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ON DISTANCE IRREGULAR LABELLING OF GRAPHS

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Abstract

Motivated by definition of distance magic labelling, we introduce a new type of irregular labelling whose evaluation is based on the neighbourhood of a vertex. We define a distance irregular labelling on a graph G with v vertices to be an assignment of positive integer labels to vertices so that the weights calculated at vertices are distinct. The weight of a vertex x in G is defined to be the sum of the labels of all the vertices adjacent to x. The distance irregularity strength of G, denoted by $dis(G)$, is the minimum value of the largest label over all such irregular assignments. We determine the distance irregularity strengths of some particular classes of graphs, such as complete graphs K_n for $n \geq 3$, paths P_n for $n \geq 4$, cycles C_n for $n \equiv 0, 1, 2, 5 \pmod 8$ and wheels W_n for $n \equiv 0, 1, 2, 5 \pmod 8$. We also present some non-existence results for other classes of graphs.

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1. Introduction

Let G be a finite, simple and undirected graph with vertex set V and edge set E, where $|V| = v$ and $|E| = e$. The labelling of a graph is a mapping that carries a set of graph elements (vertices or edges or both) onto a set of numbers (usually positive integers). There are many types of graph labellings that have been introduced (see [4] for a complete survey). One of them is *magic labelling* that was first introduced by Sedláček in 1963 [8]. A magic labelling of a graph G was defined to be a bijection f from E to a set of positive integers such that for all distinct edges with different labels, the sum of all edge labels incident to the vertex is a constant.

Another magic labelling a so-called *edge-magic total labelling* was defined to be a labelling on the vertices and edges in which the labels are the integers from 1 to $v + e$ and where the sum of the labels on an edge and its two endpoints is a constant [5]. Furthermore, *vertex-magic total labelling* was defined to be an assignment of the integers from 1 to $v + e$ to the vertices and edges of G so that at each vertex, the vertex label and the labels on the edges incident at that vertex add to a fixed constant [6]. Moreover, a graph labelling based on both magic and distance was introduced by Miller et al. [7]. In this labelling, the 1-vertex-weight of each vertex x in G under a vertex labelling was defined to be the sum of vertex labels of the vertices adjacent to x (that is, distance 1 from x). If all vertices in G have the same weight k, then we call the labelling a 1-*vertex-magic vertex labelling* [7]. Another type of graph labelling is an irregular assignment. The notion of this labelling was introduced by Chartrand et al. in 1988 [3]. In the paper, the problem was proposed, namely, what is the minimum value of the largest label over all such irregular assignments if the edges of a simple connected graph of order at least 3 are assigned by positive integer labels such that the graph becomes irregular, that is, the weight (the sum of edge labels) at each vertex is distinct? The minimum value of the largest label is called *irregularity strength* of the graph [3].

Using similar assignment, but apply to both edges and vertices of a graph, Bača et al. [2] introduced the *irregular total k-labelling*. For a graph

$G = (V, E)$ with vertex set V and edge set E, a total k-labelling was defined to be a labelling $\lambda : V \cup E \to 1, 2, ..., k$. A total k-labelling is defined to be an edge irregular total k-labelling of the graph G if for every two different edges e and f of G, there is $wt(e) \neq wt(f)$, and to be a vertex irregular total k-labelling of G if for every two distinct vertices x and y of G, there is $wt(x) \neq wt(y)$. The minimum k for which the graph G has an edge irregular total k-labelling is called the *total edge irregularity strength* of the graph G, $tes(G)$. Analogously, they defined the *total vertex irregularity strength* of G, $tvs(G)$, as the minimum k for which there exists a vertex irregular total k-labelling of G [2].

