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ON THE SYMMETRIC PROPERTIES FOR THE GENERALIZED DEGENERATE TANGENT POLYNOMIALS

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Abstract

In [6], we introduced the generalized degenerate tangent numbers and polynomials. In this paper, we study the symmetry for the generalized degenerate tangent numbers $\mathcal{T}_{n,\chi}(\lambda)$ and polynomials $\mathcal{T}_{n,\chi}(x, \lambda)$. We obtain some interesting identities of the power sums and the generalized degenerate tangent polynomials $\mathcal{T}_{n,\chi}(x, \lambda)$ using the symmetric properties for the p-adic invariant integral on $\mathbb{Z}_p$.

1. Introduction

Recently, we have studied the area of tangent numbers and polynomials (see [4-8]). In [1], Carlitz introduced the degenerate Bernoulli polynomials. Recently, Qi et al. [2] studied the partially degenerate Bernoulli polynomials of the first kind in p-adic field. In this paper, we obtain some interesting properties for generalized degenerate tangent numbers and polynomials. Throughout this paper, we use the following notations. Let p be a fixed odd

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prime number. By $\mathbb{Z}_p$, we denote the ring of p-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, $\mathbb{C}$ denotes the complex number field, $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$, $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, then q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, then one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $g \in UD(\mathbb{Z}_p)$, the Fermionic p-adic invariant q-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \text{ see [3].}$$

Note that

$$\lim_{q\to 1} I_{-q}(g) = I_{-1}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x). \tag{1.1}$$

If we take $g_n(x) = g(x+n)$ in (1.1), then we see that

$$I_{-1}(g_n) = (-1)^n I_{-1}(g) + 2\sum_{l=0}^{n-1} (-1)^{n-1-l} g(l). \tag{1.2}$$

Letting a fixed positive integer d with $(p, d) = 1$, set

$$X = X_d = \lim_{\overleftarrow{N}} (\mathbb{Z}/dp^N\mathbb{Z}), \quad X_1 = \mathbb{Z}_p,$$

$$X^* = \bigcup_{\substack{0<a<dp \\ (a, p)=1}} a + dp\mathbb{Z}_p, \quad a + dp^N \mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \le a < dp^N$. It is easy to see that

$$\int_X g(x) d\mu_{-1}(x) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x). \tag{1.3}$$

For $t, \lambda \in \mathbb{Z}_p$ such that $|\lambda t|_p < p^{-\frac{1}{p-1}}$, if we take $g(x) = \chi(x) \cdot (1 + \lambda t)^{2x/\lambda}$ in (1.2), then we easily see that

$$\int_X \chi(x)(1 + \lambda t)^{2x/\lambda} d\mu_{-1}(x) = \frac{2\sum_{a=0}^{d-1} (-1)^a \chi(a)(1 + \lambda t)^{2a/\lambda}}{(1 + \lambda t)^{2/\lambda} + 1}.$$

Let us define the degenerate generalized tangent numbers $\mathcal{T}_{n,\chi}(\lambda)$ and polynomials $\mathcal{T}_{n,\chi}(x, \lambda)$ as follows:

$$\int_X \chi(y)(1 + \lambda t)^{2y/\lambda} d\mu_{-1}(y) = \sum_{n=0}^{\infty} \mathcal{T}_{n,\chi}(\lambda) \frac{t^n}{n!}, \tag{1.4}$$

$$\int_X \chi(y)(1 + \lambda t)^{(2y+x)/\lambda} d\mu_{-1}(y) = \sum_{n=0}^{\infty} \mathcal{T}_{n,\chi}(\lambda) \frac{t^n}{n!}. \tag{1.5}$$

By (1.4) and (1.5), we obtain the Witt's formula.

