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ESTIMATING THE MEAN AND VARIANCE OF A COMPOUND POISSON PROCESS WITH THE POISSON INTENSITY OBTAINED AS EXPONENTIAL OF THE LINEAR FUNCTION

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Abstract

Estimators of the mean and variance functions of a compound Poisson process with the Poisson intensity obtained as exponential of the linear function are constructed and investigated. We consider the case when there is only a single realization of the Poisson process observed in a bounded interval. The proposed estimators are proved to be weakly and strongly consistent when the size of the interval indefinitely expands. The expected values and variances of the proposed estimators are computed.

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1. Introduction

Let $\{N(t), t \geq 0\}$ be a non-homogeneous Poisson process with (unknown) locally integrable intensity function λ. The intensity function λ is assumed to be an exponential of the linear function, that is,

$$\lambda(s) = \exp(\alpha + \beta s).$$

This intensity function can be simplified as follows:

$$\begin{aligned} \lambda(s) &= \exp(\alpha)\exp(\beta s) \\ &= \gamma \exp(\beta s), \end{aligned} \tag{1.1}$$

where γ is an unknown positive real number. We assumed that β is a known constant and $0 < \beta < \infty$.

Let $\{Z(t), t \geq 0\}$ be a compound Poisson process, that is,

$$Z(t) = \sum_{i=1}^{N(t)} X_i, \tag{1.2}$$

where $\{X_i, i \geq 1\}$ is a sequence of independent and identically distributed non-negative random variables with mean $\mu_1 < \infty$ and variance $\sigma_1^2 < \infty$, which is also independent of the process $\{N(t), t \geq 0\}$. It is assumed that $\mu_4 = E[X_1^4] < \infty$. The model presented in (1.2) is a generalization of the (well known) compound Poisson process, which assumes that $\{N(t), t \geq 0\}$ is a homogeneous Poisson process, the case when $\beta = 0$. Some applications of compound Poisson process can be found in [1, 2, 5, 6]. Some related works can be found in [3, 4, 7].

Suppose that, for some $\omega \in \Omega$, a single realization $N(\omega)$ of a Poisson process $\{N(t), t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ with intensity function λ is observed, though only within a bounded interval $[0, n]$. Furthermore, suppose that for each data point in the observed realization $N(\omega) \cap [0, n]$, say ith data point, $i = 1, 2, ..., N([0, n])$, its corresponding random variable X_i is also observed. The mean function (expected value) of $Z(t)$, denoted by $\psi(t)$, is given by

$$\psi(t) = E(Z(t)) = E\left(\sum_{i=1}^{N(t)} X_i\right) = E(N(t))E(X_1) \tag{1.3}$$

and the variance function of $Z(t)$, denoted by $V(t)$, is given by

$$V(t) = Var(Z(t)) = Var\left(\sum_{i=1}^{N(t)} X_i\right) = E(N(t))E(X_1^2). \tag{1.4}$$

Since

$$\begin{aligned} E(N(t)) &= E(N([0, t])) \\ &= \int_0^t \lambda(s)ds \\ &= \int_0^t \gamma \exp(\beta s)ds \\ &= \frac{\gamma}{\beta}(\exp(\beta t) - 1), \end{aligned} \tag{1.5}$$

the mean and variance functions of $Z(t)$ can be written as, respectively,

$$\psi(t) = \frac{\gamma}{\beta}(\exp((\beta t)) - 1)\mu_1 \tag{1.6}$$

and

$$V(t) = \frac{\gamma}{\beta}(\exp((\beta t)) - 1)\mu_2, \tag{1.7}$$

where $\mu_1 = E(X_1)$ and $\mu_2 = E(X_1^2)$.

The rest of this paper is organized as follows. The estimators and main results are presented in Section 2. Proofs of our theorems are presented in Section 3.