Motivated by the distance magic labelling and (a, d)-distance antimagic labelling as well as the irregular labelling, we introduce a new graph labelling based on both irregularity and distance. The domain of the labelling will be the set of all vertices and the codomain will be 1, 2, ..., k. We call this labelling a *vertex labelling*. We define the weight of each vertex x in G under a vertex labelling to be the sum of vertex labels of the vertices adjacent to x (distance 1 from x). If all vertices in G have different weights, then we call the labelling a *distance irregular vertex labelling*. The minimum k for which the graph G has a distance irregular vertex labelling is called the *distance irregularity strength* of the graph G, denoted by $dis(G)$. In this paper, we determine the distance irregularity strengths of some particular classes of graphs, such as complete graphs K_n for $n \geq 3$, paths P_n for $n \geq 4$, cycles C_n for $n \equiv 0, 1, 2, 5 \pmod 8$ and wheels W_n for $n \equiv 0, 1, 2, 5 \pmod 8$. We also present some non-existence results for other classes of graphs.

2. Basic Concept

In this section, we present formal definition of the distance irregular vertex labelling and two observations regarding to the non-existence of

the such labelling for graphs that contain vertices with the identical neighbourhood and almost identical neighbourhood.

Definition 1. A distance irregular vertex labelling of the graph G with v vertices is an assignment $\lambda : V \to \{1, 2, ..., k\}$ so that the weights calculated at vertices are distinct. The weight of a vertex x in G is defined as the sum of the labels of all the vertices adjacent to x (distance 1 from x), that is,

$$wt(x) = \sum_{y \in N(x)} \lambda(y).$$

The distance irregularity strength of G, denoted by $dis(G)$, is the minimum value of the largest label k over all such irregular assignments.

In this labelling, the vertex labels are not necessarily different. Figure 1 illustrates an example of a distance irregular vertex labelling of the graph G with the distance irregularity strength 2.

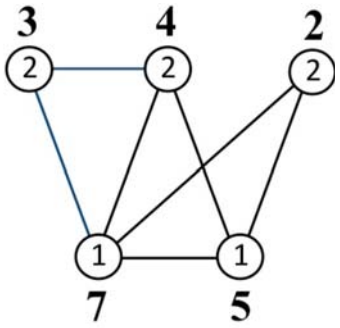


Figure 1. An example of graph G with $dis(G) = 2$.

We note that not every connected graph has a distance irregular vertex labelling as a consequence of the following observation:

Observation 1. Let u and w be any two distinct vertices in a connected graph G. If u and w have identical neighbours, i.e., $N(u) = N(w)$, then G has no distance irregular vertex labelling.

The observation can be easily seen as both vertices u and w should have the same weights. Consequently, some classes of graphs have no distance irregular vertex labelling, such as:

- Complete bipartite graphs $K_{m,n}$ for any $m, n \geq 3$.
- Complete multipartite graphs $H_{m,n}$ for any $m, n \geq 2$.
- Stars on $n+1$ vertices S_n for $n \geq 2$.
- Trees T_n for any $n \geq 3$ that contain vertex with at least two leaves.

Observation 2. Let u and w be any two adjacent vertices in a connected graph G. If $N(u) - \{w\} = N(w) - \{u\}$, then the labels of u and w must be distinct, that is, $\lambda(u) \neq \lambda(\omega)$.

Proof. Let $N(u) - \{w\} = N(w) - \{u\}$ for two adjacent vertices $u, w \in V(G)$. Then

$$wt(u) = \lambda(w) + \sum_{y \in N(u) - \{w\}} \lambda(y)$$

and

$$wt(w) = \lambda(u) + \sum_{y \in N(w) - \{u\}} \lambda(y).$$

Since $\lambda(w) + \sum_{y \in N(u) - \{w\}} \lambda(y) = \lambda(u) + \sum_{y \in N(w) - \{u\}} \lambda(y)$, thus if $\lambda(u) = \lambda(w)$, then $wt(u) = wt(w)$ which is impossible in distance irregular labelling. □

3. Main Results

We start this section with the lemma that gives the lower bound on the distance irregularity strength of connected graphs that have distance irregular vertex labelling.

