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ON HOW TO SUPERIZE THE SPACES OF DIFFERENTIAL OPERATORS ON THE CONTACT MANIFOLDS $\mathbb{R}^{2n+1}$

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Abstract

We study the existence of three types of filtrations of the space $\mathcal{D}_{\lambda\mu}(\mathbb{R}^{2l+1|n})$ of differential operators on the superspaces $\mathbb{R}^{2l+1|n}$ endowed with the standard contact structure α. On $\mathcal{D}_{\lambda\mu}(\mathbb{R}^{2l+1|n})$, we have the first filtration called canonical and because of the existence of the contact structure on superspaces $\mathbb{R}^{2l+1|n}$, we obtain the second filtration on the space $\mathcal{D}_{\lambda\mu}(\mathbb{R}^{2l+1|n})$ called filtration of Heisenberg and thus the space $\mathcal{D}_{\lambda\mu}(\mathbb{R}^{2l+1|n})$ is therefore denoted by $\mathcal{H}_{\lambda\mu}(\mathbb{R}^{2l+1|n})$. We have also a new filtration induced on $\mathcal{D}_{\lambda\mu}(\mathbb{R}^{2l+1|n})$ by the two filtrations and it is called bifiltration. Explicitly, the space $\mathcal{D}_{\lambda\mu}(\mathbb{R}^{2l+1|n})$ of differential operators is

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filtered canonically by the order of its differential operators. When it is filtered by the order of Heisenberg, the order of any differential operator is equal to $d \in \frac{1}{2}\mathbb{N}$. This study is the generalization, in super case, of the model studied by Conley and Ovsienko in [3]. Finally, we show that the $\mathfrak{spo}(2l+2|n)$ -module structure on the space $\mathcal{D}_{\lambda\mu}(\mathbb{R}^{2l+1|n})$ of differential operators is induced on the space $\mathcal{H}_{\lambda\mu}(\mathbb{R}^{2l+1|n})$ and therefore on the associated space $\mathcal{S}_{\delta}(\mathbb{R}^{2l+1|n})$ of normal symbols, and also on the space $\mathcal{P}_{\delta}(\mathbb{R}^{2l+1|n})$ of symbols of Heisenberg and on the space of fine symbol $\sum_{\delta}(\mathbb{R}^{2l+1|n})$.

1. Introduction

Based on the concept of contact supergeometry, we begin with the standard contact structure α on the supermanifold $\mathbb{R}^{2l+1|n}$. This paper is, in some way, a generalization of those formulas known in classical geometry, as done by Conley and Ovsienko in [3]. Next, we describe the concept of densities on the standard supermanifold $\mathbb{R}^{2l+1|n}$ in the classical way. We define two types of densities on $\mathbb{R}^{2l+1|n}$ and show that if d is a superdimension which is different from -1, then there is an isomorphism between the space of tensor densities $Ber_{\frac{\lambda}{d+1}}(\mathbb{R}^{2l+1|n})$ and the space of contact densities $\mathcal{F}_{\lambda}(\mathbb{R}^{2l+1|n})$, where $\lambda \in \mathbb{R}$. These spaces $Ber_{\frac{\lambda}{d+1}}(\mathbb{R}^{2l+1|n})$ and $\mathcal{F}_{\lambda}(\mathbb{R}^{2l+1|n})$ are isomorphic as $\mathcal{K}(2l+1|n)$-modules, where $\mathcal{K}(2l+1|n)$ is the Lie superalgebra of contact vector fields on the contact supermanifold $\mathbb{R}^{2l+1|n}$.

The definition of differential operators which acts between the λ-densities and μ-densities, where $\lambda, \mu \in \mathbb{R}$, on the supermanifold $\mathbb{R}^{2l+1|n}$ establishes the existence on the space $\mathcal{D}_{\lambda\mu}(\mathbb{R}^{2l+1|n})$ of two types of filtrations, i.e., the canonical filtration and the filtration of Heisenberg. These two filtrations induce another filtration called *bifiltration*. In this way, we generalize, in super case, the model described in the even case by Conley and Ovsienko in [3]. We define the space $\mathcal{S}_\delta(\mathbb{R}^{2l+1|n})$ of symbols associated to the canonical filtration, the space of symbols of Heisenberg $\mathcal{P}_\delta(\mathbb{R}^{2l+1|n})$ associated to the filtration of Heisenberg and the bigraded space $\sum_\delta(\mathbb{R}^{2l+1|n})$ of fine symbols. In the last section, we show that these spaces are $\mathfrak{spo}(2l+2|n)$-modules, where $\mathfrak{spo}(2l+2|n)$ is the Lie sub-superalgebra of $\mathcal{K}(2l+1|n)$ constituted by the contact vector fields X_f whose superfunctions *f* are of degree atmost two. In order to facilitate the computations, we use the method used by Conley and Ovsienko in [3]: we represent the symbols by the polynomials and the Lie derivatives by the differential operators. In this way, we see that the spaces $\mathcal{S}_\delta^k(\mathbb{R}^{2l+1|n})$, $\mathcal{P}_\delta^d(\mathbb{R}^{2l+1|n})$ and $\sum_\delta^{k,d}(\mathbb{R}^{2l+1|n})$ are isomorphic to some sub-spaces of

$$\mathcal{F}_\delta(\mathbb{R}^{2l+1|n}) \otimes \mathrm{Pol}(T^*\mathbb{R}^{2l+1|n}).$$

We use the symbolical notation, i.e., the moments ζ, α_i, β_i and γ_i associated to the vector fields ∂_z, $A_i = \partial_{x_i} + y_i\partial_z$, $B_i = -\partial_{y_i} + x_i\partial_z$ and $\overline{D_i} = \partial_{\theta_i} - \theta_i\partial_z$, respectively, and compute the explicit formulas of the actions of $\mathfrak{spo}(2l+2|n) \subset \mathcal{K}(2l+1|n)$ on the spaces $\mathcal{S}_\delta^k(\mathbb{R}^{2l+1|n})$, $\mathcal{P}_\delta^d(\mathbb{R}^{2l+1|n})$ and $\sum_\delta^{k,d}(\mathbb{R}^{2l+1|n})$, respectively.

