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ON THE THEORETICAL SPECIFICATION OF POISSON-AUTOREGRESSIVE MODEL FOR ANALYZING TIME SERIES COUNT DATA

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Abstract

Time series count data exhibit varying dispersion due to their sparse nature. Given the assumption on the incident rate structure, the methodology that can adequately accommodate the dispersion and inflation characteristics of count data is investigated. We develop Poisson distribution based autoregressive models that can account for dispersion and zero inflation indices in count series data. We derive the maximum likelihood estimators of the parameters in the models developed. The dispersion indices for these models depend on the structure of the incidence rate specified.

1. Introduction

In time series analysis, data that are in the form of count exhibit over or under-dispersion due to their sparse nature. A more pronounced challenge with the analysis of this data is that simple regression and autoregressive

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models have failed to evaluate appropriately the dispersion and inflation characteristics of count data. This means that the time series analysis requires procedure that could explain, measure the dispersion and inflation rate for these models. Poisson distribution is often used to model zero-inflated count data. The zero-inflated count data are found in medicine, engineering, sociology and epidemiology. Examples of these data include length of stay in hospital, number of road accidents and number of death recorded for contagious diseases within a particular period.

Poisson regression analysis has been established to be useful for modelling count variables (Venables and Ripley [9]), especially in modelling the occurrence of rare events at certain location within a study time period. The classical Poisson model has its mean equal to its variance; this is referred to as equi-dispersion. A situation when the variance of the observed counts varies with the mean leads to dispersion and the departure from equi-dispersion for count data models could lead to biased inference (Lindsay [6]; Bohning et al. [1]). But given the analogous relationship that exists between regression model and autoregressive model (Haggan and Oyetunji [4]) gives room for the extension of Poisson regression model to classical autoregressive model in time series analysis of count data. Besides, Mann and Wald [7] have shown that asymptotically, much of classical regression theory can be applied to autoregressive situations.

In this paper, the major objective of this study is to model the mean of dependent variable as a function of its lagged predictor variables using appropriate structure for the incidence rate function. This model should follow stationary property of time series models and obeys the general conditions of Poisson distribution. We develop some generalized Poisson-autoregressive models and evaluate their dispersion indices and Poisson model properties.

2. Poisson-autoregressive Model Specifications

The convectional Poisson regression involves modelling the rate (or risk) for different variables of interest. For instances, at time t, let Y_t be the observed count of incidents (e.g., number of deaths, number of cases of skin cancers, number of vehicle road traffic accidents etc.) and τ_t be the population size for the study area (say number of people exposed to risk, total number of vehicles registered and so on). The dependent variable Y_t in the Poisson regression frame work is obtained for a particular event that is described by set of predictor variables $Y_{t-1}, Y_{t-2}, ..., Y_{t-p}$. Let $\lambda(\mathbf{Y}_{t-i}, \mathbf{\Phi})$ be a function of $\mathbf{Y}_{t-i}$ and $\mathbf{\Phi}$ which represent the "rate" that measures the risk per unit of study time or follow up time such that the estimated rate or risk is simply $\lambda_t = Y_t / \tau_t$. Then the expected number at time *t* is

$$E(Y_t) = \mu_t = \tau_t \lambda(\mathbf{Y}_{(t-i)}, \mathbf{\Phi}). \tag{1}$$

We shall establish that variable Y_t obeys all properties of Poisson model, with parameters μ_t but we assume it is stationary. This literally means that the variance of Y_t is not a constant from $E(Y_t) = Var(Y_t) = \mu_t = \tau_t \lambda(Y_{t-i}; \Phi_i)$; it thus varies as a function of τ_t, Y_{t-i} and time *t*. In practice (see Kleinbaum et al. [5]). We utilize the likelihood function $L(\mathbf{Y}_{t-i}, \mathbf{\Phi})$ by specifying the particular form for the rate function $\lambda(\mathbf{Y}_{t-i}, \mathbf{\Phi})$. This enables the estimation of the parameters $\mathbf{\Phi}$ to be performed iteratively.

The specification of the rate function $\lambda_t(\mathbf{Y}_{t-i}, \mathbf{\Phi})$ to contain the stationary condition for sparse data is the focus of this study. According to Kleinbaum et al. [5], such specification should be based on the process under study and previous knowledge and experience with the relationship among variables under consideration. In this study, we assume that $\lambda(\mathbf{Y}_{t-i}, \mathbf{\Phi})$ follows an autoregressive process and consider four most appropriate representations on the certain conditions on $\lambda(\mathbf{Y}_{t-i}, \mathbf{\Phi})$. Taking the condition when $\lambda(\mathbf{Y}_{t-i}, \mathbf{\Phi}) > 0$, we have a specification for the incident rate as

$$\lambda_t(Y_{t-i}, \mathbf{\Phi}) = \sum_{i=1}^{p} \Phi_i Y_{t-i}. \tag{2}$$

In time series literature (Wei [10] and Chatfield [2]), the right hand side of equation (2) is derived from fitting the general autoregressive model $Y_t = \sum_{i=1}^{p} \Phi_i Y_{t-i} + \varepsilon_t$.

