attracting with a force inversely proportional to the distance from the particle. Stated differently, the critical points are somehow both "attracted to" and "repelled by" the roots. Except in the case of repeated roots, critical points try to stay as far away from the zeros of the polynomial as possible. For $p \in P(1, k, k)$, the repeated roots at r_1 and r_2, force the critical points not occurring at the repeated roots to be as far away from r_1 and r_2 as possible. This allows the critical points to become close to the single root at $z = 1$. In fact, as k approaches infinity, the desert region vanishes. In Figure 1, the left image corresponds to polynomials in $P(1, 7, 7)$. The interior of the white disk is the desert.

At the opposite extreme is the family of polynomials $P(k, 1, 1)$. We used Geogebra to graphically investigate the critical points of polynomials in $P(k, 1, 1)$. We set the roots in motion around the unit circle and traced the loci of the critical points. In Figure 1, the right image corresponds to polynomials in $P(7, 1, 1)$. The large number of repeated roots at $z = 1$, forces the critical points not occurring at $z = 1$ to be as far away from $z = 1$ as possible. As k approaches infinity, the desert region expands to fill the interior of the unit disk. See Corollary 10. This paper characterizes where the critical points of a $p \in P(k, 1, 1)$ can lie, and to what extent they determine p.

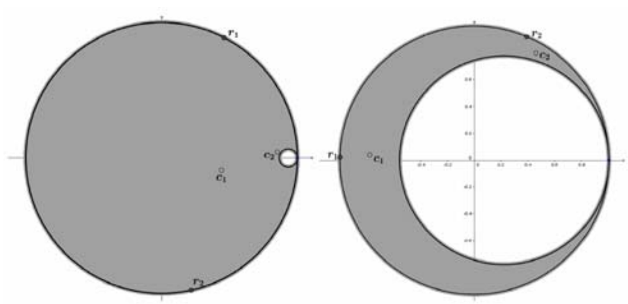

Figure 1. We set the roots in motion around the unit circle and traced the loci of the critical points in grey. Left, corresponds to polynomials in $P(1, 7, 7)$. Right, corresponds to polynomials in $P(7, 1, 1)$.

Preliminary Information

We let T_α denote the circle of diameter α passing through 1 and $1 - \alpha$ in the complex plane. That is,

$$T_\alpha = \left\{ z \in \mathbb{C} : \left| z - \left(1 - \frac{\alpha}{2}\right) \right| = \frac{\alpha}{2} \right\}.$$

For $z \in \mathbb{C}$ with $Re(z) \neq 1$, the following lemma provides a method of calculating the value of α for which $z \in T_\alpha$.

Lemma 1 [1]. *Let* $z \in \mathbb{C}$ *with* $Re(z) \neq 1$. *We have* $z \in T_\alpha$ *if and only if*

$$\frac{1}{\alpha} = Re\left(\frac{1}{1 - z}\right).$$

An additional fact of interest is related to Möbius transformations. Functions of the form $f(z) = e^{i\theta} \dfrac{z - \alpha}{\bar{\alpha} z - 1}$ with $|\alpha| < 1$ are the only one-to-one analytic mappings of the unit disk onto itself [5, p. 334]. This leads to the following useful theorem.

Theorem 2. *A Möbius transformation T sends the unit circle to the unit circle if and only if* $T(z) = \dfrac{\overline{\alpha}z - \overline{\beta}}{\beta z - \alpha}$ *for some* $\alpha, \beta \in \mathbb{C}$ *with* $\left|\dfrac{\alpha}{\beta}\right| \neq 1$.

Critical Points

A polynomial of the form

$$p(z) = (z-1)^k(z-r_1)(z-r_2)$$

with $|r_1| = |r_2| = 1$ and $k \in \mathbb{N}$ has $k+1$ critical points: $k-1$ critical points at $z = 1$, and the two additional critical points in the unit disk. Differentiation gives

$$p'(z) = (z-1)^{k-1}[(k+1)z^2 - ((k+1)(r_1+r_2)-2)z + kr_1r_2 + r_1 + r_2].$$

We define the *nontrivial critical points* of *p* to be the two roots of

$$q(z) = (k+1)z^2 - ((k+1)(r_1+r_2)-2)z + kr_1r_2 + r_1 + r_2.$$

Example 1. Let $p \in P(k, 1, 1)$ have a nontrivial critical point at $z = 1$. Then, by the Gauss-Lucas theorem, the root at $z = 1$ has multiplicity greater than *k*. Therefore, $p \in P(k, 1, 1)$ has a nontrivial critical point at $z = 1$ if and only if $p(z) = (z-1)^{k+1}(z-r)$ for some $r \in T_2$.