2. Contact Structure on $\mathbb{R}^{2l+1|n}$

We consider the supermanifold $\mathbb{R}^{2l+1|n}$, where *l* and *n* are integers. The standard contact structure on the supermanifold $\mathbb{R}^{2l+1|n}$ is defined by the kernel of the differential 1-superforms α on $\mathbb{R}^{2l+1|n}$ which, in the system of

Darboux coordinates (z, x_i, y_i, θ_j), $i = 1, \ldots, l$ and $j = 1, \ldots, n$ can be written as

$$\alpha = dz + \sum_{i=1}^{l} (x_i dy_i - y_i dx_i) + \sum_{i=1}^{n} \theta_i d\theta_i. \tag{1}$$

This differential 1-superform α is called *contact form* on $\mathbb{R}^{2l+1|n}$. We denote the space constituted of the elements T_r, $1 \leq r \leq 2l+1$ of the kernel of α by $\mathrm{Tan}(\mathbb{R}^{2l+1|n})$.

If we denote by $q^A = (z, q^r)$ the generalized coordinate where

$$q^A = \begin{cases} z & \text{if} \quad A = 0 \\ x_A & \text{if} \quad 1 \leq A \leq l \\ y_{A-l} & \text{if} \quad l+1 \leq A \leq 2l \\ \theta_{A-2l} & \text{if} \quad 2l+1 \leq A \leq 2l+n, \end{cases} \tag{2}$$

then we can write α in the following way:

$$\alpha = dz + \omega_{rs} q^r dq^s, \quad (\omega_{rs}) = \left(\begin{array}{cc|c} 0 & id_l & 0 \\ -id_l & 0 & 0 \\ \hline 0 & 0 & id_n \end{array} \right).$$

Remark 2.1. We denote the elements of the matrix (ω^{sk}) by ω^{sk}, so that $(\omega_{rs})(\omega^{sk}) = (\delta_r^k)$. We thus have

$$(\omega^{rs}) = \left(\begin{array}{cc|c} 0 & -id_l & 0 \\ id_l & 0 & 0 \\ \hline 0 & 0 & id_n \end{array} \right)$$

and $(\omega^{rs}) = -(-1)^{\tilde{r}\tilde{s}} (\omega^{sr})$.

Definition 2.2. We call the field of Reeb on $\mathbb{R}^{2l+1|n}$, the *vector field* $T_0 \in Vect(\mathbb{R}^{2l+1|n})$ which, in the system of Darboux coordinates, is written as $T_0 = \partial_z$.

We can show that the field of Reeb is the unique vector field on $\mathbb{R}^{2l+1|n}$ so that $i(T_0)\alpha = 1$ and $i(T_0)d\alpha = 0.$

As proved in [2], the vector fields $T_1, \ldots, T_{2l+n} \in \mathrm{Tan}\mathbb{R}^{2l+1|n}$ (i.e., the kernel of α) are written explicitly as follows:

$$T_r = \begin{cases} A_r & := \quad \partial_{x_r} + y_r\partial_z & \text{if} \quad 1 \le r \le 1 \\ -B_{r-l} & := \quad \partial_{y_{r-l}} - x_{r-l}\partial_z & \text{if} \quad l+1 \le r \le 2l \\ \overline{D}_{r-2l} & := \quad \partial_{\theta_{r-2l}} - \theta_{r-2l}\partial_z & \text{if} \quad 2l+1 \le r \le 2l+n. \end{cases} \tag{3}$$

If we write the vector field T_r as follows: $T_r = \partial_{q^r} - \omega_{kr}q^k\partial_z,$ then the following formulas are immediate:

$$T_r(q^k) = \delta_r^k,\ T_r(z) = -\omega_{kr}q^k,\ [T_r, T_j] = -2\omega_{rj}\partial_z,\ T_r(z^2) = -2z\omega_{kr}q^k. \tag{4}$$

3. Module of Densities on $\mathbb{R}^{2l+1|n}$

In this section, we consider the space $C^\infty(\mathbb{R}^{2l+1|n})$ of superfunctions and the space $Vect(\mathbb{R}^{2l+1|n})$ of all homogeneous vector fields on $\mathbb{R}^{2l+1|n}$.

We define an action $\mathbb{L}_X^\lambda$ of an element $X = \sum_A X^A\partial_{q^A}$ of $Vect(\mathbb{R}^{2l+1|n})$ on the space of superfunctions $C^\infty(\mathbb{R}^{2l+1|n})$ as follows:

$$\mathbb{L}_X^\lambda(g) := X(g) + \lambda div(X)g, \quad \forall g \in C^\infty(\mathbb{R}^{2l+1|n}), \tag{5}$$

where $div(X) = \sum_A (-1)^{\widetilde{X^A}\widetilde{q^A}}\partial_{q^A}X^A.$ We can show that for all $X, Y \in Vect(\mathbb{R}^{2l+1|n}),$ we have

$$[\mathbb{L}_X^\lambda, \mathbb{L}_Y^\lambda] = \mathbb{L}_{[X,Y]}^\lambda.$$

Definition 3.1. The module $Ber_\lambda(\mathbb{R}^{2l+1|n})$ of tensor densities of weight $\lambda \in \mathbb{R}$ on $\mathbb{R}^{2l+1|n}$ is the space of superfunctions $C^\infty(\mathbb{R}^{2l+1|n})$ endowed with the action (5) of $Vect(\mathbb{R}^{2l+1|n})$. One writes for $g \in C^{\infty}(\mathbb{R}^{2l+1|n})$, any λ-tensor density as $g|Dx|^{\lambda}$.

In particular, when we consider a contact structure on $\mathbb{R}^{2l+1|n}$, we can define a subspace $\mathcal{K}(2l+1|n)$ of $Vect(\mathbb{R}^{2l+1|n})$ constituted by the contact vector fields X_f on $\mathbb{R}^{2l+1|n}$. Explicitly, we can show that the elements X_f are expressed by

$$X_f = f\partial_z - \frac{1}{2}(-1)^{\tilde{f}\tilde{T}_r}\omega^{rs}T_r(f)T_s, \tag{6}$$

where ω^{rs} denotes the elements of the matrix (ω^{rs}) and T_r denotes the elements of kernel of the standard contact form α on $\mathbb{R}^{2l+1|n}$ (see (1) and (3)). In this way, we define an action of X_f on $C^{\infty}(\mathbb{R}^{2l+1|n})$ by

$$L^{\lambda}_{X_f}(g) = X_f(g) + \lambda f'g, \quad \forall g \in C^{\infty}(\mathbb{R}^{2l+1|n}), \tag{7}$$

and where $f' = \partial_z f$ and (z, x^i, y^i, θ^j), $i \in [1, l]$, $j \in [1, n]$, is the coordinates system of Darboux. It is clear that

$$[L^{\lambda}_{X_f}, L^{\lambda}_{X_g}] = L^{\lambda}_{[X_f, X_g]} = L^{\lambda}_{X_{\{f, g\}}},$$

where $\{f, g\}$ is the Lagrange bracket of the superfunctions f and g.

