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# LEARNING PROOFS, LEARNING A NEW LANGUAGE

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## Abstract

Learning a proof is similar to learning a new language: this is the assumption on which I built my current class project. It is not enough for one to read a new word's description in a dictionary in order to learn it: one needs to encounter that word in different sentences and in different contexts. I relied on this learning model when I designed the class project I describe in this paper. Once a proof style is abstracted into a category or a proof template, there is a chance that at a later stage that proof will be exported and applied onto a different context. I wanted students to build a special dictionary for the proof types they are learning, to classify them and use them and recognize what situations they would best apply to. I hoped that this activity would give the students an opportunity to take ownership of a mathematical topic. My main goal was to have them work on a long-term project where they would invest personally in mathematics and develop the tool of investigating problems. This platform that I created showed that it could provide what common problem sets and exams cannot provide.

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## Introduction

In this paper, I describe how using writing portfolios in an abstract algebra class helps students in building proof schemas. Through keeping a journal where they store and sort over time the different proofs they encounter in this class, students build stronger mathematical intuition and get placed at a higher cognitive level. One objective of this project is to help students demystify proofs and become more familiar with them. In order to complete this task, students are guided to look for the big idea in the proof and the key factors that make it work. They need to keep track of that idea that comes back in other proofs, and watch out in case it is manifested differently.

I chose abstract algebra as a vehicle because it is the first course where one goes beyond the learning imitative behavior patterns for mimicking the solution of large number of variations on a small number of themes or problems. This course requires new mature study habits that students have not experienced before. Before they enter this course, students assume that proof is a ritual with no meaning and involves mostly filling into a template that is guaranteed and tested to work (Ball et al. [1]). Whereas we all know that writing a proof requires both knowledge (ideas, methods, terms) and language (symbols, terms, notations, representations and rules of logic and syntax). When we teach this genre of mathematical text we are sometimes in denial of its complexity and the fact that the nature of this type of mathematics is different from any other discipline. We also tend to ignore the gap between mathematical rigor and students' intuitions, forgetting that it requires not only knowledge, but also acquaintance and familiarity. At the core of this complexity is that the cycles of processing a proof are multilayered since they involve different maturing stages: understanding, believing and finally justification of that belief.

Once one becomes familiar with a proof style, he or she processes it so naturally that one overlooks its many layers. A definition solely based on properties is abstract to people seeing it for the first time, but not to people who have seen many of its manifestations and have already built other definitions based on it.

Epistemological difficulty starts at the level of definitions, before students start to feel at home with the new domain they are learning. Just like when it comes to learning new words, in addition to knowing the meaning of a word in a dictionary, one needs to know how to use it in a sentence (Healy and Hoyles [3]). This would be primarily due to the multidimensionality of the meaning of arguments, which includes layers of content, epistemic and logical values (Duval [2]). Students start by believing what they read, and then they try to validate it (Selden and Selden [4]). We know that when a mathematician reads a proof, he or she looks for all common aspects between this proof and all the previous ones in his or her pool of knowledge/collection (Yang and Lin [7]). It is only normal for students to want to mimic the work of their teachers and of the text, whether this is a good thing or a bad thing. It is hoped that this writing activity would allow both. Linking arguments into logical chains depends on the status of arguments rather than their content (Duval [2]). Also remember that proofs look linear, whereas they are linear only in style (Weber [6]). Linking arguments into logical chains depends on the status of arguments rather than just their content; this is why the teaching of proofs remains a delicate pedagogy that requires extreme care by the instructor.

When writing a proof students risk sometimes thinking that the form is superior to the content since both form and content are equally foreign to them (Yang and Lin [7]).

Throughout my teaching experience, while observing students learning proofs, I noticed that it takes mimicking, sorting and eventually guessing the appropriate proof mode to succeed in understanding proofs. Although mimicking is in a way pre-intuitive, the hope is that with time, it is their intuition rather than their memory that would naturally direct learners to the successful choice of proof type. Also, research shows that being able to both summarize a proof and identify the relationships between its components constitute two dimensions of understanding it.

In this course, it is hard for students to know what is required of them; this is why they sometimes choose to memorize the proofs. Also they need to communicate their ideas rather than just have them for themselves.

## Project Description

I wanted the students to work on a long-term project where they would invest personally in mathematics; this is why I created a platform that common problem sets and exams cannot provide; I searched for an activity that would enable them to generate their own questions, create interesting viable queries, and eventually develop the tool of investigating problems.

The goal from this activity is to make learners build a special dictionary for the proof types they are learning, to classify them, use them, and recognize what situations they would best apply to. By abstracting a proof category into a template, there is a chance that at a later stage the proof will be exported and applied onto a different context. In summary, I looked for what would give the students an opportunity to take ownership of a mathematical topic and to become some sort of specialists in the area. I wanted their familiarity with the different proofs to make sense and be useful and practical. As a start, I needed to train students to read a proof with a purpose: to look at both the proof's content and form. I designed this activity by modeling what happens in successful learning cases. This is why I asked the students to keep a log where they regularly store and sort, classify or typify all the miscellaneous proof types encountered during the course, in their various manifestations. In order to come up with such a collection, one has to be an experienced observer for a long time. It is hoped that by seeking more manifestations and examples of a given proof type, students become aware and alert observers, and can then claim that they have ownership of the proof. Also, when one gets trained to elevate their finding to a higher/abstract level, there is hope that one could go beyond the context in which the proof was applied, and consequently apply the proof in a different domain where similar relations occur, but between different objects. This could be one way of building strong mathematical intuition (Thompson [5]). So in summary, the project aims at exposing learners to as many versatile manifestations of one proof type as possible in order to ensure familiarity with it: relying on the assumption that it is by highlighting a given method that one would be able to abstract it, thus making it ready for use in a different situation. Needless to say that adding items in their journal forces them to read thoroughly and to keep connecting the claim.