Since we know which $p \in P(k, 1, 1)$ have a nontrivial critical point at $z = 1$, we will assume $c \neq 1$ as necessary throughout the paper.

Example 2. Let $p \in P(k, 1, 1)$ have a nontrivial critical point at $c \in T_2$ with $c \neq 1$. Then, by the Gauss-Lucas Theorem, *c* must be a repeated root of *p*. Therefore,

$$p(z) = (z-1)^k(z-c)^2$$

is the only polynomial in $P(k, 1, 1)$ with a nontrivial critical point at $c \in T_2$

with $c \neq 1$. In this case, $p'(z) = (z-1)^{k-1}(z-c)((k+2)z-(kc+2))$, and the second nontrivial critical point is

$$c_2 = \frac{2}{k+2} + \frac{k}{k+2}c \in T_{\frac{2k}{k+2}}.$$

To characterize the critical points of $p \in P(k, 1, 1)$, we will investigate how the roots are related to a nontrivial critical point. Let $p(z) = (z-1)^k(z-r_1)(z-r_2)$ with $|r_1| = |r_2| = 1$, $k \in \mathbb{N}$, and c a nontrivial critical point of p. Then

$$0 = q(c) = (k+1)c^2 - ((k+1)(r_1+r_2)-2)c + kr_1r_2 + r_1 + r_2$$

and it follows that

$$r_1 = \frac{(1-(k+1)c)r_2 + (k+2)c^2 - 2c}{-kr_2 + (k+1)c - 1}. \tag{1}$$

Equation (1) represents the relationship between a nontrivial critical point and the roots r_1 and r_2. With this in mind, we define

$$f_c(z) = \frac{(1-(k+1)c)z + (k+2)c^2 - 2c}{-kz + (k+1)c - 1}$$

and let $S_c = f_c(T_2)$.

Let $c \in \mathbb{C}$. Then f_c is a Möbius transformation with $f_c(r_2) = r_1$. When $c = 1$, $f_1(z) = \frac{-kz+k}{-kz+k} = 1$, and f_1 is not invertible. When $c \neq 1$, f_c is invertible with

$$(f_c)^{-1}(z) = \frac{((k+1)c-1)z - ((k+2)c^2 - 2c)}{kz + 1 - (k+1)c}$$

$$= \frac{(1-(k+1)c)z + (k+2)c^2 - 2c}{-kz + (k+1)c - 1} = f_c(z).$$

Therefore, $f_c(r_1) = r_2$ implies $r_1 = f_c(r_2)$.

Theorem 3. *Suppose* $c \neq 1$ *and* $p(z) = (z-1)^k(z-r_1)(z-r_2) \in P(k, 1, 1)$. *Then* p *has a nontrivial critical point at* c *if and only if* $f_c(r_1) = r_2$ *and* $f_c(r_2) = r_1$.

Because T_2 is a circle, and Möbius transformations send circles to circles or lines, S_c is a circle or a line. In fact, S_c is a line whenever there exists a $z \in T_2$ with $-kz + (k+1)c - 1 = 0$. That is,

$$1 = |z| = \left|\frac{k+1}{k}c - \frac{1}{k}\right| \leftrightarrow \frac{k}{k+1} = \left|c - \frac{1}{k+1}\right|. \tag{2}$$

Therefore, S_c is a line if and only if $c \in T_{\frac{2k}{k+1}}$.

Example 3. Let $c \in T_{\frac{2k}{k+1}}$. Then S_c is a line passing through

$$f_c(1) = \frac{(k+2)c - 1}{k+1} \text{ and } f_c(-1) = \frac{(k+2)c^2 + (k-1)c - 1}{(k+1)c + k - 1}.$$

Algebraic manipulation gives

$$f_c(1) - f_c(-1) = \frac{-2c + 2}{(k+1)((k+1)c + k - 1)}. \tag{3}$$

Since $c \in T_{\frac{2k}{k+1}}$,

$$c = \frac{1}{k+1} + \frac{k}{k+1}e^{i\theta} = \frac{1}{k+1} + \frac{k}{k+1}\cos(\theta) + i\frac{k}{k+1}\sin(\theta)$$

for some $\theta \in [0, 2\pi]$. Substituting into equation (3), we obtain $Re(f_c(1) - f_c(-1)) = 0$. Therefore, when $c \in T_{\frac{2k}{k+1}}$, S_c is a vertical line through $f_c(1)$.