Definition 3.2. The module $\mathcal{F}_{\lambda}(\mathbb{R}^{2l+1|n})$ of λ-contact densities on $\mathbb{R}^{2l+1|n}$ is the space $C^{\infty}(\mathbb{R}^{2l+1|n})$ endowed with the action (7) of $\mathcal{K}(2l+1|n)$. One can write any λ-contact density as $g\alpha^{\lambda}$, where α is a contact 1-form on $\mathbb{R}^{2l+1|n}$ and for an arbitrary superfunction g.

We have the following result.

Proposition 3.3. *If the superdimension* $d = 2l + 1 - n$ *is different from* -1, *then the application*

$$\varphi : \mathcal{F}_\lambda(M) \to Ber_{\frac{2\lambda}{d+1}}(M) : g\alpha^\lambda \mapsto g|\,Dx\,|^{\frac{2\lambda}{d+1}}$$

is an isomorphism of $\mathcal{K}(2l+1|n)$*-modules*.

Proof. It is clear that φ is bijective. We must show that the application φ intertwines the action of X_f on the spaces $\mathcal{F}_\lambda(\mathbb{R}^{2l+1|n})$ and $Ber_{\frac{2\lambda}{d+1}}(\mathbb{R}^{2l+1|n})$, i.e.,

$$\varphi(L^\lambda_{X_f}(g\alpha^\lambda)) = \mathbb{L}^{\frac{2\lambda}{d+1}}_{X_f}(\varphi(g\alpha^\lambda)). \tag{8}$$

Indeed, on the space $\mathcal{F}_\lambda(\mathbb{R}^{2l+1|n})$, the action of $\mathcal{K}(2l+1|n)$ is given by

$$L^\lambda_{X_f}(g\alpha^\lambda) = (X_f(g) + \lambda f'g)\alpha^\lambda = \left(X_f(g) + \lambda\,\frac{2divX_f}{d+1}\,g\right)\alpha^\lambda, \tag{9}$$

where $div(X_f) = \frac{2l+2-n}{2} f'$. The second member of (8) can be written as

$$\left(X_f(g) + \frac{2\lambda}{d+1}\,div(X_f)g\right)|\,Dx\,|^{\frac{2\lambda}{d+1}}. \tag{10}$$

If we apply the isomorphism φ to the density given by (9), we obtain (10). $\square$

In particular, if we denote by $T\mathbb{R}^{2l+1|n}$, the supertangent sheaf on $\mathbb{R}^{2l+1|n}$, then we can define the application

$$X_\cdot : \mathcal{F}_{-1}(\mathbb{R}^{2l+1|n}) \to T\mathbb{R}^{2l+1|n} : f\alpha^{-1} \mapsto X_f,$$

and this application establishes the intertwine of the representations $\mathcal{F}_{-1}(\mathbb{R}^{2l+1|n})$ and $T\mathbb{R}^{2l+1|n}$ of $\mathcal{K}(2l+1|n)$.

If we denote by $\mathrm{Tan}\mathbb{R}^{2l+1|n}$ the space of all vector fields on $\mathbb{R}^{2l+1|n}$ which preserves the contact structure, then we can also show that the space

$T\mathbb{R}^{2l+1|n}$ is the direct sum of two spaces as follows:

$$T\mathbb{R}^{2l+1|n} = \mathrm{Tan}\mathbb{R}^{2l+1|n} \oplus \mathcal{K}(2l+1|n).$$

The space $\mathrm{Tan}\mathbb{R}^{2l+1|n}$ is a $C^\infty(\mathbb{R}^{2l+1|n})$-module but is not a Lie superalgebra and the space $\mathcal{K}(2l+1|n)$ is not a module but it possesses a structure of Lie superalgebra.

4. Module of Differential Operators on $\mathbb{R}^{2l+1|n}$

We begin here by the classical definitions of differential operators between the spaces of densities.

Definition 4.1. If we denote by $(q^A) = (z, x^i, y^i, \theta^j)$ $(i \in [1, l],\ j \in [1, n])$ the coordinates system of Darboux on $\mathbb{R}^{2l+1|n}$, we call the differential operator D of order k on $\mathbb{R}^{2l+1|n}$, the application which maps $\mathcal{F}_\lambda(\mathbb{R}^{2l+1|n})$ to $\mathcal{F}_\mu(\mathbb{R}^{2l+1|n})$. It is written in coordinates by

$$D : f\alpha^\lambda \mapsto \left(\sum_{I:|I|\leq k} D_I \partial_{q^I} f\right)\alpha^\mu,$$

where $I = (i_0, i_1, \ldots, i_{2l+n})$ is a multi-index, $|I| = i_0 + \cdots + i_{2l+n}$ and D_I is a superfunction for all I.

More explicitly, a differential operator D can be written in coordinates by

$$\sum_{I:|I|\leq k} D_I (\partial_z)^{i_0} (\partial_{x_1})^{i_1} \ldots (\partial_{x_l})^{i_l} (\partial_{y_1})^{i_{l+1}} \ldots (\partial_{y_l})^{i_{2l}} (\partial_{\theta_1})^{i_{2l+1}} \ldots (\partial_{\theta_n})^{i_{2l+n}}. \tag{11}$$

Because $\partial^2_{\theta_i} = 0$, the exponents $i_{2l+1}, \ldots, i_{2l+n}$ in the expression (11) are atmost equal to 1.

The differential operators of order 0 are simply the multiplication by $(\mu - \lambda)$-densities. We define the space of differential operators as follows:

$$\mathcal{D}_{\lambda\mu}(\mathbb{R}^{2l+1|n}) = \bigcup_{k=0}^{\infty} \mathcal{D}_{\lambda\mu}^{k}(\mathbb{R}^{2l+1|n}).$$

If $D \in \mathcal{D}_{\lambda\mu}(\mathbb{R}^{2l+1|n})$ and $X \in \mathcal{K}(2l+1|n)$, then the action of X on $D_{\lambda\mu}(\mathbb{R}^{2l+1|n})$ is defined by the Lie derivative $\mathcal{L}_X^{\lambda\mu}$ via the following supercommutator:

$$\mathcal{L}_X^{\lambda\mu} D = L_X^{\mu} \circ D - (-1)^{\widetilde{X}\widetilde{D}} D \circ L_X^{\lambda}, \tag{12}$$

where L_X^{λ} and L_X^{μ} are defined by (7). We have thus a $\mathcal{K}(2l+1|n)$-module structure on the space $\mathcal{D}_{\lambda\mu}(\mathbb{R}^{2l+1|n})$.

