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TRAJECTORY OF AN EARTH SURFACE POINT AROUND SUN

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Abstract

This paper serves to guide calculus students deriving a parametric function portraying a GPS location's solar orbit.

1. Introduction

Calculus textbooks (see [1] for example) give a unified treatment of all three types of conic sections in terms of a fixed point F (focus) and a fixed line l (directrix): the set of all points in the plane such that the ratio between their distance to F and their distance to l is a constant e (eccentricity). A conic is an ellipse if $0 < e < 1$, a parabola if $e = 1$ and a hyperbola if $e > 1$.

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If we place the focus at the origin, then a conic section has a simple polar equation which provides a convenient description of the motion of planets, satellites, and comets. If we place the origin in the center of the Sun, and the polar axis on the major axis of the elliptic orbit of the Earth, then the equation of this elliptic orbit in polar coordinates (r, t) is:

$$r = f(t) = \frac{a(1-e^2)}{1-e\cos t},$$

where a is the semi-major axis (approximately $1.49 \times 10^8 \text{km}$) and $e = 0.017$.

2. Approach and Resolution of the Problem

In advanced calculus in engineering, one of the applications of polar equations of conics is to obtain the orbits of the planets around the Sun. Some students are interested in finding a curve that describes the trajectory of a point on the Earth surface during its rotation and translation motions. Figure 1 shows a handwritten sketch, realized by a student when asking about this question in the classroom.

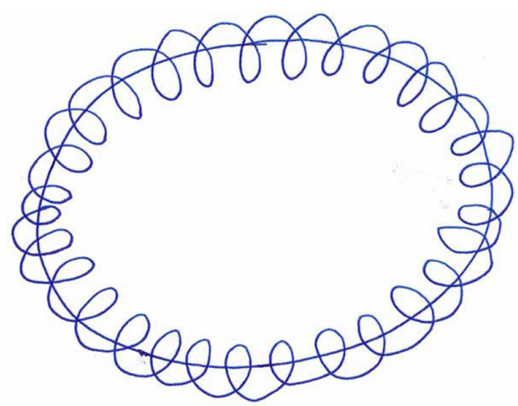

Figure 1

To solve this question, we set ourselves the objective of obtaining a mathematical formula that describes this trajectory. To do this, we showed students Figure 2.

In this figure, we have considered a point P located on the Earth surface in the north latitude L (similarly for a south latitude). We are assuming that $z = 0$ is the ecliptic plane. The vector $r(t)$ that joins the center of the Sun with P describes the path to be calculated. This vector is obtained as the sum of three other vectors:

$$\mathbf{r}(t) = \mathbf{X}(t) + \mathbf{V}_1 + \mathbf{V}_2(t).$$

(a) $\mathbf{X}(t) = (f(t)\cos t,\ f(t)\sin t,\ 0)$ is the vector from center of the Sun to Earth.

(b) $\mathbf{V}_1$ is an orthogonal vector to the equatorial plane joining the center of the Earth with the center of the parallel of latitude L.

(c) $\mathbf{V}_2(t)$ is a vector joining the extreme of $\mathbf{V}_1$ with the point P.

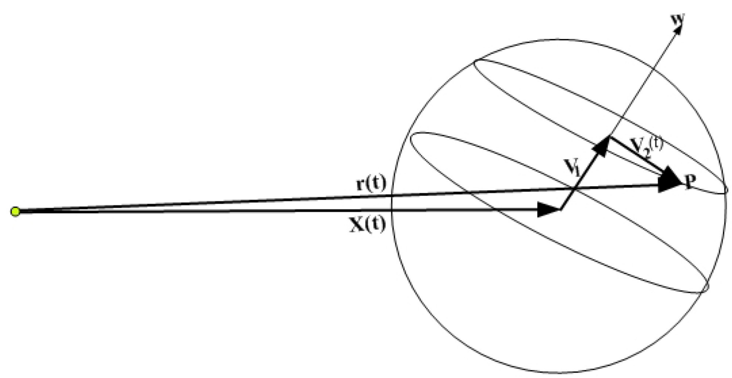

Figure 2. Earth.