Lemma 1. *Let G be a connected graph on v vertices with minimum degree δ and maximum degree Δ containing no vertices with identical neighbours. Then*

$$dis(G) \geq \left\lceil \frac{v + \delta - 1}{\Delta} \right\rceil.$$

Proof. Let G be a connected graph on v vertices containing no vertices with identical neighbours. Let δ be the minimum degree of vertices in G and Δ be the maximum degree of vertices in G. The smallest weight value among the weights of vertices of G is δ. Since the weight of every vertex must be distinct and G has v vertices, the largest weight value among the weights of vertices of G is at least $v + \delta - 1$. This weight is obtained from the sum of at most Δ integers. Thus, the largest label that contributes to this weight must be at least $\left\lceil \frac{v + \delta - 1}{\Delta} \right\rceil$. □

In the next theorems, we present the distance irregularity strengths of some natural classes of graphs, namely, complete graphs, paths, cycles, and wheels.

Theorem 1. *Let K_n be a complete graph with $n \geq 3$ vertices. Then $dis(K_n) = n$.*

Proof. Let K_n be a complete graph with $n \geq 3$ vertices. Let $u, w \in V(K_n)$ and $u \neq w$. Since every vertex of K_n is adjacent to all other vertices, $N(u) - w = N(w) - u$. By Observation 2, $\lambda(u) \neq \lambda(w)$. Thus, the labels of all vertices in K_n must be distinct. Consequently, $dis(K_n) \geq n$. Assigning the n vertices of K_n with the consecutive integers 1, 2, ..., n as the labels result in different vertex-weights:

$$\frac{n(n-1)}{2} - 1, \frac{n(n-1)}{2} - 2, ..., \frac{n(n-1)}{2} - n.$$

This labelling gives the largest label n, that is, $dis(K_n) \leq n$. Combining with $dis(K_n) \geq n$, we conclude that $dis(K_n) = n$. □

By Observation 1, path on 3 vertices P_3 has no distance irregular vertex labelling. However, for $n \geq 4$, path P_n admits the distance irregular vertex labelling and its distance irregularity strength is determined below.

Theorem 2. *Let* P_n *be a path with* $n \geq 4$ *vertices. Then* $dis(P_n) = \left\lceil \frac{n}{2} \right\rceil$.

Proof. Let P_n be a path with $n \geq 4$ vertices. First, we show the lower bound on the distance irregularity strength of path P_n. By Lemma 1, $dis(P_n) \geq \left\lceil \frac{n+1-1}{2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil$. To show the upper bound, we consider 3 cases. Let x_i be the vertices of P_n and $x_i x_{i+1}$ be the edges of P_n.

Case 1. For $n \equiv 0 \bmod 4$.

Label the vertices and edges using the formula:

$$\lambda(x_i) = \begin{cases} \frac{n}{2} - 2\left\lfloor \frac{i}{4} \right\rfloor & \text{for } i = 1, 3, ..., n-1, \\ \frac{i}{2} & \text{for } i = 2, 4, ..., n. \end{cases}$$

This labelling provides vertex-weights:

$$wt(x_i) = \begin{cases} i, & \text{for } i = 1, 3, ..., n-1, \\ n + 2 - i, & \text{for } i = 2, 4, ..., n. \end{cases}$$

Case 2. For $n \equiv 2 \bmod 4$.

Label the vertices and edges using the formula:

$$\lambda(x_i) = \begin{cases} \frac{n}{2} - \left\lfloor \frac{i}{4} \right\rfloor & \text{for } i = 1, 3, \text{ and } 5, \\ \frac{n}{2} - 2\left\lfloor \frac{i-2}{4} \right\rfloor - 1 & \text{for } i = 7, 9, ..., n-1, \\ \frac{i}{2} & \text{for } i = 2, 4, ..., n-2, \\ \frac{n}{2} - 1 & \text{for } i = n. \end{cases}$$

This labelling provides vertex-weights:

$$wt(x_i) = \begin{cases} i & \text{for } i = 1, 3, ..., n-3, \\ n+1-\dfrac{i}{2} & \text{for } i = 2, 4, \\ n+1-i & \text{for } i = 6, 8, ..., n-2, \\ 2 & \text{for } i = n. \end{cases}$$

Case 3. For $n \equiv 1 \bmod 2$.