Remark 4.2. The construction given above can be done with the representations $Ber_{\lambda}(\mathbb{R}^{2l+1|n})$ and $Ber_{\mu}(\mathbb{R}^{2l+1|n})$ of $Vect(\mathbb{R}^{2l+1|n})$. We obtain the representation of $Vect(\mathbb{R}^{2l+1|n})$ on the space of differential operators and it induces the representation of $\mathcal{K}(2l+1|n)$. Proposition 3.3 allows us to show that these representations of $\mathcal{K}(2l+1|n)$ are isomorphic, modulo a change of an adequate weight.

5. Canonical Filtration of $\mathcal{D}_{\lambda\mu}(\mathbb{R}^{2l+1|n})$

Firstly, we show that the space $\mathcal{D}_{\lambda\mu}(\mathbb{R}^{2l+1|n})$ has the canonical filtration. Secondly, we show that the action of $\mathcal{K}(2l+1|n)$ on $\mathcal{D}_{\lambda\mu}(\mathbb{R}^{2l+1|n})$ induces an action on the associated graded space $\mathcal{S}_{\delta}(\mathbb{R}^{2l+1|n})$ that we call the space of symbols on $\mathbb{R}^{2l+1|n}$. The subspaces $\mathcal{D}_{\lambda\mu}^{l}(\mathbb{R}^{2l+1|n})$ are stable under the action of $\mathcal{K}(2l+1|n)$.

Proposition 5.1. *If* $X \in \mathcal{D}^k_{\lambda\mu}(\mathbb{R}^{2l+1|n})$ *and* $X \in \mathcal{K}(2l+1|n)$, *then* $\mathcal{L}^{\lambda\mu}_X(D) \in \mathcal{D}^k_{\lambda\mu}(\mathbb{R}^{2l+1|n})$.

Proof. We compute the terms of order $k+1$ in the expression of $\mathcal{L}^{\lambda\mu}_X(D)$. Because of definition of the action of *X*, we must evaluate the terms of order $k+1$ of $L^{\mu}_X \circ D$ and $D \circ L^{\lambda}_X$. We remark that these terms are identical. Therefore, $\mathcal{L}^{\lambda\mu}_X(D)$ is a differential operator of order *k*. □

It is easy to see that the following inclusions are immediate:

$$\mathcal{D}^0_{\lambda\mu}(\mathbb{R}^{2l+1|n}) \subset \mathcal{D}^1_{\lambda\mu}(\mathbb{R}^{2l+1|n}) \subset \mathcal{D}^2_{\lambda\mu}(\mathbb{R}^{2l+1|n}) \subset \cdots$$
$$\subset \mathcal{D}^{l-1}_{\lambda\mu}(\mathbb{R}^{2l+1|n}) \subset \mathcal{D}^l_{\lambda\mu}(\mathbb{R}^{2l+1|n}) \subset \cdots,$$

and, therefore, we deduce that the spaces $\mathcal{D}^l_{\lambda\mu}(\mathbb{R}^{2l+1|n})$ define a filtration of the module of differential operators $\mathcal{D}_{\lambda\mu}(\mathbb{R}^{2l+1|n})$.

Remark 5.2. If $\lambda = \mu$, then the space $\mathcal{D}_{\lambda\lambda}(\mathbb{R}^{2l+1|n})$ is an associative and filtered superalgebra by the composition of differential operators. It is therefore a filtration of algebra, i.e.,

$$\mathcal{D}^k_{\lambda\lambda}(\mathbb{R}^{2l+1|n})\mathcal{D}^l_{\lambda\lambda}(\mathbb{R}^{2l+1|n}) \subseteq \mathcal{D}^{k+l}_{\lambda\lambda}(\mathbb{R}^{2l+1|n}).$$

We can now define the graded space which is associated to above filtration of $\mathcal{D}^k_{\lambda\lambda}(\mathbb{R}^{2l+1|n})$.

Definition 5.3. The space $\mathcal{S}_\delta(\mathbb{R}^{2l+1|n})$ is called the *space of principal symbols of order k* defined by

$$\mathcal{S}^k_\delta(\mathbb{R}^{2l+1|n}) := \mathcal{D}^k_{\lambda\mu}(\mathbb{R}^{2l+1|n})/\mathcal{D}^{k-1}_{\lambda\mu}(\mathbb{R}^{2l+1|n}), \quad \delta = \mu - \lambda.$$

We define on $\mathcal{D}^k_{\lambda\mu}(\mathbb{R}^{2l+1|n})$ the following surjective application σ_k :

$$\sigma_k : \mathcal{D}^k_{\lambda\mu}(\mathbb{R}^{2l+1|n}) \to \mathcal{S}^k_{\delta}(\mathbb{R}^{2l+1|n}) : D \mapsto [D], \tag{13}$$

where $[D]$ means the equivalence class of D.

The action L^{δ}_X of $\mathcal{K}(2l+1|n)$ on $\mathcal{S}^k_{\delta}(\mathbb{R}^{2l+1|n})$ is induced by the action of $\mathcal{K}(2l+1|n)$ on $\mathcal{D}^k_{\lambda\mu}(\mathbb{R}^{2l+1|n})$, i.e., if $S = [D]$, where D is given by the formula (11), then we have

$$L^{\delta}_X(S) := [\mathcal{L}^{\lambda\mu}_X(D)].$$

Remark 5.4. The previous constructions are also valid for the representations of $Vect(\mathbb{R}^{2l+1|n})$ if we consider the spaces $Ber_{\lambda}(\mathbb{R}^{2l+1|n})$ instead of the spaces $\mathcal{F}_{\lambda}(\mathbb{R}^{2l+1|n})$. The induced representations of $\mathcal{K}(2l+1|n)$ are isomorphic to those presented here, modulo a change of an adequate weight.

6. Filtration of Heisenberg of $\mathcal{D}_{\lambda\mu}(\mathbb{R}^{2l+1|n})$

If the superspace $\mathbb{R}^{2l+1|n}$ is endowed with the standard contact form α, then the superderivations on $C^{\infty}(\mathbb{R}^{2l+1|n})$ are generated by the field of Reeb ∂_z and by the vector fields $T_1, \ldots, T_{2l+n} \in \operatorname{Tan}\mathbb{R}^{2l+1|n}$. We can therefore define another filtration on $\mathcal{D}_{\lambda\mu}(\mathbb{R}^{2l+1|n})$.

Proposition 6.1. *If* $K = (i_1, \ldots, i_{2l+n})$ *is a multi-index of length* $|K| = i_1 + i_2 + \cdots + i_{2l+n}$ *and if D is a differential operator of order k, then D can be written in the following unique form*:

$$\sum_{K : c + |K| \le k} D_{cK} \partial_z^c T^K, \tag{14}$$

where D_{cK} *is a superfunction and* $T^K = T_1^{i_1} \cdots T_{2l+n}^{i_{2l+n}}$.