(I) Given the explanation made on the estimated risk above and using the quantities in equation (2) when $\left\{\sum \Phi_i Y_{t-i}\right\} > 0$, the first Poisson-autoregressive model is

$$Y_t^{(1)} = \tau_t \left(\sum_{i=1}^{p} \Phi_i Y_{t-i} \right) + \varepsilon_t^{(1)}, \tag{3}$$

ε_t's are independent and identically distributed with mean zero and variance σ_ε^2. Assuming independence for variables in equation (3), the expected value of the model $Y_t^{(1)}$ is given by

$$E(Y_t^{(1)}) = \tau_t \sum \Phi_i Y_{t-i}, \quad \forall t = 1, ..., n. \tag{4}$$

Since τ_t and Y_{t-i} vary with time (t), the variance of model (3) is

$$Var(Y_t^{(1)}) = E\left[\left(Y_t - \tau_t \sum_{i=1}^{p} \Phi_i Y_{t-i}\right)^2\right] = E[(\varepsilon_t^2)] = \sigma^2_{\varepsilon^{(1)}}. \tag{5}$$

(II) Assume that $\sum_{i=1}^{p} \Phi_i Y_{t-i} > 0$ and according to Kleinbaum et al. [5], we can stipulate $\lambda_t(Y_{t-i}\Phi_i) = \ln \sum_{i=1}^{p} \Phi_i Y_{t-i}$. This means that the second Poisson-autoregressive model will be of the form

$$Y_t^{(2)} = \tau_t \ln \sum_{i=1}^{p} \Phi_i Y_{t-i} + \varepsilon_t^{(2)}. \tag{6}$$

Since $\ln \sum_{i=1}^{p} \Phi_i Y_{t-i} = f(y)$ is concave, it is well established that $\ln \sum_{i=1}^{p} \Phi_i Y_{(t-i)} \geq \sum_{i=1}^{p} \ln \Phi_i Y_{t-i}$ and thus we have the linear approximation

$$\sum_{i=1}^{p} \ln(\Phi_i Y_{t-i}) = \sum_{i=1}^{p} \ln \Phi_i + \sum_{i=1}^{p} \ln Y_{t-i}, \quad \forall t = 1, 2, ..., n. \tag{7}$$

Using equation (7) in (6) gives another form of Poisson-autoregressive model as

$$Y_t^{(2)} = \tau_t \sum_{i=1}^{p} \ln \Phi_i + \tau_t \sum_{i=1}^{p} \ln Y_{t-i} + \varepsilon_t^{(2)} = \tau_t \sum_{i=1}^{p} \Phi_i^* + \tau_t \sum_{i=1}^{p} Y_{t-i}^* + \varepsilon_t^{(2)}, \tag{8}$$

where $\Phi_i^* = \ln \Phi_i$ and $Y_{t-i}^* = \ln Y_{t-i}$. The expected value of $Y_t^{(2)}$ is

$$E(Y_t^{(2)}) = \tau_t \sum_{i=1}^{p} \ln \Phi_i + \tau_t \sum_{i=1}^{p} \ln Y_{t-i}. \tag{9}$$

The variance of $Y_t^{(2)}$ is

$$Var(Y_t^{(2)}) = \sigma^2_{\varepsilon^{(2)}}. \tag{10}$$

(III) Suppose that we use the linear approximation via Taylor series expansion under the condition that $\sum_{i=1}^{p} \Phi_i Y_{t-i} > 0$ and such that $\lambda_t(\mathbf{Y}, \mathbf{\Phi}) = e^{\sum_{i=1}^{p} \Phi_i Y_{t-i}} \cong 1 + \sum_{i=1}^{p} \Phi_i Y_{t-i}$ to have the third Poisson-autoregressive model as

$$Y_t^{(3)} = \tau_t \left(1 + \sum \hat{\Phi}_i Y_{t-i}\right) + \varepsilon_t. \tag{11}$$

Assuming the independence of τ_t and Y_{t-i} for all t and that Y_{t-i} are pairwise independent for all i, the expected value of $Y_t^{(3)}$ is

$$E(Y_t^{(3)}) = \tau_t\left(1 + \sum_{i=1}^{p} \Phi_i Y_{t-i}\right). \tag{12}$$