2.1. Computation of $\mathbf{V}_1$

To obtain $\mathbf{V}_1$, it is necessary to calculate the vector w (Figure 2), which is a unitary vector that points upwards and is orthogonal to the equatorial plane. This vector is obtained with the vector product $\mathbf{w} = \mathbf{u} \times \mathbf{v}$, where $\{\mathbf{u},\ \mathbf{v}\}$ is an orthonormal basis of the equatorial plane.

To calculate $\{\mathbf{u},\ \mathbf{v}\}$, we use Figure 3.

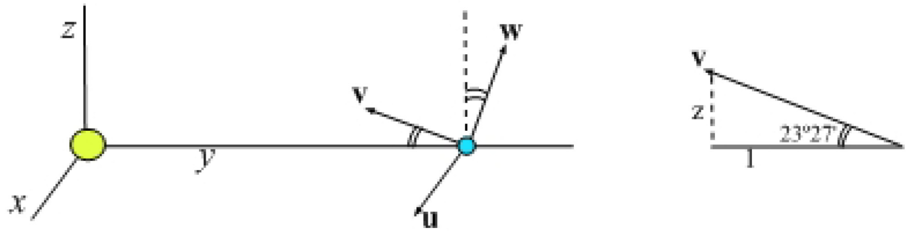

Figure 3. Vectors.

Since the equatorial plane is $z = 0$, we can take $\mathbf{u} = (1, 0, 0)$.

To obtain *v*, we use the Earth obliquity, 23°27′. Therefore, from the equation $\cos 23^\circ 27' = \dfrac{1}{\sqrt{1+z^2}}$, we obtain $z = 0.4335$. As a consequence, **v** is parallel to the vector $(0, -1, 0.4335)$.

In order to remove the decimals and for **w** to point upwards, we take the unit vector $\mathbf{v} = \left(0, \dfrac{10000}{\sqrt{\gamma}}, \dfrac{-4335}{\sqrt{\gamma}}\right)$, where $\gamma = 118792225$. Thus

$$\mathbf{w} = \mathbf{u} \times \mathbf{v} = \left(0, \frac{4335}{\sqrt{\gamma}}, \frac{10000}{\sqrt{\gamma}}\right).$$

Any point *P* located in the north latitude *L* is associated to the pair (x_L, y_L):

$x_L = 6378 \cos L$ is the distance from *P* to the Earth’s axis (note that the Earth’s equatorial radius is 6378kms).

$y_L = 6357 \sin L$ is the distance from *P* to the equatorial plane (note that the Earth’s polar radius is 6357kms).

Then $\mathbf{V}_1$ is a positive multiple of **w**, with length y_L, that is:

$$\mathbf{V}_1 = \mathbf{y}_L \mathbf{w}.$$

In the case of a south latitude, $\mathbf{V}_1 = -\mathbf{y}_L \mathbf{w}$.

2.2. Computation of $\mathbf{V_2}(t)$

$\mathbf{V_2}(t)$ is a vector whose length is the distance from P to the Earth's axis, that is x_L. Notice that when t varies, the point P runs through the parallel L. This parallel is located in the plane whose orthonormal basis is $\{\mathbf{u}, \mathbf{v}\}$. Therefore,

$$\mathbf{V_2}(t) = x_L \mathbf{u} \cos 365.24t + x_L \mathbf{v} \sin 365.24t.$$

The coefficient 365.24 indicates the number of times that the point P turns around the Earth axis in a year.

Finally, the desired equation is:

$$\mathbf{r}(t) = \mathbf{X}(t) + y_L \mathbf{w} + x_L \mathbf{u} \cos 365.24t + x_L \mathbf{v} \sin 365.24t, \quad t \in [0, 2\pi].$$

2.3. Examples

Let us take the point P located at the north latitude 51′47° (Royal Observatory Greenwich). In the following pictures, it is compared the orbit described by the center of the Earth around the Sun (discontinuous line) with the trajectory of the point P during the rotation and translation motions (continuous line). The difference between both trajectories is hard to appreciate if we look it up over a whole year (Figure 4). The next figures have been created with Mathematica 11.0.