For $n = 5, 7, 9$ and 11, label the vertices and edges as follows:

$P_5 = \{2, 1, 2, 2, 3\}$ that provides vertex-weights $\{1, 4, 3, 5, 2\}$.

$P_7 = \{2, 1, 3, 2, 3, 2, 4\}$ that provides vertex-weights $\{1, 5, 3, 6, 4, 7, 2\}$.

$P_9 = \{1, 1, 2, 3, 5, 3, 5, 2, 4\}$ that provides vertex-weights $\{1, 3, 4, 7, 6$ $10, 5, 9, 2\}$.

$P_{11} = \{2, 1, 2, 2, 6, 3, 4, 4, 5, 2, 6\}$ that provides vertex-weights $\{1, 4, 3, 8, 5, 10, 7, 9, 6, 11, 2\}$.

For $n \geq 13$, label the vertices and edges by using the formula:

$$\lambda(x_i) = \begin{cases} \dfrac{n+1}{2} - \left\lfloor \dfrac{i+1}{4} \right\rfloor & \text{for } i = 1, 3, ..., 2\left\lfloor \dfrac{n+13}{4} \right\rfloor + 1, \\ \dfrac{n+1}{2} - \left\lfloor \dfrac{i+1}{4} \right\rfloor - 1 & \text{for } i = 2\left\lfloor \dfrac{n+13}{4} \right\rfloor + 3, 2\left\lfloor \dfrac{i+13}{4} \right\rfloor + 5, ..., n, \\ \left\lfloor \dfrac{i+2}{2} \right\rfloor & \text{for } i = 2, 4, ..., n-5, \\ \dfrac{n-5}{2} & \text{for } i = n-3, \\ 2 & \text{for } i = n-1. \end{cases}$$

This labelling provides vertex-weights:

$$wt(x_i) = \begin{cases} 1 & \text{for } i = 1, \\ n+1-i/2 & \text{for } i = 2, 4, ..., 2\left\lfloor \frac{n+13}{4} \right\rfloor, \\ n-i/2 & \text{for } i = 2\left\lfloor \frac{n+13}{4} \right\rfloor + 2, 2\left\lfloor \frac{n+13}{4} \right\rfloor + 4, ..., n-1, \\ \frac{i+3}{2} & \text{for } i = 3, 5, ..., n-6, \\ \left\lceil \frac{3n-13}{4} \right\rceil & \text{for } i = n-4, \\ \frac{n-1}{2} & \text{for } i = n-2, \\ 2 & \text{for } i = n. \end{cases}$$

For each case, the labellings provide different weights for each vertex and the largest label is $\left\lceil \frac{n}{2} \right\rceil$ which leads to $dis(P_n) \le \left\lceil \frac{n}{2} \right\rceil$. Combining with the lower bound, we conclude that $dis(P_n) = \left\lceil \frac{n}{2} \right\rceil$ for $n \ge 4$. □

The following theorem presents the distance irregularity strength of a cycle with n vertices C_n for $n \ge 5$. For $n = 3$, $C_3 = K_3$, thus $dis(C_3) = dis(K_3) = 3$. For $n = 4$, by Observation 1, C_4 has no distance irregular vertex labelling.