Proof. The existence is proven by the decomposition

$$T\mathbb{R}^{2l+1|n} = \mathrm{Tan}\mathbb{R}^{2l+1|n} \oplus \mathcal{K}(2l+1|n),$$

and because of the explicit form of vector fields T_i, we prove that the form is unique. □

Definition 6.2. A differential operator D of the form (14) is called to be of an order of Heisenberg equal to d if $c + \frac{1}{2}|K| \leq d$ for all c, K. We denote by $\mathcal{H}^d_{\lambda\mu}(\mathbb{R}^{2l+1|n})$ the space of differential operators of order of Heisenberg equal to d on $\mathbb{R}^{2l+1|n}$.

We can see that this space of differential operators is therefore filtered by the order of Heisenberg. Indeed, the total space $\mathcal{H}_{\lambda\mu}(\mathbb{R}^{2l+1|n})$ is the union of the spaces $\mathcal{H}^d_{\lambda\mu}(\mathbb{R}^{2l+1|n})$, i.e.,

$$\mathcal{H}_{\lambda\mu}(\mathbb{R}^{2l+1|n}) = \bigcup_{d\in\frac{1}{2}\mathbb{N}} \mathcal{H}^d_{\lambda\mu}(\mathbb{R}^{2l+1|n}).$$

Because $\mathcal{H}^d_{\lambda\mu}(\mathbb{R}^{2l+1|n}) \subset \mathcal{H}^{d+\frac{1}{2}}_{\lambda\mu}(\mathbb{R}^{2l+1|n})$, we have the following inclusions:

$$\mathcal{H}^0_{\lambda\mu}(\mathbb{R}^{2l+1|n}) \subset \mathcal{H}^{\frac{1}{2}}_{\lambda\mu}(\mathbb{R}^{2l+1|n}) \subset \mathcal{H}^1_{\lambda\mu}(\mathbb{R}^{2l+1|n})$$

$$\subset \cdots \subset \mathcal{H}^d_{\lambda\mu}(\mathbb{R}^{2l+1|n}) \subset \mathcal{H}^{d+\frac{1}{2}}_{\lambda\mu}(\mathbb{R}^{2l+1|n}) \subset \cdots$$

for all $d \in \frac{1}{2}\mathbb{N}$.

Definition 6.3. The graded space associated to the space $\mathcal{H}^d_{\lambda\mu}(\mathbb{R}^{2l+1|n})$ is denoted by $\mathcal{P}_\delta(\mathbb{R}^{2l+1|n})$. We have

$$\mathcal{P}_\delta(\mathbb{R}^{2l+1|n}) := \bigoplus_{d\in\frac{1}{2}\mathbb{N}} \mathcal{P}_\delta^d(\mathbb{R}^{2l+1|n}) := \sum_{d\in\frac{1}{2}\mathbb{N}} \mathcal{H}_{\lambda\mu}^d(\mathbb{R}^{2l+1|n}) / \mathcal{H}_{\lambda\mu}^{d-\frac{1}{2}}(\mathbb{R}^{2l+1|n}), \tag{15}$$

where $\delta = \mu - \lambda$.

The canonical projection defines the application $h\sigma$ called *Heisenberg symbol map* as follows:

$$h\sigma : \mathcal{H}_{\lambda\mu}^d(M) \to \mathcal{P}_\delta^d(M) : D \mapsto [D],$$

where $[D]$ means the equivalence class of D in the quotient $\mathcal{H}_{\lambda\mu}^d(\mathbb{R}^{2l+1|n})/$ $\mathcal{H}_{\lambda\mu}^{d-\frac{1}{2}}(\mathbb{R}^{2l+1|n})$.

7. Bifiltration of $\mathcal{D}_{\lambda\mu}(\mathbb{R}^{2l+1|n})$ and the Associated Bigraded Space $\sum_\delta(\mathbb{R}^{2l+1|n})$

In this section, we show that the canonical filtration and the filtration of Heisenberg induce a particular filtration called *bifiltration* of $D_{\lambda\mu}(\mathbb{R}^{2l+1|n})$. We generalize in super case, the model used by Conley and Ovsienko in [3] in the even case.

Definition 7.1. We define a bifiltration on $D_{\lambda\mu}(\mathbb{R}^{2l+1|n})$ by

$$\mathcal{D}_{\lambda\mu}^{k,d}(\mathbb{R}^{2l+1|n}) := \mathcal{D}_{\lambda\mu}^k(\mathbb{R}^{2l+1|n}) \cap \mathcal{H}_{\lambda\mu}^d(\mathbb{R}^{2l+1|n}).$$

The bigraded space $\sum_\delta(\mathbb{R}^{2l+1|n})$ associated to the bifiltration $\mathcal{D}_{\lambda\mu}^{k,d}(\mathbb{R}^{2l+1|n})$ is defined by

$$\sum_{\delta}(\mathbb{R}^{2l+1|n}) = \bigoplus_{k=0}^{\infty} \bigoplus_{d\in\frac{1}{2}\mathbb{N}} \sum_{\delta}^{k,d}(\mathbb{R}^{2l+1|n})$$

$$= \bigoplus_{k=0}^{\infty} \bigoplus_{d\in\frac{1}{2}\mathbb{N}} \mathcal{D}_{\lambda\mu}^{k,d}(\mathbb{R}^{2l+1|n})/(\mathcal{D}_{\lambda\mu}^{k-1,d}(\mathbb{R}^{2l+1|n}) + \mathcal{D}_{\lambda\mu}^{k,d-\frac{1}{2}}(\mathbb{R}^{2l+1|n})). \quad (16)$$

The elements of $\sum_{\delta}(\mathbb{R}^{2l+1|n})$ are called *fine symbols*.

We define accordingly the fine symbol map by

$$f\sigma_{k,d} : \mathcal{D}_{\lambda\mu}^{k,d}(\mathbb{R}^{2l+1|n}) \to \sum_{\delta}^{k,d}(\mathbb{R}^{2l+1|n}) : D \mapsto [D],$$

where the bracket means the equivalence class of D in $D_{\lambda\mu}^{k,d}(\mathbb{R}^{2l+1|n})/$ $(\mathcal{D}_{\lambda\mu}^{k-1,d}(\mathbb{R}^{2l+1|n}) + \mathcal{D}_{\lambda\mu}^{k,d-\frac{1}{2}}(\mathbb{R}^{2l+1|n}))$. To justify the terminology of fine symbol, we refer to [3].