Theorem 1. *For $n \in \mathbb{Z}_+$, we have*

$$\int_{\mathbb{Z}_p} \chi(x)(2x \mid \lambda)_n d\mu_{-1}(x) = \mathcal{T}_{n,\chi}(\lambda),$$

$$\int_{\mathbb{Z}_p} \chi(y)(x + 2y \mid \lambda)_n d\mu_{-1}(y) = \mathcal{T}_{n,\chi}(x, \lambda).$$

Theorem 2. *For* $n \geq 0$, *we have*

$$\mathcal{T}_{n,\chi}(x, \lambda) = \sum_{l=0}^{n} \binom{n}{l} \mathcal{T}_{l,\chi}(\lambda)(x|\lambda)_{n-l}.$$

The generalized falling factorial $(x|\lambda)_n$ with increment λ is defined by

$$(x|\lambda)_n = \prod_{k=0}^{n-1}(x - \lambda k) \tag{1.6}$$

for positive integer n, with the convention $(x|\lambda)_0 = 1$ (see [9]). We also need the binomial theorem: for a variable x,

$$(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty}(x|\lambda)_n \frac{t^n}{n!}. \tag{1.7}$$

2. Symmetry for the Generalized Degenerate Tangent Polynomials

In this section, we assume that $q \in \mathbb{C}_p$. We obtain some interesting identities of alternating generalized falling factorial sums and generalized degenerate tangent polynomials $\mathcal{T}_{n,\chi}(x)$ using the symmetric properties for the p-adic invariant integral on $\mathbb{Z}_p$. If n is odd from (1.2), then we obtain

$$I_{-1}(g_n) + I_{-1}(g) = 2\sum_{k=0}^{n-1}(-1)^k g(k). \tag{2.1}$$

It will be more convenient to write (2.1) as the equivalent integral form

$$\int_{\mathbb{Z}_p} g(x+n)d\mu_{-1}(x) + \int_{\mathbb{Z}_p} g(x)d\mu_{-1}(x) = 2\sum_{k=0}^{n-1}(-1)^k g(k). \tag{2.2}$$

Substituting $g(x) = \chi(x)(1+\lambda t)^{2x/\lambda}$ into the above, we obtain

$$\int_X \chi(x+n)(1+\lambda t)^{(2x+2n)/\lambda} d\mu_{-1}(x) + \int_X \chi(x)(1+\lambda t)^{2x/\lambda} d\mu_{-1}(x)$$

$$= 2\sum_{j=0}^{n-1} (-1)^j \chi(j)(1+\lambda t)^{2j/\lambda}. \tag{2.3}$$

For $k \in \mathbb{Z}_+$, let us define the alternating generalized falling factorial sums $S_{k,\chi}(n, \lambda)$ as follows:

$$S_{k,\chi}(n, \lambda) = \sum_{l=0}^{n} (-1)^l \chi(l)(2l \mid \lambda)_k. \tag{2.4}$$

After some calculations, we have

$$\int_X \chi(x)(1+\lambda t)^{2x/\lambda} d\mu_{-1}(x) = \frac{2\sum_{a=0}^{d-1} \chi(a)(-1)^a (1+\lambda t)^{2a/\lambda}}{(1+\lambda t)^{2d/\lambda} + 1},$$

$$\int_X \chi(x)(1+\lambda t)^{(2x+2n)/\lambda} d\mu_{-1}(x)$$

$$= (1+\lambda t)^{2n/\lambda} \frac{2\sum_{a=0}^{d-1} \chi(a)(-1)^a (1+\lambda t)^{2a/\lambda}}{(1+\lambda t)^{2d/\lambda} + 1}. \tag{2.5}$$

By using (2.5), we have

$$\int_X \chi(x)(1+\lambda t)^{(2x+2nd)/\lambda} d\mu_{-1}(x) + \int_X \chi(x)(1+\lambda t)^{2x/\lambda} d\mu_{-1}(x)$$

$$= (1 + (1+\lambda t)^{2nd/\lambda}) \frac{2\sum_{a=0}^{d-1} \chi(a)(-1)^a (1+\lambda t)^{2a/\lambda}}{(1+\lambda t)^{2d/\lambda} + 1}.$$

From the above, we get

$$\int_X \chi(x)(1+\lambda t)^{(2x+2nd)/\lambda} d\mu_{-1}(x) + \int_X \chi(x)(1+\lambda t)^{2x/\lambda} d\mu_{-1}(x)$$

$$= \frac{2\int_X \chi(x)(1+\lambda t)^{2x/\lambda} d\mu_{-1}(x)}{\int_X (1+\lambda t)^{2ndx/\lambda} d\mu_{-1}(x)}. \tag{2.6}$$