Theorem 3. *Let C_n be a cycle with $n \ge 5$ vertices for $n \equiv 0, 1, 2, 5 \pmod 8$. Then $dis(C_n) = \left\lceil \frac{n+1}{2} \right\rceil$.*

Proof. Let C_n be a cycle with $n \ge 5$ vertices for $n \equiv 0, 1, 2, 5 \pmod 8$. First, we show the lower bound on the distance irregularity strength of a cycle C_n. Since C_n is a regular graph of degree 2, by Lemma 1,

$dis(C_n) \geq \left\lceil \frac{n+1}{2} \right\rceil$. To show the upper bound, we consider cases, for $n \equiv 0, 1, 2, 5 \pmod 8$. Let x_i be the vertices of C_n and $x_i x_{i+1}$ for $i = 1, 2, .., n-1$ and $x_n x_1$ be edges of C_n.

Case 1. For $n \equiv 1 \bmod 4$ ($n \equiv 1, 5 \bmod 8$).

Label the vertices and edges by using the formula:

$$\lambda(x_i) = \begin{cases} \frac{n+1}{2} - 2\left\lfloor \frac{i}{4} \right\rfloor & \text{for } i = 1, 3, ..., n-2, \\ \frac{i}{2} & \text{for } i = 2, 4, ..., n-1, \\ 1 & \text{for } i = n. \end{cases}$$

This labelling provides vertex-weights:

$$wt(x_i) = \begin{cases} 2 & \text{for } i = 1, \\ i & \text{for } i = 3, 5, ..., n, \\ n+3-i & \text{for } i = 2, 4, ..., n-1. \end{cases}$$

Case 2. For $n \equiv 0 \bmod 8$.

Label the vertices and edges using the formula:

$$\lambda(x_i) = \begin{cases} \frac{n}{4} + 1 & \text{for } i = 1, \\ \frac{n}{4} + \frac{i+1}{2} - 1 & \text{for } i = 3, 5, ..., \frac{n}{2} + 1, \\ \frac{n}{2} - 2\left\lfloor \frac{i-5-n/2}{4} \right\rfloor - 1 & \text{for } i = \frac{n}{2} + 3, \frac{n}{2} + 5, ..., n-1, \\ \frac{i}{2} & \text{for } i = 2, 4, ..., \frac{n}{2} + 2, \\ \frac{n}{4} - 2\left\lfloor \frac{i-3-n/2}{4} \right\rfloor - 1 & \text{for } i = \frac{n}{2} + 4, \frac{n}{2} + 6, ..., n-2, \\ 1 & \text{for } i = n. \end{cases}$$

This labelling provides vertex-weights:

$$wt(x_i) = \begin{cases} 2 & \text{for } i = 1, \\ \frac{n}{2} + 2 & \text{for } i = 2, \\ i & \text{for } i = 3, 5, ..., \frac{n}{2} + 1, \\ \frac{n}{2} - 1 + i & \text{for } i = 4, 6, ..., \frac{n}{2} + 2, \\ n + 3 - i & \text{for } i = \frac{n}{2} + 3, \frac{n}{2} + 5, ..., n - 1, \\ \frac{3n}{2} + 4 - i & \text{for } i = \frac{n}{2} + 4, \frac{n}{2} + 6, ..., n. \end{cases}$$

Case 3. For $n \equiv 2 \bmod 8$.

Label the vertices and edges using the formula:

$$\lambda(x_i) = \begin{cases} \frac{n+2}{4} + \frac{i+1}{2} & \text{for } i = 1, 3, ..., \frac{n}{2}, \\ \frac{n}{2} - 2\left\lfloor \frac{i - n/2}{4} \right\rfloor + 1 & \text{for } i = \frac{n}{2} + 2, \frac{n}{2} + 4, ..., n - 1, \\ \frac{i}{2} & \text{for } i = 2, 4, ..., \frac{n}{2} + 1, \\ \frac{n+2}{4} - 2\left\lfloor \frac{i - n/2}{4} \right\rfloor & \text{for } i = \frac{n}{2} + 3, \frac{n}{2} + 5, ..., n. \end{cases}$$