Remark 7.2. By the definition of action, the applications $f\sigma_{k,d}$ and σ_k are $\mathcal{K}(2l+1|n)$-equivariant, i.e.,

$$L_{X_f}^{\delta,\mathcal{S}} \circ \sigma_k = \sigma_k \circ \mathcal{L}_{X_f}^{\lambda\mu} \text{ on } D_{\lambda\mu}^{k}(\mathbb{R}^{2l+1|n})$$

and

$$L_{X_f}^{\delta,\Sigma} \circ f\sigma_{k,d} = f\sigma_{k,d} \circ \mathcal{L}_{X_f}^{\lambda\mu} \text{ on } \mathcal{D}_{\lambda\mu}^{k,d}(\mathbb{R}^{2l+1|n}),$$

where $L_{X_f}^{\delta,\mathcal{S}}$ and $L_{X_f}^{\delta,\Sigma}$ denote, respectively, the actions $L_{X_f}^{\delta}$ on $\mathcal{S}(\mathbb{R}^{2l+1|n})$ and $L_{X_f}^{\delta}$ on $\Sigma(\mathbb{R}^{2l+1|n})$.

We have thus the following main result.

Proposition 7.3. *The action of* $\mathcal{K}(2l+1|n)$ *preserves the filtrations of* $\mathcal{D}_{\lambda\mu}(\mathbb{R}^{2l+1|n})$, $\mathcal{H}_{\lambda\mu}(\mathbb{R}^{2l+1|n})$ *and* $\mathcal{D}_{\lambda\mu}^{k,d}(\mathbb{R}^{2l+1|n})$.

Proof. We must verify that the form of differential operator defined by the formulae (14) is not modified by the action of $\mathcal{K}(2l+1|n)$. Therefore, we consider a differential operator D of order k and compute $\mathcal{L}^{\lambda\mu}_{X_f}(D)$ as follows:

$$L^{\lambda\mu}_{X_f}(D) = (X_f + \mu f')D - (-1)^{\widetilde{f}\widetilde{D}} D.(X_f + \lambda f')$$

$$= [X_f, D] + D_k,$$

where $D_k = \mu f'D - (-1)^{\widetilde{f}\widetilde{D}} \lambda D.f'$. We can see that the term D_k is a differential operator of the same order as D. Because of

$$[X_f, T_I] \in \langle T_1, \ldots, T_{2l+n}\rangle,$$

the term $[X_f, D]$ is a differential operator of order k. Therefore, the filtrations of $\mathcal{D}_{\lambda\mu}(\mathbb{R}^{2l+1|n})$, $\mathcal{H}_{\lambda\mu}(\mathbb{R}^{2l+1|n})$ and $\mathcal{D}^{k,d}_{\lambda\mu}(\mathbb{R}^{2l+1|n})$ are preserved by the canonical action of X_f. $\square$

The action of $\mathcal{K}(2l+1|n)$ on the space $\mathcal{H}_{\lambda\mu}(\mathbb{R}^{2l+1|n})$ induces the structure of $\mathcal{K}(2l+1|n)$-module on $\mathcal{P}^d_\delta(\mathbb{R}^{2l+1|n})$. If $L^{\mathcal{H}}_{X_f}$ denotes the action of X_f on $\mathcal{P}^d_\delta(\mathbb{R}^{2l+1|n})$ and $[D] \in \mathcal{P}^d_\delta(\mathbb{R}^{2l+1|n})$, then we can see that

$$L^{\mathcal{H}}_{X_f}[D] = [\mathcal{L}^{\lambda\mu}_{X_f} D].$$

8. $\mathfrak{spo}(2l+2|n)$-modules on Spaces $\mathcal{S}_\delta$, $\mathcal{P}_\delta$ and Σ_δ

In this section, we consider the Lie sub-superalgebra $\mathfrak{spo}(2l+2|n)$ of $\mathcal{K}(2l+1|n)$ constituted by the contact vector fields X_f whose superfunctions f are of degree atmost two. We show that the $\mathfrak{spo}(2l+2|n)$-modules on spaces of differential operators $\mathcal{D}_{\lambda\mu}(\mathbb{R}^{2l+1|n})$, $\mathcal{H}_{\lambda\mu}(\mathbb{R}^{2l+1|n})$ and $\mathcal{D}_{\lambda\mu}^{k,d}(\mathbb{R}^{2l+1|n})$ are induced on the symbol spaces $\mathcal{S}_{\delta}(\mathbb{R}^{2l+1|n})$, $\mathcal{P}_{\delta}(\mathbb{R}^{2l+1|n})$ and $\sum_{\delta}(\mathbb{R}^{2l+1|n})$. We give the explicit formulas of those actions. To facilitate the computations, we use the symbolical notations by defining the isomorphisms between the spaces $\mathcal{S}_{\delta}(\mathbb{R}^{2l+1|n})$ and $\mathcal{P}_{\delta}(\mathbb{R}^{2l+1|n})$ and some subspaces of the space $\mathcal{F}_{\delta}\otimes\mathrm{Pol}(T^{*}\mathbb{R}^{2l+1|n})$ (see also [3] in the even case). Therefore, we deduce the form of the elements of $\sum_{\delta}^{k,d}(\mathbb{R}^{2l+1|n})$.

First, we denote by ξ_0 the moment associated to the vector fields ∂_z and by ξ_r the moment associated to the vector fields T_r. We obtain the following isomorphism:

$$\mathcal{S}_{\delta}^{k}(\mathbb{R}^{2l+1|n})\cong\langle\alpha^{\delta}\xi_0^{c}\xi^{I},\ c+|I|=k\rangle, \tag{17}$$

where $I=(i_1,\ldots,i_{2l+n})$ is a multi-index of length $|I|=i_1+\cdots+i_{2l+n}$ and $\xi^{I}=\xi_1^{i_1}\cdots\xi_{2l+n}^{i_{2l+n}}$.

Denoting the isomorphism defined by the formula (17) by φ and using (14), we obtain

$$\varphi:\sum_{\substack{c,I\\ c+|I|\le k}}D_I\partial_z^{c}T_1^{i_1}\ldots T_{2l+n}^{i_{2l+n}}+\mathcal{D}_{\lambda,\mu}^{k-1}(\mathbb{R}^{2l+1|n})\mapsto\sum_{\substack{c,I\\ c+|I|=k}}D_I\alpha^{\delta}\xi_0^{c}\xi_1^{i_1}\ldots\xi_{2l+n}^{i_{2l+n}}. \tag{18}$$

From now, we represent ξ_0 by ζ and the notation ξ_i means the unified notation of the moments α_i, β_i and γ_i. The notation T_i (see the formula

(3)) means the unified notation of vector fields:

$$A_i = \partial_{x_i} + y_i\partial_z, \quad B_i = -\partial_{y_i} + x_i\partial_z, \quad \overline{D}_i = \partial_{\theta_i} - \theta_i\partial_z. \tag{19}$$

If the multi-indexes I, J, T are, respectively, given by $I = (i_1, i_2, \ldots, i_l)$, $J = (j_i, j_2, \ldots, j_l)$ and $T = (t_1, t_2, \ldots, t_n)$, then the quantities $A_1^{i_1} \cdots A_l^{i_l}$, $B_1^{j_1} \cdots B_l^{j_l}$ and $\overline{D}_1^{t_1} \cdots \overline{D}_n^{t_n}$ are represented by A^I, B^J and $\overline{D}^T$, respectively. We represent by K the multi-index (I, J, T) and by $|K|$, $|I|$, $|J|$ and $|T|$ their lengths, respectively.