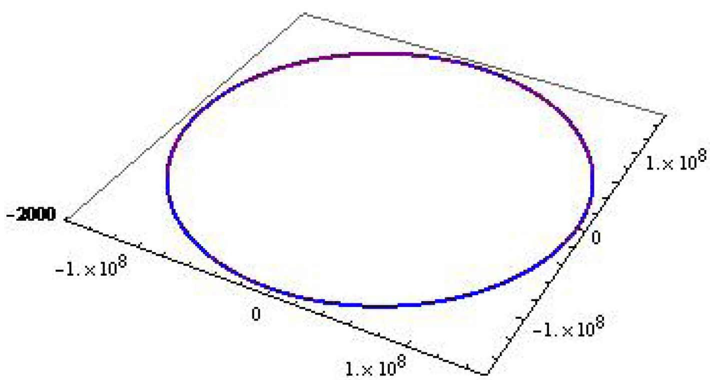

Figure 4. Orbit.

Figure 4 shows that the orbit of the center of Earth around Sun, is an ellipse with an eccentricity so low, that it seems a circumference.

We note that the trajectory of the point and the orbit of the center of Earth around Sun, from afar, are practically the same, they seem to overlap each other. The turns of the point around the elliptical orbit cannot be perceived.

This difference is more appreciable when we consider a short period of time. For a period of H hours, the value of t varies in the interval $\left[0, \frac{2\pi H}{24(365.24)}\right]$ (Figures 5 and 6). Over the course of an hour (Figure 5), both paths seem parallel lines. Increasing the number of hours, we see that they are not parallel (Figure 6).

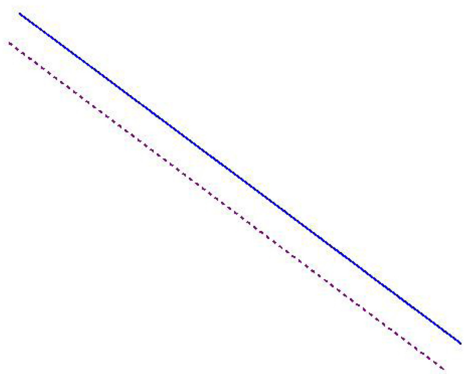

Figure 5. $H = 1$ and 365.24 turns.

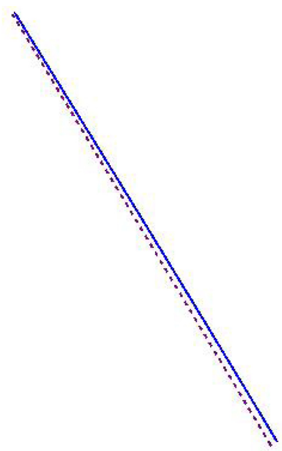

Figure 6. $H = 4$ and 365.24 turns.

If the point P turned 10000 times around the Earth axis in a year, then the difference between both trajectories would be more noticeable (Figures 7 and 8).

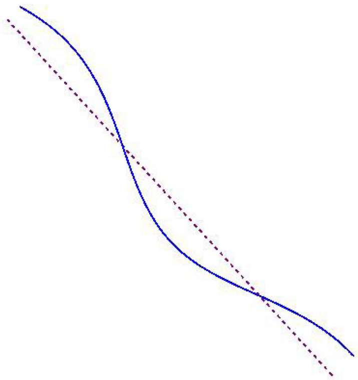

Figure 7. $H = 1$ and 10000 turns.

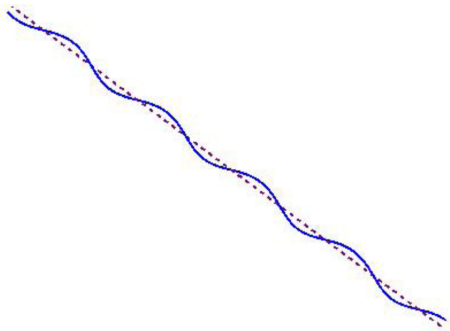

Figure 8. $H = 4$ and 10000 turns.

For the actual trajectory to be similar to that expected by students, the number of rotations of the Earth in a year would have to be greater.

Acknowledgement

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Reference

[1] J. Stewart, Calculus, Cengage Learning, 2015.