This labelling provides vertex-weights:

$$wt(x_i) = \begin{cases} 2 & \text{for } i = 1, \\ \frac{n}{2} + 2 + i & \text{for } i = 2, 4, ..., \frac{n}{2} - 1, \\ i & \text{for } i = 3, 5, ..., \frac{n}{2}, \\ \frac{3(n+2)}{2} - i & \text{for } i = \frac{n}{2} + 1, \frac{n}{2} + 3, ..., n, \\ n + 3 - i & \text{for } i = \frac{n}{2} + 2, \frac{n}{2} + 4, ..., n - 1. \end{cases}$$

For each case, the labellings provide different weights for each

vertex and the largest label is $\left\lceil \frac{n+1}{2} \right\rceil$ which leads to $dis(C_n) \le \left\lceil \frac{n+1}{2} \right\rceil$. Combining with the lower bound, we conclude that $dis(C_n) = \left\lceil \frac{n+1}{2} \right\rceil$ for $n \equiv 0, 1, 2, 5 \pmod 8$. □

Theorem 4. *Let W_n be a wheel with $n \ge 5$ rim vertices for $n \equiv 0, 1, 2, 5 \pmod 8$. Then $dis(W_n) = \left\lceil \frac{n+1}{2} \right\rceil$.*

Proof. Let W_n be a wheel with $n \ge 5$ rim vertices for $n \equiv 0, 1, 2, 5 \pmod 8$. Let k be the largest label of W_n and c be the center of W_n. The optimal weights of vertices W_n are $\lambda(c) + 2, \lambda(c) + 3, ..., \lambda(c) + n + 1, wt(c)$. Thus, $dis(W_n) = k \ge \max\left\{\left\lceil \frac{\lambda(c)+n+1}{3} \right\rceil, \left\lceil \frac{wt(c)}{n} \right\rceil\right\}$. Since k is the largest label which is the average of the sum of labels of center and 2 rim vertices, it implies that $\lambda(c) \le \left\lceil \frac{n+1}{2} \right\rceil \le k$. As $n \ge 5$, it follows that $k \ge \left\lceil \frac{\lambda(c)+n+1}{3} \right\rceil > \left\lceil \frac{w(c)}{n} \right\rceil$. Thus, $k \ge \left\lceil \frac{\lambda(c)+n+1}{3} \right\rceil$:

$$k \ge \left\lceil \frac{\lambda(c)+n+1}{3} \right\rceil$$

$$\ge \left\lceil \frac{\left\lceil \frac{n+1}{2} \right\rceil + n + 1}{3} \right\rceil$$

$$\ge \left\lceil \frac{\frac{n+1}{2} + n + 1}{3} \right\rceil$$

$$\ge \left\lceil \frac{\frac{3n+3}{2}}{3} \right\rceil$$

$$= \left\lceil \frac{n+1}{2} \right\rceil.$$

Therefore, $k \geq \left\lceil \frac{n+1}{2} \right\rceil$.

Assigning the n rim vertices of W_n as the labels of vertices of cycle C_n presented in the proof of Theorem 3 and k to the center results in $dis(W_n) \leq \left\lceil \frac{n+1}{2} \right\rceil$. Combining with $dis(W_n) \geq \left\lceil \frac{n+1}{2} \right\rceil$, we conclude that $dis(W_n) = \left\lceil \frac{n+1}{2} \right\rceil$. □

4. Conclusion

We conclude this paper with the following open problem:

Open Problem 1. Determine the distance irregularity strengths of some particular families of graphs.

Open Problem 2. Generalize the distance irregular labelling of graphs where the weight sum of each vertex includes the label of the vertex itself (closed version).

Open Problem 3. Expand the distance irregular labelling of graphs to the distance at least 2.

Open Problem 4. Characterize the relationship between distance irregular labelling and (a, d)-distance antimagic labelling.

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