The space $\mathcal{P}_\delta^d(\mathbb{R}^{2l+1|n})$ defined by the formula (15) is isomorphic to the subspace of $\mathcal{F}_\delta \otimes \mathrm{Pol}(T^*\mathbb{R}^{2l+1|n})$. More precisely, we have

$$\mathcal{P}_\delta^d(\mathbb{R}^{2l+1|n}) \cong \langle D_{c,K}\alpha^\delta\zeta^c\alpha^I\beta^J\gamma^T,\ c + \frac{1}{2}|K| = d\rangle$$

for all c, I, J, K and this isomorphism is explicitly given by

$$\varphi: \sum_{\substack{c,K \\ c+\frac{1}{2}|K|\leq d}} D_{c,K}\partial_z^c A^I B^J \overline{D}^T + \mathcal{H}_{\lambda,\mu}^{d-\frac{1}{2}}(\mathbb{R}^{2l+1|n}) \mapsto \sum_{\substack{c,K \\ c+\frac{1}{2}|K|=d}} D_{c,K}\alpha^\delta\zeta^c\alpha^I\beta^J\gamma^T. \tag{20}$$

In the same way, can see that the space $\sum_\delta^{k,d}(\mathbb{R}^{2l+1|n})$ is also isomorphic to the subspace of $\mathcal{F}_\delta \otimes \mathrm{Pol}(T^*\mathbb{R}^{2l+1|n})$. We have

$$\sum\nolimits_\delta^{k\,d}(\mathbb{R}^{2l+1|n}) \cong \left\langle D_{c,K}\alpha^\delta\zeta^c\alpha^I\beta^J\gamma^T,\ c + \frac{1}{2}|K| = d,\ c + |K| = k\right\rangle,$$

and more explicitly,

$$\varphi : \sum_{\substack{c, J \\ c+\frac{1}{2}|K| \le d \\ c+|K| \le k}} D_{c,K} \partial_z^c A^I B^J \overline{D}^T + (\mathcal{D}_{\lambda,\mu}^{k, d-\frac{1}{2}}(\mathbb{R}^{2l+1|n}) + \mathcal{D}_{\lambda,\mu}^{k-1, d}(\mathbb{R}^{2l+1|n}))$$

$$\mapsto \sum_{\substack{c, K \\ c+\frac{1}{2}|K| = d \\ c+|K| = k}} D_{c,K} \alpha^\delta \zeta^c \alpha^I \beta^J \gamma^T.$$

It is now possible to write the space $\mathcal{P}_\delta(\mathbb{R}^{2l+1|n})$ as a direct sum of the spaces $\sum_\delta^{k,d}(\mathbb{R}^{2l+1|n})$ as follows:

$$\mathcal{P}_\delta(\mathbb{R}^{2l+1|n}) = \bigoplus_{d \in \frac{1}{2}\mathbb{N}} \mathcal{H}^d(\mathbb{R}^{2l+1|n}) / \mathcal{H}^{d-\frac{1}{2}}(\mathbb{R}^{2l+1|n})$$

$$= \bigoplus_{d \in \frac{1}{2}\mathbb{N}} \bigoplus_{k=\lceil d \rceil}^{2d} \sum\nolimits_\delta^{k,d}(\mathbb{R}^{2l+1|n}), \tag{21}$$

where $\lceil x \rceil := \inf\{n \in \mathbb{N} : n \ge x\}$.

We recall that if $D \in \mathcal{H}_{\lambda,\mu}^d(\mathbb{R}^{2l+1|n})$ and if X_f is the contact vector field, then the Lie derivative in the direction of X_f, denoted by $L_{X_f}^{\mathcal{H}} D$, is given by

$$(L_{X_f}^{\mathcal{H}} D)(\psi) := L_{X_f}(D\psi) - (-1)^{\widetilde{f}\widetilde{D}} D(L_{X_f}\psi),$$

which is an element of $H_{\lambda,\mu}^d(\mathbb{R}^{2l+1|n})$, due to the fact that the action of X_f preserves $\mathrm{Tan}\mathbb{R}^{2l+1|n}$. Therefore, the canonical action of X_f preserves the bifiltered space $\mathcal{D}_{\lambda,\mu}^{k,d}(\mathbb{R}^{2l+1|n})$.

If we consider the Lie sub-superalgebra $\mathfrak{spo}(2l+2|n)$ of $\mathcal{K}(2l+1|n)$ constituted by the contact vector fields X_f whose superfunctions f are of degrees atmost two, then we can see that the action of contact vector fields X_f on $\mathcal{H}^d_{\lambda,\mu}(\mathbb{R}^{2l+1|n})$ and $\mathcal{D}^{k,d}_{\lambda,\mu}(\mathbb{R}^{2l+1|n})$ induces the $\mathfrak{spo}(2l+2|n)$-actions on the spaces $\mathcal{P}^d_\delta(\mathbb{R}^{2l+1|n})$ and $\sum^{k,d}_\delta(\mathbb{R}^{2l+1|n})$.

The following theorem gives the explicit formulas of the actions of the Lie superalgebra $\mathfrak{spo}(2l+2|n)$ on the spaces $\mathcal{S}_\delta(\mathbb{R}^{2l+1|n})$, $\mathcal{P}_\delta(\mathbb{R}^{2l+1|n})$ and on the fine symbol space $\sum_\delta(\mathbb{R}^{2l+1|n})$.