The variance of $Y_t^{(3)}$ is

$$Var(Y_t^{(3)}) = \sigma^2_{\varepsilon^{(3)}}. \tag{13}$$

(IV) Consider the Poisson regression model which is a function of time predictors used in measuring the incident rate then the fourth model is given as

$$Y_t^{(4)} = \tau_t \sum_{i=1}^{p} \Phi_i t_i^i + \varepsilon_t^{(4)}, \quad t = 1, \ldots, n. \tag{14}$$

The expectation of this model when τ_t and time (t) are independent, also t_i are pair wise independent is

$$E(Y_t^{(4)}) = \tau_t \sum_{i=1}^{p} \Phi_i t^i. \tag{15}$$

The variance of $Y_t^{(4)}$ is

$$Var(Y_t^{(4)}) = \sigma^2_{\varepsilon^{(4)}}. \tag{16}$$

The mean and variance of the expected number of incidence (Y_t) developed in equations (3), (6), (11) and (14) depend largely on the choice of the structure of $\lambda(Y_{t-i}, \Phi)$. The expected value of Y at time t goes to zero when $\tau_t \to 0$; in real life situation such as in epidemiology, when the population at risk tends to extinction, it is an indication that occurrence of such epidemic is no longer expected. Provided that $\tau_t\lambda(\Phi_i, Y_{t-i}) \simeq \sigma^2_\varepsilon$, we have equi-dispersion models; the measures for this indicator are discussed in the next section.

3. Evaluation of Basic Properties of the Poisson-autoregressive Model

3.1. Poisson-autoregressive model characterization

The models proposed in equations (3), (8), (11) and (16) evidently follow Poisson mass function from:

$$\sum_{Y_t=0}^{\infty} p(Y_t) = \sum_{Y_t=0}^{\infty} \frac{e^{-\tau_t\lambda(\Phi_i,\, Y_{t-i})}[(\tau_t\lambda(\Phi_i,\, Y_{t-i}))^{Y_t}]}{Y_t!}$$

$$= e^{-\tau_t\lambda(\Phi_i,\, Y_{t-i})}\left[\sum_{Y_t=0}^{\infty}\left(\frac{(\tau_t\lambda(\Phi_i,\, Y_{t-i}))^{Y_t}]}{Y_t}\right)\right] = 1. \quad (17)$$

By using this legal probability condition, we have for all the models discussed above that

$$E(Y_t) = \sum_{Y_t=0}^{\infty} Y_t\, p(Y_t) = \sum_{Y_t=0}^{\infty} \frac{Y_t e^{-\tau_t\lambda(\Phi_i,\, Y_{t-i})}[(\tau_t\lambda(\Phi_i,\, Y_{t-i}))^{Y_t}]}{Y_t!}$$

$$= \tau_t\lambda(\Phi_i,\, Y_{t-i})e^{-\tau_t\lambda(\Phi_i,\, Y_{t-i})}\left[\sum_{Y_t=1}^{\infty}\left(\frac{(\tau_t\lambda(\Phi_i,\, Y_{t-i}))^{Y_t-1}]}{(Y_t-1)!}\right)\right]$$

$$= \tau_t\lambda(\Phi_i,\, Y_{t-i}),$$

we observed that the zero inflation index (ZII) relative to Poisson (Puig and Valero [8]) is satisfied, since for all these models we have

$$1 + \frac{\log p(Y_t = 0)}{E(Y_t)} = 0.$$

3.2. Dispersion indices

The major characteristics of count time series data are greater variation than the predicted standard Poisson distribution (over dispersion). The fisher's dispersion index $D(Y)$ for a given model Y_t is

$$D(Y) = 1 + \frac{S(Y)}{E(Y)}, \text{ where } S(Y) = Var(Y) - E(Y). \quad (18)$$

The Fisher's dispersion index for the Poisson-autoregressive model defined in equation (3) is

$$DPI_{PAR_{(1)}} = 1 + \sigma^2_{\varepsilon(1)}\left[\tau_t \sum_{i=1}^{p} \Phi_i Y_{t-i}\right]^{-1}. \tag{19}$$

And the associated measure of dispersion is

$$SD_{PAR_{(1)}} = \sigma^2_{\varepsilon(1)} - \tau_t \sum_{i=1}^{p} \Phi_i Y_{t-i}. \tag{20}$$

The dispersion index for model defined in equation (8) is

$$DPI_{PAR_{(2)}} = 1 + \sigma^2_{\varepsilon(2)}\left[\tau_t \sum_{i=1}^{p} \ln \Phi_i + \tau_t \sum_{i=1}^{p} \ln Y_{t-i}\right]^{-1}. \tag{21}$$

The associated measure of dispersion is

$$SD_{PAR_{(2)}} = \sigma^2_{\varepsilon(2)} - \tau_t \sum_{i=1}^{p} \ln \Phi_i - \tau_t \sum_{i=1}^{p} \ln Y_{t-i}. \tag{22}$$