By (1.7) and (2.3), we obtain

$$\sum_{m=0}^{\infty}\left(\int_X \chi(x)(2x+2nd\,|\,\lambda)_m d\mu_{-1}(x) + \int_X \chi(x)(2x\,|\,\lambda)_m d\mu_{-1}(x)\right)\frac{t^m}{m!}$$

$$= \sum_{m=0}^{\infty}\left(2\sum_{j=0}^{nd-1}(-1)^j \chi(j)(2j\,|\,\lambda)_m\right)\frac{t^m}{m!}.$$

By comparing coefficients $\frac{t^m}{m!}$ in the above equation, we obtain

$$\sum_{k=0}^{m}\binom{m}{k}(2nd\,|\,\lambda)_{m-k}\int_X \chi(x)(2x\,|\,\lambda)_k d\mu_{-1}(x) + \int_X \chi(x)(2x\,|\,\lambda)_m d\mu_{-1}(x)$$

$$= 2\sum_{j=0}^{nd-1}(-1)^j \chi(j)(2j\,|\,\lambda)_m.$$

By using (2.4), we have

$$\sum_{k=0}^{m}\binom{m}{k}(2nd\,|\,\lambda)_{m-k}\int_X \chi(x)(2x\,|\,\lambda)_k d\mu_{-1}(x) + \int_X \chi(x)(2x\,|\,\lambda)_m d\mu_{-1}(x)$$

$$= 2S_{m,\chi}(nd-1,\lambda). \tag{2.7}$$

By using (2.6) and (2.7), we obtain the following theorem:

Theorem 3. *Let n be an odd positive integer*. *Then*

$$\frac{2\int_X \chi(x)(1+\lambda t)^{2x/\lambda} d\mu_{-1}(x)}{\int_X \chi(x)(1+\lambda t)^{2ndx/\lambda} d\mu_{-1}(x)} = \sum_{m=0}^{\infty}(2S_{m,\chi}(nd-1,\lambda))\frac{t^m}{m!}.$$

Let w_1 and w_2 be odd positive integers. Then we set

$$S(w_1, w_2) = \frac{\int_X \int_X \chi(x_1)\chi(x_2)(1+\lambda t)^{(2w_1x_1+2w_2x_2+w_1w_2x)/\lambda} d\mu_{-1}(x_1) d\mu_{-1}(x_2)}{\int_X (1+\lambda t)^{2w_1w_2dx/\lambda} d\mu_{-1}(x)}. \quad (2.8)$$

By Theorem 3 and (2.8), after some calculations, we obtain

$$\begin{aligned} S(w_1, w_2) &= \left(\frac{1}{2}\int_X \chi(x_1)(1+\lambda t)^{(2w_1x_1+w_1w_2x)/\lambda} d\mu_{-1}(x_1)\right) \\ &\quad \times \left(\frac{2\int_X \chi(x_2)(1+\lambda t)^{2x_2w_2/\lambda} d\mu_{-1}(x_2)}{\int_X (1+\lambda t)^{2w_1w_2dx/\lambda} d\mu_{-1}(x)}\right) \\ &= \left(\frac{1}{2}\sum_{m=0}^{\infty} \mathcal{T}_{m,\chi}\left(w_2x, \frac{\lambda}{w_1}\right) w_1^m \frac{t^m}{m!}\right) \\ &\quad \times \left(2\sum_{m=0}^{\infty} S_{m,\chi}\left(w_1d-1, \frac{\lambda}{w_2}\right) w_2^m \frac{t^m}{m!}\right). \end{aligned} \quad (2.9)$$

By using Cauchy product in the above, we have

$$S(w_1, w_2) = \sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j} T_{j,\chi}\left(w_2x, \frac{\lambda}{w_1}\right) w_1^j S_{m-j,\chi}\left(w_1d-1, \frac{\lambda}{w_2}\right) w_2^{m-j}\right)\frac{t^m}{m!}. \quad (2.10)$$