Since $r_1, r_2 \in R_2$, $f_c(r_1) = r_2 \in S_c$ and $f_c(r_2) = r_1 \in S_c$. Therefore, $\{r_1, r_2\} \subseteq S_c \cap T_2$. This fact combined with Theorem 3 leads to the following result. Our proof generalizes Theorem 7 in [2].

Theorem 4. *Suppose* $c \neq 1$.

(1) *If* $S_c \cap T_2 = \varnothing$, *then no* $p \in P(k, 1, 1)$ *has a critical point at* c.

(2) *If* $|S_c \cap T_2| \in \{1, 2\}$, *then there is a unique* $p \in P(k, 1, 1)$ *with a nontrivial critical point at* c.

(3) *If* $S_c \cap T_2 = T_2$, *then* c *is a nontrivial critical point of* $p(z) = (z-1)^k(z-r)(z-f_c(r))$ *for each* $r \in T_2$.

Proof. Suppose $c \neq 1$. In the first case, if $S_c \cap T_2 = \varnothing$, there are no candidates for r_1 and r_2. Hence, no $p \in P(k, 1, 1)$ has a critical point at c.

If $S_c \cap T_2 = \{r\}$, then it follows from the definition of f_c and S_c that $f_c(r) = r$. By Theorem 3, c is a nontrivial critical point of $p(z) = (z-1)^k(z-r)^2$. Furthermore, as $S_c \cap T_2 = \{r\}$, no other $p \in P(k, 1, 1)$ has a nontrivial critical point at c.

Suppose $S_c \cap T_2 = \{a, b\}$ with $a \neq b$. There are two possibilities: $f_c(a) = a$ or $f_c(a) = b$. If $f_c(a) = a$, then by definition of S_c, $f_c(b) = b$. By Theorem 3, c is a nontrivial critical point of $p_1(z) = (z-1)^k(z-a)^2$ and $p_2(z) = (z-1)^k(z-b)^2$, which contradicts the Gauss-Lucas theorem. Therefore, $f_c(a) = b$. Since $(f_c)^{-1} = f_c$, we have $f_c(b) = a$, and Theorem 3 implies that c is a nontrivial critical point of $p(z) = (z-1)^k(z-a)(z-b)$. Moreover, as $S_c \cap T_2 = \{a, b\}$, no other $p \in P(k, 1, 1)$ has a nontrivial critical point at c.

Lastly, suppose $S_c \cap T_2 = T_2$ and $r \in T_2$. By Theorem 3, c is a nontrivial critical point of $p(z) = (z-1)^k(z-r)(z-f_c(r))$. □

Center and Radius of S_c

To further characterize critical points of $p \in P(k, 1, 1)$, we need a better understanding of S_c. For $c \neq 1$, we will determine the center and radius of S_c. By definition of S_c, $z \in S_c$ if and only if there exists a $w \in T_2$ with $f_c(w) = z$. As $(f_c)^{-1} = f_c$, $z \in S_c$ if and only if there exists a $w \in T_2$ with $| f_c(z) | = | w | = 1$. That is,

$$\left| \frac{(1-(k+1)c)z + (k+2)c^2 - 2c}{-kz + (k+1)c - 1} \right| = 1. \tag{4}$$

For $\lambda \neq 1$, it follows from introductory complex analysis that the solution set of

$$| z - u | = \lambda | z - v |$$

is a circle with center $C = v + \dfrac{v-u}{\lambda^2 - 1}$ and radius R satisfying $R^2 = |C|^2 - \dfrac{\lambda^2 |v|^2 - |u|^2}{\lambda^2 - 1}$. Manipulating equation (4) gives

$$\left| z - \left(\frac{(k+1)c - 1}{k} \right) \right| = \left| \frac{1-(k+1)c}{k} \right| \left| z - \left(\frac{2c - (k+2)c^2}{1-(k+1)c} \right) \right|.$$

Applying the change of variables $z = W + \dfrac{(k+2)c - 1}{k+1}$ yields

$$\left| W - \frac{c-1}{k(k+1)} \right| = \left| \frac{1-(k+1)c}{k} \right| \left| W - \frac{1-c}{(k+1)(1-(k+1)c)} \right|. \tag{5}$$