The Fisher's dispersion index for the Poisson-autoregressive model in (11) is

$$DPI_{PAR_{(3)}} = 1 + \sigma^2_{\varepsilon(3)}\left[\tau_t^{-1}\left(1 + \sum \Phi_i Y_{t-i}\right)^{-1}\right]. \tag{23}$$

The associated measure of dispersion is

$$SD_{PAR_{(3)}} = \sigma^2_{\varepsilon(3)} - \tau_t\left(1 + \sum_{i=1}^{p} \Phi_i Y_{t-i}\right). \tag{24}$$

The Fisher's dispersion index for model defined in equation (14) is

$$DPI_{PAR_{(4)}} = 1 + \sigma^2_{\varepsilon(4)}\left[\tau_t \sum_{i=1}^{p} \Phi_i t^i\right]^{-1}. \tag{25}$$

And the associated measure of dispersion is

$$SD_{PAR_{(4)}} = \sigma^2_{\varepsilon(4)} - \tau_t \sum_{i=1}^{p} \Phi_i t^i. \tag{26}$$

Analytically, in the derived equations (19) through (26): if $\sigma^2_{\varepsilon(k)} > E(Y_t^{(k)})$, $\forall k = 1, 2, 3, 4,$ then it means that $DPI > 1$ and $SD > 0$ causing over dispersion. And if $\sigma^2_{\varepsilon(k)} = E(Y_t^{(k)})$, $\forall k = 1, 2, 3, 4,$ then it means that DPI will equal to unity. It implies that the model gives an equi-dispersion. But if $\sigma^2_{\varepsilon(k)} < E(Y_t^{(k)})$, $k = 1, 2, 3, 4,$ then it means that DPI will be less than unity. This implies that SD will be negative, thus we have under-dispersion.

4. Maximum Likelihood Estimators of the Parameters of Poisson-autoregressive Models

The general notion about regression analysis is that it permits the modelling of mean of dependent variable under consideration as a function of certain predicted variables as expressed in equations (3), (8), (11) and (14) in Section 3 above. In this section, we develop the likelihood function that can be used to estimate the coefficient of Φ_i's under the Poisson assumption made about the dependent variable Y_t.

Let us assume that $Y_1, Y_2, ..., Y_n$ constitute a mutually independent set of Poisson random variables with Y_t having the probability mass function:

$$p(Y_t, \Phi_i) = \begin{cases} \dfrac{e^{-\tau_t \lambda(\Phi_i, Y_{t-i})} [(\tau_t \lambda_t(\Phi_i, Y_{t-i}))^{Y_t}]}{Y_t!}, & Y_t = 0, 1, ..., \ t = 1, 2, ..., n. \\ 0, & \text{elsewhere} \end{cases} \tag{27}$$

The general form of the likelihood function for Poisson-autoregressive model is

$$L(Y_t, \Phi_i)$$

$$= \prod_{t=1}^{n} p(Y_t, \Phi_i) = \frac{\left\{\prod_{t=1}^{n} (\tau_t \lambda(Y_{t-1}, \Phi_i))^{Y_t}\right\} e^{-\sum_{t=1}^{n} (\tau_t (\lambda_t (Y_{(t-i)}, \Phi_i)))}}{\prod_{t=1}^{n} Y_t!}. \quad (28)$$

Using equation (28), the respective likelihood functions and the estimators of the parameters for the Poisson-autoregressive models described in equations (3), (8), (11) and (14) are discussed in the following cases:

Case 1. Taking the natural logarithm of the likelihood function $L_1(Y_t, \Phi_i) = \frac{\left\{\prod_{t=1}^{n} \left(\tau_t \sum_{i=1}^{p} \Phi_i Y_{t-i}\right)^{Y_t}\right\} \ell^{-\sum_{t=1}^{n}\left\{\tau_t \sum_{i=1}^{p} \Phi_i Y_{t-i}\right\}}}{\prod_{t=1}^{n} Y_t!}$ and differentiate with respect to Φ_i gives $\frac{\partial \ln L(Y_t, \Phi_i)}{\partial \Phi_i} = \left(\frac{\sum_{t=1}^{n} Y_t}{\tau_t \sum_{i=1}^{p} \hat{\Phi}_i Y_{t-i}} - n\right) \tau_t Y_{t-i}$, equating this to zero gives $\frac{\sum_{t=1}^{n} Y_t}{n\tau_t} = \sum_{i=1}^{p} \hat{\Phi}_i Y_{t-i}$. Without loss of generality, take the sum to n on both sides of this expression to have $\sum_{t=1}^{n} \left(\frac{\bar{Y}}{\tau_t}\right) = \sum_{t=1}^{n} \sum_{i=1}^{p} \hat{\Phi}_i Y_{t-i}$, this reduces to

$$\sum_{t=1}^{n} \sum_{i=1}^{p} \hat{\Phi}_i Y_{t-i} = \bar{Y} \sum_{t=1}^{n} \tau_t^{-1}. \quad (29)$$