From the symmetry of $S(w_1, w_2)$ in w_1 and w_2, we also see that

$$\begin{aligned} S(w_1, w_2) &= \left(\frac{1}{2}\int_X \chi(x_2)(1+\lambda t)^{(2w_2x_2+w_1w_2x)/\lambda} d\mu_{-1}(x_2)\right) \\ &\quad \times \left(\frac{2\int_X \chi(x_1)(1+\lambda t)^{2x_1w_1/\lambda} d\mu_{-1}(x_1)}{\int_X (1+\lambda t)^{2w_1w_2dx/\lambda} d\mu_{-1}(x)}\right) \end{aligned}$$

$$= \left(\frac{1}{2} \sum_{m=0}^{\infty} \mathcal{T}_{m,\chi}\left(w_1 x, \frac{\lambda}{w_2} \right) w_2^m \frac{t^m}{m!} \right)$$

$$\times \left(2 \sum_{m=0}^{\infty} S_{m,\chi}\left(w_2 d - 1, \frac{\lambda}{w_1} \right) w_1^m \frac{t^m}{m!} \right).$$

Thus, we have

$$S(w_1, w_2)$$

$$= \sum_{m=0}^{\infty} \left(\sum_{j=0}^{m} \binom{m}{j} \mathcal{T}_{j,\chi}\left(w_1 x, \frac{\lambda}{w_2} \right) w_2^j S_{m-j,\chi}\left(w_2 d - 1, \frac{\lambda}{w_1} \right) w_1^{m-j} \right) \frac{t^m}{m!}. \quad (2.11)$$

By comparing coefficients $\frac{t^m}{m!}$ in both the sides of (2.10) and (2.11), we obtain the following theorem:

Theorem 4. *Let w_1 and w_2 be odd positive integers. Then*

$$\sum_{j=0}^{m} \binom{m}{j} w_1^{m-j} \omega_2^j \mathcal{T}_{j,\chi}\left(w_1 x, \frac{\lambda}{w_2} \right) S_{m-j,\chi}\left(w_2 d - 1, \frac{\lambda}{w_1} \right)$$

$$= \sum_{j=0}^{m} \binom{m}{j} w_1^j w_2^{m-j} \mathcal{T}_{j,\chi}\left(w_2 x, \frac{\lambda}{w_1} \right) S_{m-j,\chi}\left(w_1 d - 1, \frac{\lambda}{w_2} \right),$$

where $\mathcal{T}_{k,\chi}(x, \lambda)$ and $\mathcal{T}_{m,\chi}(k, \lambda)$ denote the generalized degenerate tangent polynomials and the alternating generalized falling factorial sums, respectively (*see* [6, 9]).

By Theorem 2, we have the following corollary:

Corollary 5. *Let w_1 and w_2 be odd positive integers. Then*

$$\sum_{j=0}^{m} \sum_{k=0}^{j} \binom{m}{j} \binom{j}{k} w_1^{m-j} w_2^j \left(w_1 x | \frac{\lambda}{w_2} \right)_{j-k} \mathcal{T}_{k,\chi}\left(\frac{\lambda}{w_2} \right) S_{m-j,\chi}\left(w_2 d - 1, \frac{\lambda}{w_1} \right)$$

$$= \sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k} w_1^j w_2^{m-j}\left(w_2 x \mid \frac{\lambda}{w_1}\right)_{j-k} \mathcal{T}_{k,\chi}\left(\frac{\lambda}{w_1}\right) S_{m-j,\chi}\left(w_1 d - 1, \frac{\lambda}{w_2}\right).$$