When $\lambda = \left| \dfrac{1-(k+1)c}{k} \right| = 1$, equation (2) implies $c \in T_{\frac{2k}{k+1}}$ and it follows that S_c is a line. When $c \in T_\alpha$ with $\alpha \neq \dfrac{2k}{k+1}$, $\lambda \neq 1$ and the solution set of (5) is a circle with center

$$C = \frac{1-c}{(k+1)(1-(k+1)c)} + \frac{\dfrac{1-c}{(k+1)(1-(k+1)c)} + \dfrac{1-c}{k(k+1)}}{\left| \dfrac{1-(k+1)c}{k} \right|^2 - 1}$$

$$= \frac{\left|\frac{1-(k+1)c}{k}\right|^2 \frac{1-c}{(k+1)(1-(k+1)c)} + \frac{1-c}{k(k+1)}}{\left|\frac{1-(k+1)c}{k}\right|^2 - 1}$$

$$= \frac{\left(\frac{1}{k}\right)^2 |1-(k+1)c|^2 \frac{1-c}{(k+1)(1-(k+1)c)} + \frac{1-c}{k(k+1)}}{\left|\frac{1-(k+1)c}{k}\right|^2 - 1}.$$

Using the fact that $|1-(k+1)c|^2 = (1-(k+1)c)\overline{(1-(k+1)c)}$ yields

$$C = \frac{\frac{\overline{(1-(k+1)c)}(1-c) + k(1-c)}{k^2(k+1)}}{\left|\frac{1-(k+1)c}{k}\right|^2 - 1}$$

$$= \frac{\frac{(1-c)(1-(k+1)\bar{c} + k)}{k^2(k+1)}}{\left|\frac{1-(k+1)c}{k}\right|^2 - 1}$$

$$= \frac{\frac{1}{k^2}|1-c|^2}{\left|\frac{1-(k+1)c}{k}\right|^2 - 1}$$

$$= \frac{1}{\left|k+1-\frac{k}{1-c}\right|^2 - \left|\frac{k}{1-c}\right|^2}.$$

By Lemma 1, $\frac{k}{1-c} = \frac{k}{\alpha} + iky,$ and it follows that

$$C = \frac{\alpha}{(k+1)^2\alpha - 2k(k+1)}.$$

Furthermore,

$$R^2 = |C|^2 - \frac{\left|\frac{1-(k+1)c}{k}\right|^2 \left|\frac{1-c}{(k+1)(1-(k+1)c)}\right|^2 - \left|\frac{c-1}{k(k+1)}\right|^2}{\left|\frac{1-(k+1)c}{k}\right|^2 - 1} = |C|^2$$

implies that $R = |C|$. Resubstituting $z = W + \frac{(k+2)c-1}{k+1}$ establishes the following result.

Lemma 5. *Let* $1 \neq c \in T_\alpha$ *with* $\alpha \neq \frac{2k}{k+1}$. *Then* S_c *is a circle with center* γ *and radius r given by*

$$\gamma = \frac{(k+2)c-1}{k+1} + \frac{\alpha}{(k+1)^2\alpha - 2k(k+1)} \text{ and } r = \left|\frac{\alpha}{(k+1)^2\alpha - 2k(k+1)}\right|.$$

We now investigate several examples for future reference.

Example 4. For $1 \neq c \in T_2$, S_c is a circle with center

$$\gamma = \frac{(k+2)c-1}{k+1} + \frac{2}{2(k+1)^2 - 2k(k+1)} = \frac{k+2}{k+1}c$$

and radius

$$r = \left|\frac{2}{2(k+1)^2 - 2k(k+1)}\right| = \frac{1}{k+1}.$$

Therefore, when $1 \neq c \in T_2$, S_c is externally tangent to T_2 at *c*.

Example 5. For $c \in \mathbb{C}$, $S_c = T_2$ whenever f_c satisfies the conditions of Theorem 2. Since

$$f_c(z) = \frac{(1-(k+1)c)z + (k+2)c^2 - 2c}{-kz + (k+1)c - 1},$$

Theorem 2 implies $\overline{(k+2)c^2 - 2c} = k$ and $\overline{1-(k+1)c} = 1-(k+1)c$.