## General Advantages of Writing Activities

By having students write, we make sure they are not emotionally disengaged from their mathematical thoughts. Through expository writing they discover that they know more than they thought. We know that by writing students participate in the work in an authentic dialogue with a virtual person, namely the reader of the journal. Also, it is good for the instructor to read what the students write: this way they see how the learners react with the material, hope they make it theirs. Also, when students are asked to work on long-term projects on a topic, they are able to invest personally in mathematics in a manner that problem sets and exams cannot provide.

## Prerequisites and Class Composition

At the university where this study was conducted, the math majors are not offered a transition course that would prepare them for proof writing. Most of the students in the class had taken the discrete mathematical structures course together with linear algebra, where they get introduced to proofs. In my previous abstract algebra classes, I witnessed that students enjoyed ― watching || those proofs being performed by someone else. They appreciated specially the dual nature of their simplicity and complexity. But in general, students were frustrated when it came to writing their own proofs.

When I decided to conduct this study/experiment, the class included thirteen students (unlike the regular smaller size class with barely four mathematics majors). This time, there were seven computer science and economics students seeking a minor in mathematics, one actuarial studies student seeking a math elective and 4 math majors, and one chemistry major.

This class is taught in a semi interactive fashion, with continual discussion, tying ideas together, constantly tying parts of the material in a meaningful way, motivating any new concept, pointing to new proof styles and techniques, alarming the learners whenever a new trend is introduced, reminding of the larger picture where all the parts would eventually make sense.

In general, I would introduce every new topic by motivating the material, and connecting it to sometimes, seemingly irrelevant past subject, making sure to justify the introduction of any new definitions, so that in this way, any new work would make sense. So, for instance when I introduce cyclic groups, I make it a point to mention the analogy/metaphor of the prime numbers as the building blocks of numbers.

There was a two-hour brainstorming session to discuss and then share their findings as far as proof templates; both style (form) and context are important.

## Project Characteristics and Guidelines

Throughout the project, grades are attributed to writing style, coherence, organization and structure, clarity, mathematical content, insightful connections, correctness, appropriate examples. Only wide directions and guidelines are provided. Substantial revision of the journal is required. The first draft was due ten days prior to the real deadline and was given extensive feedback. Students were required to use their personal language, first person singular. "I can find.." instead of "there exists..", to confirm ownership and establish a good rapport with the material. They were advised to look for similarities between proofs, to compare proofs at all levels: proofs same on the surface or similar ingredients-wise, or connections-wise, or reach the same objects eventually. After that, they had to decide on the parameters that would guide their classification: would they classify according to proof structure? According to format? or big idea? They were also advised to keep a chapter in their journal entitled: "what used to be a misconception but no more". They had to decide on the sorting criteria themselves.

**Samples from Students on Proof Categories**

- **Universal, discipline-free?**

  It was interesting that almost all the students included in this category the proof about the uniqueness of the identity element and that of the inverse. They had encountered the same proof in linear algebra when it comes to vector spaces.

- **"Tool rather than goal" category: their value is as a stepping-stone toward the same Bigger goal**

  In this category, most students named the different lemma that they had encountered.

- **Definition unpacking: A----→B**

  Some students referred out these proofs as "decoding proofs". Sometimes by paraphrasing the definitions in statement A, the claim B is concluded in an automatic way. To name a few examples: to show that a group with the property that $(ab)^{-1} = a^{-1}b^{-1}$ for all elements is Abelian, one starts with the given and ends in the conclusion. Same applies to groups with the property that $(ab)^2 = a^2b^2$.

- **Proofs that share the Big Idea (or are lead by the same big idea)**

  All the proofs that involve a greatest common divisor (gcd) shared one common idea that if $\gcd(m, n) = d$, then we can find integers $s$ and $t$ such that $d = sm + tn$.

- **ELEGANT category (a form style)**

  These proofs included construction proofs such as the Lagrange theorem as well as the Sylow theorem. Also at a lower level, proofs that are apparent by making an observation applicable to the Cayley table of a group.

- **Proofs inspired/lead/motivated or even exemplified by the same set of examples**

  All proofs about properties of cosets fit in that category.

- **Proofs where each step taken in isolation is simple, but the proof as a whole is not simple**

  This applies to the majority of advanced proofs.

- **Proofs that have a similar structure of complexity**

  All proofs around isomorphisms fit in this category.

## Conclusion

This activity has a therapeutic edge because students get elevated to the role of a “meta-creator”, or “re-creator” of the proof schema, because they have been forced to look at the proof’s content and form at the same time, and at each stage.

Students have been through this activity of classifying methods earlier on in their mathematics education. Students themselves had noted that this activity reminded them of the way they dealt with various types of problems in probability: they had to categorize problems according to type in order to succeed in the absence of a clear algorithm there. This helps them fear the unknown much less, and automatically try to find the best-fit template, or the most appropriate model.

One possible drawback is that this method may encourage students to memorize proof types and go about linearly seeking the appropriate method by going blindly through the list of proof templates they have compiled.

I noticed that stronger students tend to mimic the proofs that they have already encountered and witnessed and can recall the appropriate hints that make them look for the relevant proof. In other words, this technique that requires mimicking is pre-intuitive. Completing the project rigorously will be a stamp checking that the concepts have been encapsulated and the schemas constructed.

Students who took the journal seriously showed improvement all along. Language evolves so clearly from page to page within a journal.

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