Now we will derive other interesting identities for the generalized degenerate tangent polynomials using the symmetric property of $S(w_1, w_2)$,

$$S(w_1, w_2) = \left(\frac{1}{2}\int_X \chi(x_1)(1+\lambda t)^{(2w_1x_1 + w_1w_2x)/\lambda} d\mu_{-1}(x_1)\right)$$

$$\times \left(\frac{2\int_X \chi(x_2)(1+\lambda t)^{2x_2w_2/\lambda} d\mu_{-1}(x_2)}{\int_X (1+\lambda t)^{2w_1w_2dx/\lambda} d\mu_{-1}(x)}\right)$$

$$= \left(\frac{1}{2}(1+\lambda t)^{w_1w_2x/\lambda}\int_X \chi(x_1)(1+\lambda t)^{2x_1w_1/\lambda} d\mu_{-1}(x_1)\right)$$

$$\times \left(2\sum_{j=0}^{w_1d-1}(-1)^j \chi(j)(1+\lambda t)^{2jw_2/\lambda}\right)$$

$$= \sum_{j=0}^{w_1d-1}(-1)^j \chi(j)\int_X \chi(x_1)(1+\lambda t)^{\left(2x_1 + w_2x + \frac{2jw_2}{w_1}\right)w_1/\lambda} d\mu_{-1}(x_1)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{w_1d-1}(-1)^j \chi(j)\mathcal{T}_{n,\chi}\left(w_2x + \frac{2jw_2}{w_1}, \frac{\lambda}{w_1}\right) w_1^n\right)\frac{t^n}{n!}. \tag{2.12}$$

By using the symmetry property in (2.12), we also have

$$S(w_1, w_2) = \left(\frac{1}{2}(1+\lambda t)^{w_1w_2x/\lambda}\int_X \chi(x_2)(1+\lambda t)^{2x_2w_2/\lambda} d\mu_{-1}(x_2)\right)$$

$$\times \left(\frac{2\int_X \chi(x_1)(1+\lambda t)^{2x_1w_1/\lambda} d\mu_{-1}(x_1)}{\int_X (1+\lambda t)^{2w_1w_2dx/\lambda} d\mu_{-1}(x)}\right)$$

$$= \left(\frac{1}{2}(1+\lambda t)^{w_1 w_2 x/\lambda}\int_X \chi(x_2)(1+\lambda t)^{2x_2 w_2/\lambda}\, d\mu_{-1}(x_2)\right)$$

$$\times\left(2\sum_{j=0}^{w_2 d-1}(-1)^j\chi(j)(1+\lambda t)^{2jw_1/\lambda}\right)$$

$$= \sum_{j=0}^{w_2 d-1}(-1)^j\chi(j)\int_X \chi(x_2)(1+\lambda t)^{\left(2x_2+w_1x+\frac{2jw_1}{w_2}\right)w_2/\lambda}\, d\mu_{-1}(x_1)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{w_2-1}(-1)^j\chi(j)\mathcal{T}_{n,\chi}\left(w_1x+\frac{2jw_1}{w_2}, \frac{\lambda}{w_2}\right)w_2^n\right)\frac{t^n}{n!}. \tag{2.13}$$

By comparing coefficients $\frac{t^n}{n!}$ in both the sides of (2.12) and (2.13), we have the following theorem:

Theorem 6. *Let* w_1 *and* w_2 *be odd positive integers. Then*

$$\sum_{j=0}^{w_1 d-1}(-1)^j\chi(j)\mathcal{T}_{n,\chi}\left(w_2x+\frac{2jw_2}{w_1}, \frac{\lambda}{w_1}\right)w_1^n$$

$$= \sum_{j=0}^{w_2 d-1}(-1)^j\chi(j)\mathcal{T}_{n,\chi}\left(w_1x+\frac{2jw_1}{w_2}, \frac{\lambda}{w_2}\right)w_2^n. \tag{2.14}$$

If we take $x = 0$ in Theorem 6, then we also derive the interesting identity for the generalized degenerate tangent numbers as follows:

$$\sum_{j=0}^{w_1 d-1}\sum_{l=0}^{m}\binom{m}{l}(-1)^j\chi(j)\mathcal{T}_{l,\chi}\left(\frac{\lambda}{w_1}\right)(2jw_2\,|\,\lambda)_{m-l}\,w_2^l$$

$$= \sum_{j=0}^{w_2 d-1}\sum_{l=0}^{m}\binom{m}{l}(-1)^j\chi(j)\mathcal{T}_{l,\chi}\left(\frac{\lambda}{w_2}\right)(2jw_1\,|\,\lambda)_{m-l}\,w_1^l.$$

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