The second equation implies $c \in \mathbb{R}$, which reduces the first equation to $(k+2)c^2 - 2c - k = 0$. Therefore,

$$0 = (k+2)c^2 - 2c - k = ((k+2)c + k)(c-1)$$

and $c \in \left\{1, \frac{-k}{k+2}\right\}$. When $c = 1$, $f_1(z) = 1$ and hence the hypotheses of Theorem 2 are not satisfied. Therefore, $S_c = T_2$ if and only if $c = \frac{-k}{k+2}$.

Where not to find the critical points

We continue our analysis of S_c by determining when S_c is tangent to T_2. For $1 \neq c \in T_\alpha$ with $\alpha \in (0, 2]$, if S_c is internally tangent to T_2, then

$$|\gamma| + r = 1. \tag{6}$$

See Figure 2.

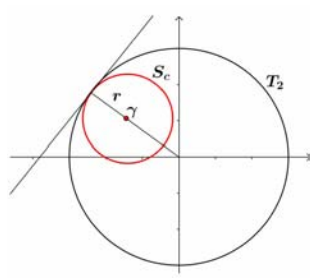

Figure 2. If S_c is internally tangent to T_2, then $|\gamma| + r = 1$.

For $1 \neq c \in T_\alpha$ and $R = \frac{\alpha}{(k+1)^2\alpha - 2k(k+1)}$, S_c is a circle with center $\gamma = \frac{(k+1)c - 1}{k+1} + R$ and radius $r = |R|$. Substituting into equation (6) and setting $c = x + iy$ gives

$$((k+2)x - 1 + (k+1)R)^2 + (k+2)^2 y^2 = (k+1)^2(1 - |R|)^2. \tag{7}$$

We denote equation (7) by I_α. Since $r > 0$, equation (6) is satisfied if and only if S_c is internally tangent to T_2 or $S_c = T_2$. Recalling that $S_c = T_2$ if and only if $c = \frac{-k}{k+2}$ leads to the following lemma.

Lemma 6. *Let* $c \notin \left\{1, \frac{-k}{k+2}\right\}$ *and* $\alpha \in (0, 2]$. *Then* S_c *is internally tangent to* T_2 *if and only if* $c \in I_\alpha \cap T_\alpha$.

Observe that R is undefined when $\alpha = \frac{2k}{k+1}$, positive when $\alpha > \frac{2k}{k+1}$, and negative when $\alpha < \frac{2k}{k+1}$. With this in mind, we consider three cases:

(1) $0 < \alpha < \frac{2k}{k+1}$,

(2) $\alpha = \frac{2k}{k+1}$,

(3) $\frac{2k}{k+1} < \alpha \leq 2$.

In the first case, R is negative, which implies $|R| = -R$. Equation (7) becomes

$$\left(x - \frac{-1+(k+1)R}{k+2}\right)^2 + y^2 = \left(\frac{(k+1)+(k+1)R}{k+2}\right)^2,$$

which is equivalent to

$$\left(x - \left(1 - \frac{(k+1)+(k+1)R}{k+2}\right)\right)^2 + y^2 = \left(\frac{(k+1)+(k+1)R}{k+2}\right)^2. \tag{8}$$

Therefore, I_α is a circle tangent to the line $x = 1$, centered at $x = 1 - \frac{(k+1)+(k+1)R}{k+2}$ with radius $\left|\frac{(k+1)+(k+1)R}{k+2}\right|$. Substituting $R = \frac{\alpha}{(k+1)^2\alpha - 2k(k+1)}$ gives

$$\frac{(k+1)+(k+1)R}{k+2} = \frac{(k+1)+(k+1)\dfrac{\alpha}{(k+1)^2\alpha - 2k(k+1)}}{k+2}$$

$$= \frac{(k^2+2k+2)\alpha - 2k(k+1)}{(k+1)(k+2)\alpha - 2k(k+2)},$$

which equals zero when $\alpha = \dfrac{2k(k+1)}{(k+1)^2+1}$.

- $0 < \alpha < \dfrac{2k(k+1)}{(k+1)^2+1} \Rightarrow \dfrac{(k+1)+(k+1)R}{k+2} > 0$ and I_α is tangent to

$x = 1$ on the left. Therefore, I_α and T_α intersect (at $c \neq 1$) precisely when $I_\alpha = T_\alpha$. This occurs when the two circles have the same radius. That is, when

$$\frac{(k^2+2k+2)\alpha - 2k(k+1)}{(k+1)(k+2)\alpha - 2k(k+2)} = \frac{\alpha}{2}. \tag{9}$$

After simplification, this becomes $0 = ((k+2)\alpha - 2k)(\alpha - 2)$, which implies $\alpha = \dfrac{2k}{k+2}$ or $\alpha = 2 \notin \left(0, \dfrac{2k(k+1)}{(k+1)^2+1}\right)$. By Lemma 6, when $c \in T_{\frac{2k}{k+2}}$, S_c is internally tangent to T_2.