Theorem 8.1. *If* $X_f \in \mathfrak{spo}(2l+2|n)$ *and if we denote by* $L^{\mathcal{P}}_{X_f}$ (*resp.* $L^{\Sigma}_{X_f}, L^{\mathcal{S}}_{X_f}$) *the actions of* X_f *on* $\mathcal{P}_\delta(\mathbb{R}^{2l+1|n})$ (*resp.* $\Sigma_\delta(\mathbb{R}^{2l+1|n})$; $\mathcal{S}_\delta(\mathbb{R}^{2l+1|n})$), *then these actions are given by*

(i)

$$L^{\Sigma}_{X_f} = f\partial_z + \partial_z(f)(\delta - \mathcal{E}_\zeta) - \frac{1}{2}(-1)^{\tilde{f}\tilde{T}_r}\omega^{rs}T_r(f)T_s + \frac{1}{2}(-1)^{\tilde{f}(\tilde{T}_i+\tilde{T}_r)}\omega^{rs}T_iT_r(f)\xi_s\partial_{\xi_i}, \tag{22}$$

(ii) $L^{\mathcal{P}}_{X_f} = L^{\Sigma}_{X_f}$,

(iii)

$$L^{\mathcal{S}}_{X_f} = L^{\Sigma}_{X_f} + \frac{1}{2}(-1)^{\tilde{f}\tilde{T}_r}\omega^{rs}T_r(f')\xi_s\partial_\zeta, \tag{23}$$

where the notation $\mathcal{E}_\zeta$ *denotes the Euler operator* $\zeta\partial_\zeta$ *and the notations* ∂_z *and* T_i *denote the action of the vector fields* ∂_z *and* T_i *on the coefficients of the symbol.*

Proof. The spaces $\sum(\mathbb{R}^{2l+1|n}) := \bigcup_{\delta\in\mathbb{R}} \Sigma_\delta(\mathbb{R}^{2l+1|n})$, $\mathcal{P}(\mathbb{R}^{2l+1|n}) := \bigcup_{\delta\in\mathbb{R}} \mathcal{P}_\delta(\mathbb{R}^{2l+1|n})$ and $\mathcal{S}(\mathbb{R}^{2l+1|n}) := \bigcup_{\delta\in\mathbb{R}} \mathcal{S}_\delta(\mathbb{R}^{2l+1|n})$ are actually algebras for the canonical product of symbols. We can consider the operators $\widetilde{L}^{\Sigma}_{X_f}$, $\widetilde{L}^{\mathcal{P}}_{X_f}$ and $\widetilde{L}^{\mathcal{S}}_{X_f}$ acting, respectively, on $\sum(\mathbb{R}^{2l+1|n})$, $\mathcal{P}(\mathbb{R}^{2l+1|n})$ and $\mathcal{S}(\mathbb{R}^{2l+1|n})$ whose restrictions on the spaces $\sum_\delta(\mathbb{R}^{2l+1|n})$, $\mathcal{P}_\delta(\mathbb{R}^{2l+1|n})$ and $\mathcal{S}_\delta(\mathbb{R}^{2l+1|n})$ are given by $L^{\Sigma}_{X_f}$, $L^{\mathcal{P}}_{X_f}$ and $L^{\mathcal{S}}_{X_f}$ for all $\delta \in \mathbb{R}$. The operators $\widetilde{L}^{\Sigma}_{X_f}$, $\widetilde{L}^{\mathcal{P}}_{X_f}$ and $\widetilde{L}^{\mathcal{S}}_{X_f}$ are actually derivations of the spaces $\sum(\mathbb{R}^{2l+1|n})$, $\mathcal{P}(\mathbb{R}^{2l+1|n})$ and $\mathcal{S}(\mathbb{R}^{2l+1|n})$. We can compute the actions $L^{\Sigma}_{X_f}$ and $L^{\mathcal{S}}_{X_f}$ on the generators ζ, ξ_i and $g\alpha^{\delta}$ of the spaces $\sum_\delta(M)$ and $\mathcal{S}_\delta(M)$.

If the application of $L^{\Sigma}_{X_f}$ and $L^{\mathcal{S}}_{X_f}$ on those generators coincides with the actions of the second members of equations (22) and (23) on the same generators, then due to the fact that the right members of those equations are the derivation operators, equations (22) and (23) hold.

From the isomorphism φ defined by (20), we compute the operators $L^{\Sigma}_{X_f}$ and $L^{\mathcal{S}}_{X_f}$ on the generators ζ, ξ_i and $g\alpha^{\delta}$ of the spaces $\sum_\delta(M)$ and $\mathcal{S}_\delta(M)$ by using, respectively, the Lie derivative of differential operators ∂_z, T_i and $g\alpha^{\delta}$.

We obtain

$$L^{\Sigma}_{X_f}(T_i) = [X_f, T_i] + \delta f' T_i$$

$$= [f\partial_z, T_i] + \delta f' T_i - \frac{1}{2}(-1)^{\widetilde{f}\widetilde{T}_r}\omega^{rs}(-(-1)^{\widetilde{f}\widetilde{T}_i}[T_i, T_r(f)T_s])$$

$$= -(-1)^{\tilde{f}\tilde{T}_i} T_i(f)\partial_z + \delta f' T_i$$

$$+ \frac{1}{2}(-1)^{(\tilde{T}_r+\tilde{T}_i)\tilde{f}} \omega^{rs}(T_i T_r(f)T_s + (-1)^{\tilde{T}_i(\tilde{T}_r+\tilde{f})} T_r(f)[T_i, T_s]).$$

Since the commutator $[T_i, T_s]$ is equal to $-2\omega_{is}\partial_z$, using the isomorphism φ given by (20), we obtain

$$L^{\Sigma}_{X_f}(\xi_i) = \left(\delta f' Id + \frac{1}{2}(-1)^{\tilde{f}(\tilde{T}_r+\tilde{T}_i)}\omega^{rs} T_i T_r(f)\xi_s\partial_{\xi_i}\right)(\xi_i). \tag{24}$$

Computing $L^{\Sigma}_{X_f}(\partial_z)$ and using the isomorphism φ given by (20), we obtain

$$L^{\Sigma}_{X_f}(\zeta) = (\delta - 1) f' Id(\zeta), \tag{25}$$

and finally the same condition

$$L^{\Sigma}_{X_f}(g\alpha^{\delta}) = \left(f\partial_z + \delta f' Id - (-1)^{\tilde{f}\tilde{T}_r}\frac{1}{2}\omega^{rs} T_r(f) T_s\right)(g\alpha^{\delta}). \tag{26}$$

We can see that if we restrict the formula (22) to the generators T_i, ∂_z and $g\alpha^{\delta}$, then we obtain, respectively, formulas (24), (25) and (26). The proof is similar. □

Remark 8.2. The spaces of symbols $\mathcal{S}_\delta(\mathbb{R}^{2l+1|n})$, $\mathcal{P}_\delta(\mathbb{R}^{2l+1|n})$ and $\sum_\delta(\mathbb{R}^{2l+1|n})$ are $\mathfrak{spo}(2l+2|n)$-modules.

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