To solve equation (29) for the estimators of Φ_i, $\forall i = 1, \ldots, p$, we shall utilize an approach similar to the algorithm proposed in Durbin [3]. Start the recursive process by taking $p = 1$ in equation (29), then we have:

$$\hat{\Phi}_1 = \frac{\bar{Y}\sum_{t=1}^{n} \tau_t^{-1}}{\sum_{t=1}^{n} Y_{t-1}}. \tag{30}$$

To obtain other estimators $\hat{\Phi}_i$, $\forall i = 2, \ldots, p$ we use the following equation recursively:

$$\hat{\Phi}_i = \frac{\bar{Y}\sum_{t=1}^{n} \tau_t^{-1}}{\sum_{t=1}^{n} Y_{t-1}} - \sum_{t=1}^{n}\sum_{j=1}^{i-1} \hat{\Phi}_j Y_{t-j}, \ \forall i = 2, \ldots, p. \tag{31}$$

The variance of $\hat{\Phi}_i$, $\forall i = 1, \ldots, p$ is:

$$Var(\hat{\Phi}_i) = Var\left\{\frac{\bar{Y}\sum_{t=1}^{n} \tau_t^{-1}}{\sum_{t=1}^{n} Y_{t-i}} - \sum_{t=1}^{n}\sum_{j=1}^{i-1} \hat{\Phi}_j Y_{t-j}\right\}, \quad \forall i = 1, \ldots, p, \tag{32}$$

$$Var(\hat{\Phi}_i) = Var\left[\bar{Y}\,\frac{\sum_{t=1}^{n} \tau_t^{-1}}{\sum_{t=1}^{n} Y_{t-i}} - \sum_{t=1}^{n}\sum_{j=1}^{i-1} \hat{\Phi}_j Y_{t-j}\right]$$

$$= Var\{f(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)\};\ f(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3) = \bar{Y}\,\frac{\hat{\theta}_1}{\hat{\theta}_2} - \hat{\theta}_3, \text{ where}$$

$$\hat{\theta}_1 = \left(\sum_{t=1}^{n} \tau_t^{-1}/n\right),\ \hat{\theta}_2 = \sum_{t=1}^{n} Y_{t-i}/n,\ \hat{\theta}_3 = \sum_{t=1}^{n}\sum_{j=1}^{i-1} \hat{\Phi}_j Y_{t-j},$$

$$Var\{f(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)\}$$

$$= \left(\frac{\partial f}{\partial \hat{\theta}_1}\right)^2 \Bigg|_{E(\hat{\theta})=\theta} Var(\hat{\theta}_1) + \left(\frac{\partial f}{\partial \hat{\theta}_2}\right)^2 \Bigg|_{E(\hat{\theta})=\theta} Var(\hat{\theta}_2)$$

$$+ \left(\frac{\partial f}{\partial \hat{\theta}_3}\right)^2 \Bigg|_{E(\hat{\theta})=\theta} Var(\hat{\theta}_3) + 2\left(\frac{\partial f}{\partial \hat{\theta}_1}\frac{\partial f}{\partial \hat{\theta}_2}\right)\Bigg|_{E(\hat{\theta})=\theta} Cov(\hat{\theta}_1\hat{\theta}_2)$$

$$+ 2\left(\frac{\partial f}{\partial \hat{\theta}_1}\frac{\partial f}{\partial \hat{\theta}_3}\right)\Bigg|_{E(\hat{\theta})=\theta} Cov(\hat{\theta}_1\hat{\theta}_3) + 2\left(\frac{\partial f}{\partial \hat{\theta}_2}\frac{\partial f}{\partial \hat{\theta}_3}\right)\Bigg|_{E(\hat{\theta})=\theta} Cov(\hat{\theta}_2\hat{\theta}_3)$$

$$= \left(\bar{\gamma}\frac{1}{\theta_2}\right)^2 V(\hat{\theta}_1) + \left(\bar{\gamma}\frac{\theta_1}{\theta_2^2}\right)^2 V(\hat{\theta}_2) + V(\hat{\theta}_3) + 2\left(\bar{\gamma}\frac{1}{\theta_2}\bar{\gamma}\frac{\theta_1}{\theta_2^2}\right) Cov(\hat{\theta}_1, \hat{\theta}_2)$$

$$- 2\left(\bar{\gamma}\frac{1}{\theta_2}\right) Cov(\hat{\theta}_1, \hat{\theta}_3) - 2\left(\bar{Y}\frac{\theta_1}{\theta_2^2}\right) Cov(\hat{\theta}_2, \hat{\theta}_3)$$

$$= \bar{Y}^2\left[\frac{V(\hat{\theta}_1)}{\theta_2^2} + \frac{\theta_1^2}{\theta_2^4}V(\hat{\theta}_2) + \frac{1}{\bar{Y}^2}V(\hat{\theta}_3) + 2\frac{\theta_1}{\theta_2^3}Cov(\hat{\theta}_1, \hat{\theta}_2)\right.$$

$$\left. - \frac{2}{\theta_2\bar{Y}}Cov(\hat{\theta}_1, \hat{\theta}_3) - 2\frac{\theta_1}{\bar{Y}\theta_2^2}Cov(\hat{\theta}_2, \hat{\theta}_3)\right].$$