- $\dfrac{2k(k+1)}{(k+1)^2+1} \le \alpha < \dfrac{2k}{k+1} \Rightarrow \dfrac{(k+1)+(k+1)R}{k+2} \le 0$ and I_α is tangent

to $x = 1$ on the right. See Figure 3. By Lemma 6, when $1 \neq c \in T_\alpha$ with $\dfrac{2k(k+1)}{(k+1)^2+1} \le \alpha < \dfrac{2k}{k+1}$, S_c is not internally tangent to T_2.

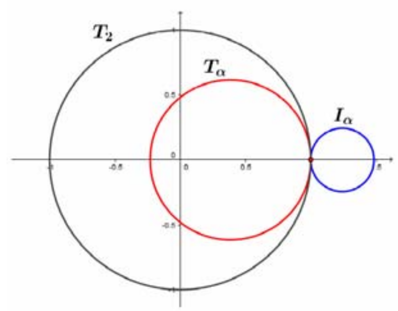

Figure 3. When $\frac{2k(k+1)}{(k+1)^2+1} < \alpha < \frac{2k}{k+1}$, I_α is tangent to $x = 1$ on the right.

In the second case, $\alpha = \frac{2k}{k+1}$ and S_c is a vertical line passing through $f_c(1) = \frac{(k+2)c-1}{k+1}$ which is not tangent to T_2. See Example 3.

In the third case, R is positive, so $|R| = R$. Equation (7) becomes

$$\left(x - \frac{1-(k+1)R}{k+2}\right)^2 + y^2 = \left(\frac{(k+1)-(k+1)R}{k+2}\right)^2,$$

which is equivalent to

$$\left(x - \left(\frac{-k}{k+2} + \frac{(k+1)-(k+1)R}{k+2}\right)\right)^2 + y^2 = \left(\frac{(k+1)-(k+1)R}{k+2}\right)^2.$$

Therefore, I_α is a circle tangent to the line $x = \frac{-k}{k+2}$, centered at $x = \frac{-k}{k+2} + \frac{(k+1)-(k+1)R}{k+2}$ with radius $\left|\frac{(k+1)-(k+1)R}{k+2}\right|$. Substituting $R = \frac{\alpha}{(k+1)^2\alpha - 2k(k+1)}$ gives

$$\frac{(k+1)-(k+1)R}{k+2} = \frac{(k+1)-(k+1)\frac{\alpha}{(k+1)^2\alpha - 2k(k+1)}}{k+2}$$

$$= \frac{k(k+2)\alpha - 2k(k+1)}{(k+1)(k+2)\alpha - 2k(k+2)},$$

which equals zero when $\alpha = \dfrac{2k+2}{k+2}$.

- $\dfrac{2k}{k+1} < \alpha < \dfrac{2k+2}{k+2} \Rightarrow \dfrac{(k+1)-(k+1)R}{k+2} < 0$ and I_α is tangent to $x = \dfrac{-k}{k+2}$ on the left. When $\alpha = \dfrac{2k+2}{k+2}$, T_α intersects the negative real axis at $x = \dfrac{-k}{k+2}$. Therefore, for $\dfrac{2k}{k+1} < \alpha < \dfrac{2k+2}{k+1}$, $I_\alpha \cap T_\alpha = \varnothing$, and by Lemma 6, S_c is not internally tangent to T_2.

- $\alpha = \dfrac{2k+2}{k+2} \Rightarrow \dfrac{(k+1)-(k+1)R}{k+2} = 0$ and I_α is the single point $x = \dfrac{-k}{k+2}$. Therefore, $I_\alpha \cap T_\alpha = \left\{\dfrac{-k}{k+2}\right\}$. By Example 5, $S_{\frac{-k}{k+2}} = T_2$ and it follows that S_c is not internally tangent to T_2.