Case 2. The likelihood of interest is

$$L_2(Y_t, \Phi_i) = \frac{\left\{\prod_{t=1}^{n}\left(\tau_t \sum_{i=1}^{p}\Phi_i^* + Y_{t-i}^*\right)^{Y_t}\right\} e^{-\sum_{t=1}^{n}\left\{\tau_t \sum_{i=1}^{p}(\Phi_i^* + Y_{t-i}^*)\right\}}}{\prod_{t=1}^{n} Y_t!}$$

and taking the natural logarithm with necessary algebraic manipulations we have the expression:

$$\frac{\partial \ln L_2(Y_t,\ \Phi_i)}{\partial \Phi_i} = \frac{\sum_{t=1}^{n} Y_t}{\tau_t \left\{ \sum_{i=1}^{n} \ln \Phi_i + \sum_{i=1}^{n} \ln Y_{t-i} \right\}} - n = 0.$$

Taking the sum to n of both sides of this partial derivatives and rearranging gives:

$$\sum_{i=1}^{p} \ln \hat{\Phi}_i = \frac{\overline{Y}}{n}\left(\sum_{t=1}^{n} \tau_t^{-1} \right) - \frac{\sum_{t=1}^{n}\sum_{i=1}^{p} \ln Y_{t-i}}{n}. \tag{33}$$

Setting $p = 1$ in equation (33) gives the estimator of ln $\hat{\Phi}_i$ as:

$$\ln \hat{\Phi}_1 = n^{-1}\left\{ \overline{Y} \sum_{t=1}^{n} \tau_t^{-1} - \sum_{t=1}^{n} \ln Y_{t-1} \right\}. \tag{34}$$

Using equation (33) with equation (34) as starting point, we can recursively derive the estimators of ln $\Phi_i,\ i = 2, \ldots, p$ from:

$$\ln \hat{\Phi}_i = n^{-1}\left\{ \overline{Y} \sum_{t=1}^{n} \tau_t^{-1} - \sum_{t=1}^{n}\sum_{j=1}^{i-1} \ln Y_{t-j} - \sum_{j=1}^{i-1} \ln \hat{\Phi}_j \right\}. \tag{35}$$

We derive the approximate estimator of $\hat{\Phi}_i$ using the equation $\hat{\Phi}_i = \exp(\ln \hat{\Phi}_i) \mathrel{\hat{=}} 1 + \ln \hat{\Phi}_i$, neglecting higher orders. Therefore, we obtain the variance of $\hat{\Phi}_i$ using the variance of $\ln \hat{\Phi}_i$ as:

$$Var(\ln \hat{\Phi}_i) = Var\left[n^{-1}\left\{ \overline{Y} \sum_{t=1}^{n} \tau_t^{-1} - \sum_{t=1}^{n}\sum_{j=1}^{i-1} \ln Y_{t-j} - \sum_{j=1}^{i-1} \ln \hat{\Phi}_j \right\} \right], \tag{36}$$

$$Var(\ln \hat{\Phi}_i) = Var\left[\bar{Y}\left(\sum_{t=1}^{n} \tau_t^{-1}/n\right) - \left(\sum_{t=1}^{n}\sum_{j=1}^{i-1} \ln Y_{t-i}/n\right) - \left(\sum_{j=1}^{i-1} \ln \hat{\Phi}_j/n\right)\right]$$

$$= Var(\bar{Y}\hat{\theta}_1 - \hat{\theta}_2 - \hat{\theta}_3);\ \hat{\theta}_1 = \left(\sum_{t=1}^{n} \tau_t^{-1}/n\right);$$

$$\hat{\theta}_2 = \left(\sum_{t=1}^{n}\sum_{j=1}^{i-1} \ln Y_{t-i}/n\right);$$

$$\hat{\theta}_3 = \left(\sum_{j=1}^{i-1} \ln \hat{\Phi}_j/n\right)$$

$$= \bar{Y}^2 V(\hat{\theta}_1) + V(\hat{\theta}_2) + V(\hat{\theta}_3) - 2\bar{Y}Cov(\hat{\theta}_1, \hat{\theta}_2)$$

$$- 2\bar{Y}Cov(\hat{\theta}_1, \hat{\theta}_3) + 2Cov(\hat{\theta}_2, \hat{\theta}_3).$$

Case 3. The likelihood function of the model developed in equation (11) is $L_3(Y_t, \Phi_i) = \dfrac{\left\{\prod_{t=1}^{n}\left(\tau_t\left(1 + \sum \Phi_i Y_{t-i}\right)\right)^{Y_t}\right\} e^{-\sum_{t=1}^{n}\left\{\tau_t\left(1+\sum \hat{\Phi}_i Y_{t-i}\right)\right\}}}{\prod_{t=1}^{n} Y_t!}$ and taking the derivative of the natural logarithm of this function with respect to $\hat{\Phi}_i$ gives $\dfrac{\partial \ln L_2(Y_t, \Phi_i)}{\partial \Phi_i} = \left\{\dfrac{\sum_{t=1}^{n} Y_t}{\tau_t\left[1 + \sum_{i=1}^{n} \Phi_i Y_{t-i}\right]}\right\}(\tau_t Y_{t-i}) = 0,$ this reduces to $\tau_t\left(1 + \sum_{i=1}^{n} \Phi_i Y_{t-i}\right) = \bar{Y},$ summing both sides of this expression to n and rearrange gives:

$$\sum_{t=1}^{n}\sum_{i=1}^{p}\hat{\Phi}_i \tau_t Y_{t-i} = \sum_{t=1}^{n} Y_t - \sum_{t=1}^{n} \tau_t. \tag{37}$$

To solve equation (37), start with $p = 1$ to get the estimator of Φ_1 as:

$$\hat{\Phi}_1 = \frac{\sum_{t=1}^{n} Y_t - \sum_{t=1}^{n} \tau_t}{\sum \tau_t Y_{t-1}}. \tag{38}$$

The other estimators $\hat{\Phi}_i$ could recursively be derived from the equation:

$$\hat{\Phi}_i = \frac{\sum_{t=1}^{n} Y_t - \sum_{t=1}^{n} \tau_t - \sum_{t=1}^{n}\sum_{j=1}^{i-1} \hat{\Phi}_j Y_{t-j}}{\sum_{t=1}^{n} \tau_t Y_{t-j}}, \quad \forall i = 2, ..., p. \tag{39}$$

The variance of $\hat{\Phi}_i$ could be obtained using:

$$Var(\hat{\Phi}_i) = Var\left[\frac{\sum_{t=1}^{n} Y_t - \sum_{t=1}^{n} \tau_t - \sum_{t=1}^{n}\sum_{j=1}^{i-1} \hat{\Phi}_j Y_{t-j}}{\sum_{t=1}^{n} \tau_t Y_{t-j}}\right]. \tag{40}$$

We neglect the quantity

$$\frac{\sum_{t=1}^{n}\sum_{j=1}^{i-1} \hat{\Phi}_j Y_{t-j}}{\sum_{t=1}^{n} \tau_t Y_{t-i}}$$

when $i = 1$.

$$Var(\hat{\Phi}_i) = Var\left[\frac{\left(\sum_{t=1}^{n} y_t/n\right) - \left(\sum_{t=1}^{n} \tau_i/n\right) - \left(\sum_{i=1}^{n}\sum_{j=1}^{i-1} \hat{\Phi}_j y_{t-j}/n\right)}{\sum_{i=1}^{n} \tau_t Y_{t-i}/n}\right]$$

$$= Var[f(\hat{\varphi}/\hat{\theta}_4)];\ \hat{\varphi} = \left(\sum_{t=1}^{n} y_t/n\right) - \left(\sum_{t=1}^{n} \tau_i/n\right) - \left(\sum_{i=1}^{n}\sum_{j=1}^{i-1} \hat{\Phi}_j y_{t-j}/n\right);$$

$$\hat{\theta}_4 = \sum_{i=1}^{n} \tau_t y_{t-i}/n$$

$$= \left(\frac{\partial f}{\partial \hat{\varphi}}\right)^2 \Bigg|_{\hat{\varphi}=\varphi,\, \hat{\theta}_4=\theta_4} Var(\hat{\varphi}) + \left(\frac{\partial f}{\partial \hat{\theta}_4}\right)^2 \Bigg|_{\hat{\varphi}=\varphi,\, \hat{\theta}_4=\theta_4} Var(\hat{\theta}_4)$$

$$- 2\left(\frac{\partial^2 f}{\partial \hat{\varphi} \partial \hat{\theta}_4}\right)\Bigg|_{\hat{\varphi}=\varphi,\, \hat{\theta}_4=\theta_4} Cov(\hat{\varphi},\, \hat{\theta}_4)$$

$$= \left(\frac{1}{\theta_4}\right)^2 V(\hat{\varphi}) + \left(\frac{\varphi}{\theta_4^2}\right)^2 Var(\hat{\theta}_4) - 2\frac{1}{\theta_4^2} Cov(\hat{\varphi},\, \hat{\theta}_4).$$