- $\dfrac{2k+2}{k+2} < \alpha < 2 \Rightarrow 0 < \dfrac{(k+1)-(k+1)R}{k+2} < \dfrac{k}{k+2}$ and I_α is tangent to $x = \dfrac{-k}{k+2}$ on the right. When $\alpha = \dfrac{2k+2}{k+2}$, T_α intersects the negative real axis at $x = \dfrac{-k}{k+2}$ and $I_\alpha = \left\{\dfrac{-k}{k+2}\right\}$. When $\alpha = 2$, I_α is centered at the origin with radius $\dfrac{k}{k+2}$. Therefore, for $\dfrac{2k+2}{k+2} < \alpha < 2$, I_α lies inside T_α. See Figure 4. By Lemma 6, when $c \in T_\alpha$ with $\dfrac{2k+2}{k+2} < \alpha < 2$, S_c is not internally tangent to T_2.

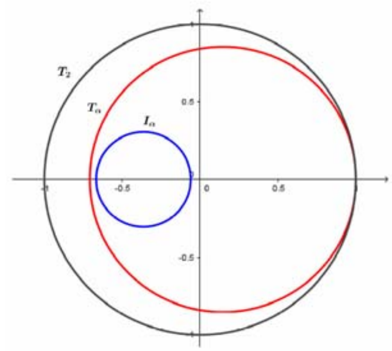

Figure 4. When $\dfrac{2k+2}{k+2} < \alpha < 2$, I_α lies inside T_α.

Our analysis of I_α has established the following result.

Lemma 7. *Let* $1 \neq c \in \mathbb{C}$. *Then* S_c *is internally tangent to* T_2 *if and only if* $c \in T_{\frac{2k}{k+1}}$.

Furthermore, for $c \in T_\alpha$ with $0 < \alpha \leq 2$, S_c will be externally tangent to T_2 if and only if

$$|\gamma| - r = 1.$$

A similar, but less involved, analysis leads to the following result. See Example 3.

Lemma 8. *Let* $1 \neq c \in \mathbb{C}$. *Then* S_c *is externally tangent to* T_2 *if and only if* $c \in T_2$.

The unit disk contains a desert region in which the critical points of $p \in P(k, 1, 1)$ cannot occur.

Theorem 9. *No polynomial in* $P(k, 1, 1)$ *has a critical point strictly inside* $T_{\frac{2k}{k+1}}$.

Proof. Let $c \in T_\alpha$ with $0 < \alpha < \dfrac{2k}{k+2}$. Then, from equations (8) and

(9), $I_\alpha = T_\beta$ with

$$\beta = \frac{2[(k^2 + 2k + 2)\alpha - 2k(k+1)]}{(k+1)(k+2)\alpha - 2k(k+2)}.$$

Furthermore, $\beta = \alpha$ when $\alpha = \frac{2k}{k+2}$ or $\alpha = 2$, and β is undefined when $\alpha = \frac{2k}{k+1}$. When $\alpha = 0$, $\beta = \frac{2(k+1)}{k+2} > 0$ and it follows that $\beta > \alpha$ when $0 < \alpha < \frac{2k}{k+2}$. Therefore, when $0 < \alpha < \frac{2k}{k+2}$, T_α lies inside I_α. See Figure 5. Setting $c = x + iy$, it follows from equation (8) that

$$\left(x - \left(1 - \frac{(k+1) + (k+1)R}{k+2}\right)\right)^2 + y^2 < \left(\frac{(k+1) + (k+1)R}{k+2}\right)^2.$$

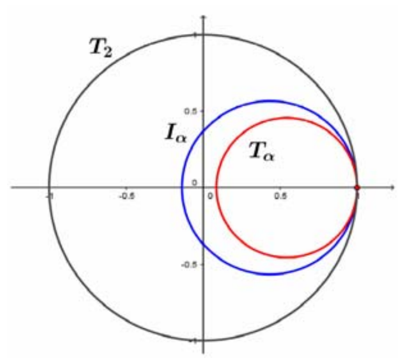

Figure 5. When $0 < \alpha < \frac{2k}{k+2}$, T_α lies inside I_α.

Equivalently, equations (6) and (7) imply that $|\gamma| + r < 1$. Therefore, $S_c \cap T_2 = \varnothing$ and by Theorem 4, no $p \in P(k, 1, 1)$ has a critical point strictly inside $T_{\frac{2k}{k+2}}$. $\square$

Observe that as k approaches infinity, the diameter of $T_{\frac{2k}{k+2}}$ approaches 2.

Corollary 10. *The desert region bounded by* $T_{\frac{2k}{k+2}}$ *expands to fill the interior of the unit disk as k approaches infinity.*

Main Result

We are now able to characterize the critical points of $p \in P(k, 1, 1)$.