Now

$$Cov(\hat{\varphi},\, \hat{\theta}_4) = Cov\left[\hat{\theta}_4,\, \left\{\left(\sum_{t=1}^{n} y_t/n\right) - \left(\sum_{t=1}^{n} \tau_i/n\right) - \left(\sum_{i=1}^{n}\sum_{j=1}^{i-1} \hat{\Phi}_j y_{t-j}/n\right)\right\}\right].$$

Case 4. The likelihood of the model in equation (16) is $L_4(Y_t,\, \Phi_i) =$ $\dfrac{\left\{\prod_{t=1}^{n}\left(\tau_t \sum_{i=1}^{p} \Phi_i t_i^i\right)^{Y_t}\right\} e^{-\sum_{t=1}^{n}\left\{\tau_t \sum_{i=1}^{p} \hat{\Phi}_i t_i^i\right\}}}{\prod_{t=1}^{n} Y_t!}$, taking the derivative of the natural logarithm of this function with respect to $\hat{\Phi}_i$ gives $\frac{\partial \ln L_4(Y_t, \Phi_i)}{\partial \Phi_i} = \left\{ \frac{\sum_{t=1}^{n} Y_t}{\tau_t \sum_{i=1}^{p} \hat{\Phi}_i t^i} - n \right\} \tau_t t^i = 0.$ This reduces to $\frac{\bar{Y}}{\tau_t} = \sum_{i=1}^{p} \hat{\Phi}_i t^i$ and taking sum to n on both sides of this expression and rearrange to have:

$$\sum_{t=1}^{n} \sum_{i=1}^{p} \hat{\Phi}_i t^i = \bar{Y} \sum_{t=1}^{n} \tau_t^{-1}. \tag{41}$$

To solve equation (41), start by setting $p = 1$ to get the estimator of $\hat{\Phi}_1$ as:

$$\hat{\Phi}_1 = \frac{\bar{Y} \sum_{t=1}^{n} \tau_t^{-1}}{\sum_{t-1}^{n} t} = \bar{Y}\bar{\tau}^{-1}. \tag{42}$$

The other estimators $\hat{\Phi}_i$, $\forall i = 2, \ldots, p$ are obtained recursively from:

$$\hat{\Phi}_i = \frac{\bar{Y} \sum_{t=1}^{n} \tau_t^{-1} - \sum_{t=1}^{n} \sum_{j=1}^{i-1} \hat{\Phi}_j t^i}{\sum_{t-1}^{n} t^i}, \ \forall i = 2, \ldots, p. \tag{43}$$

The variance $\hat{\Phi}_i$ is derived from

$$Var(\hat{\Phi}_i) = Var \left[\frac{\bar{Y} \sum_{t=1}^{n} \tau_t^{-1} - \sum_{t=1}^{n} \sum_{j=1}^{i-1} \hat{\Phi}_j t^i}{\sum_{t-1}^{n} t^i} \right], \tag{44}$$

$$Var(\hat{\Phi}_i) = Var\left[\frac{\bar{Y}\left(\sum_{t=1}^{n} \tau_t^{-1}/n\right) - \left(\sum_{i=1}^{n}\sum_{j=1}^{i-1} \hat{\Phi}_j t^i/n\right)}{\sum_{t=1}^{n} t^i/n}\right] = Varf(\hat{\varphi}_2/\hat{\theta}_5),$$

$$\hat{\varphi}_2 = \bar{Y}\left(\sum_{t=1}^{n} \tau_t^{-1}/n\right) - \left(\sum_{i=1}^{n}\sum_{j=1}^{i-1} \hat{\Phi}_j t^i/n\right),$$

$$\hat{\theta}_5 = \sum_{t=1}^{n} t^i/n = \frac{1}{\theta_5^2} V(\hat{\varphi}_2) + \frac{\hat{\varphi}_2^2}{\theta_5^4} V(\hat{\theta}_5) - 2\frac{1}{\theta_5^2} Cov(\hat{\varphi}_2, \hat{\theta}_5).$$

5. Conclusion

The derived expected values in equations (4), (9), (12) and (15) are directly proportional to the structure of the incident rate used. Also, the variances derived in equations (5), (10), (13) and (16) are dependent on the error structures modeled. The nature of the dispersion for these models could be practically checked using the condition that if $SD > 0$, it is over dispersion, if $SD = 0$, it implies equi-dispersion, and if we have the condition $-E(Z) \leq SD < 0$, then it is under-dispersion. Depending on the incident rate structure chose for a particular dataset, the recursive approach provided in Section 4 will give useful estimates of the parameters of the model.

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