Theorem 11. *Let* $c \in \mathbb{C}$.

(1) *If* c *lies strictly inside* $T_{\frac{2k}{k+2}}$ *or strictly outside* T_2, *then no* $p \in P(k, 1, 1)$ *has a critical point at* c.

(2) $p \in P(k, 1, 1)$ *has a nontrivial critical point at* $c = 1$ *if and only if* $p(z) = (z-1)^{k+1}(z-r)$ *for some* $r \in T_2$.

(3) $p \in P(k, 1, 1)$ *has a nontrivial critical point at* $c = \frac{-k}{k+2}$ *if and only if* $p(z) = (z-1)^k(z-r)(z - f_{\frac{-k}{k+2}}(r))$ *for some* $r \in T_2$.

(4) *If* $c \notin \left\{1, \frac{-k}{k+2}\right\}$ *lies on* T_α *for some* $\alpha \in \left[\frac{2k}{k+2}, 2\right]$, *then there is a unique* $p \in P(k, 1, 1)$ *with a nontrivial critical point at c.*

Proof. Let c lie strictly inside $T_{\frac{2k}{k+2}}$ or strictly outside T_2. Theorem 9 and the Gauss-Lucas theorem imply that no $p \in P(k, 1, 1)$ has a critical point at c.

By Example 1, $p \in P(k, 1, 1)$ has a nontrivial critical point at $c = 1$ if and only if $p(z) = (z-1)^{k+1}(z-r)$ for some $r \in T_2$.

By Example 5, $f_c(T_2) = T_2$ if and only if $c = \frac{-k}{k+2}$. By Theorems 3 and 4, $p \in P(k, 1, 1)$ has a nontrivial critical point at $c = \frac{-k}{k+2}$ if and only if $p(z) = (z-1)^k (z-r)(z - f_{\frac{-k}{k+2}}(r))$ for some $r \in T_2$.

If $c \in T_\alpha$ with $\alpha = \frac{2k}{k+2}$ or $\alpha = 2$, it follows by Lemmas 7 and 8 that S_c is tangent to T_2. By Theorem 3, there is a unique $p \in P(k, 1, 1)$ with a nontrivial critical point at c. Now, suppose $c \notin \left\{1, \frac{-k}{k+2}\right\}$ lies on T_α for some $\alpha \in \left(\frac{2k}{k+2}, 2\right)$. Then $|S_c \cap T_2| \in \{0, 2\}$. Without loss of generality, suppose $S_c \cap T_2 = \varnothing$ and S_c lies inside T_2. As we drag c to T_2 along a line segment contained in $\left\{c \in T_\alpha \,|\, \alpha \in \left(\frac{2k}{k+2}, 2\right]\right\} \setminus \left\{1, \frac{-k}{k+2}\right\}$, S_c is continuously transformed into a circle externally tangent to T_2. See Example 4. By the Intermediate Value Theorem, there exists a c_0 on the line segment with S_c internally tangent to T_2. As c does not cross $T_{\frac{2k}{k+2}}$, this contradicts Lemma 7. Therefore, $|S_c \cap T_2| = 2$, and by Theorem 4, there exists a unique $p \in P(k, 1, 1)$ with a nontrivial critical point at c. □

For $m \in \mathbb{N}$, one can easily extend Theorem 11 to the family $P(k, m, m)$. The main result restated for this general case is as follows.

Theorem 12. *Let* $c \in \mathbb{C}$.

(1) *If* c *lies strictly inside* $T_{\frac{2k}{2m+k}}$ *or strictly outside* T_2, *then no* $p \in P(k, m, m)$ *has a critical point at c.*

(2) $p \in P(k, m, m)$ *has a nontrivial critical point at* $c = 1$ *if and only if* $p(z) = (z-1)^{k+m}(z-r)^m$ *for some* $r \in T_2$.

(3) $p \in P(k, m, m)$ *has a nontrivial critical point at* $c = \frac{-k}{2m+k}$ *if and only if* $p(z) = (z-1)^k (z-r)^m (z - f_{\frac{-k}{2m+k}}(r))^m$ *for some* $r \in T_2$.

(4) *If* $c \notin \left\{1, \frac{-k}{2m+k}\right\}$ *lies on* T_α *for some* $\alpha \in \left[\frac{2k}{2m+k}, 2\right]$, *then there is a unique* $p \in P(k, m, m)$ *with a nontrivial critical point at c.*

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