2. The Estimators and Main Results

The estimators of functions $\psi(t)$ and $V(t)$ using the available data set at hand are given, respectively, by

$$\hat{\psi}_{n,\beta}(t) = \frac{\hat{\gamma}_{n,\beta}}{\beta}(\exp(\beta t) - 1)\hat{\mu}_{1,n} \tag{2.1}$$

and

$$\hat{V}_{n,\beta}(t) = \frac{\hat{\gamma}_{n,\beta}}{\beta}(\exp(\beta t) - 1)\hat{\mu}_{2,n}, \tag{2.2}$$

where

$$\hat{\gamma}_{n,\beta} = \frac{\beta}{(\exp(\beta n) - 1)} N([0, n]), \tag{2.3}$$

$$\hat{\mu}_{1,n} = \frac{1}{N([0, n])} \sum_{i=1}^{N([0,n])} X_i \tag{2.4}$$

and

$$\hat{\mu}_{2,n} = \frac{1}{N([0, n])} \sum_{i=1}^{N([0,n])} X_i^2, \tag{2.5}$$

with the understanding that $\hat{\psi}_{n,\beta}(t) = \hat{V}_{n,\beta}(t) = 0$ when $N([0, n]) = 0.$

Next, we describe the idea behind the construction of the estimator in (2.3). By (1.1), (1.5) and the fact that $E(N([0, n])) = \int_0^n \lambda(s)ds,$ we have

$$E(N([0, n])) = \frac{\gamma}{\beta}(\exp(\beta n) - 1) \Leftrightarrow \gamma = \frac{\beta}{(\exp(\beta n) - 1)} E(N([0, n])). \tag{2.6}$$

Replacing $E(N([0, n]))$ by $N([0, n]),$ then we obtain the estimator in (2.3).

Our main results are presented in the following four theorems. The first theorem is about expected values of $\hat{\psi}_{n,\beta}(t)$ and $\hat{V}_{n,\beta}(t)$ while the second theorem is about variances of $\hat{\psi}_{n,\beta}(t)$ and $\hat{V}_{n,\beta}(t)$. The weak and strong consistencies of $\hat{\psi}_{n,\beta}(t)$ are presented in Theorem 3 while the weak and strong consistencies of $\hat{V}_{n,\beta}(t)$ are presented in Theorem 4.

Theorem 1 (The expected values of $\hat{\psi}_{n,\beta}(t)$ and $\hat{V}_{n,\beta}(t)$). *Suppose that the intensity function* λ *satisfies* (1.1) *and is locally integrable*. *If* $Z(t)$ *satisfies* (1.2), *then*

$$E(\hat{\psi}_{n,\beta}(t)) = \psi(t)$$

and

$$E(\hat{V}_{n,\beta}(t)) = V(t).$$

Theorem 2 (The variances of $\hat{\psi}_{n,\beta}(t)$ and $\hat{V}_{n,\beta}(t)$). *Suppose that the intensity function* λ *satisfies* (1.1) *and is locally integrable*. *If* $Z(t)$ *satisfies* (1.2), *then*

$$Var(\hat{\psi}_{n,\beta}(t)) = \frac{1}{(\exp(\beta n) - 1)} \frac{\gamma\mu_2(\exp(\beta t) - 1)^2}{\beta}$$

and

$$Var(\hat{V}_{n,\beta}(t)) = \frac{1}{(\exp(\beta n) - 1)} \frac{\gamma\mu_4(\exp(\beta t) - 1)^2}{\beta}.$$

Since $Bias(\hat{\psi}_{n,\beta}(t)) = 0$ and $Bias(\hat{V}_{n,\beta}(t)) = 0,$ we have that $MSE(\hat{\psi}_{n,\beta}(t)) = Var(\hat{\psi}_{n,\beta}(t))$ and $MSE(\hat{V}_{n,\beta}(t)) = Var(\hat{V}_{n,\beta}(t))$.

Theorem 3 (Consistency of $\hat{\psi}_{n,\beta}(t)$). *Suppose that the intensity function* λ *satisfies* (1.1) *and is locally integrable*. *If* $Z(t)$ *satisfies* (1.2), *then*

$$\hat{\psi}_{n,\beta}(t) \xrightarrow{P} \psi(t) \tag{2.7}$$

and

$$\hat{\psi}_{n,\beta}(t) \xrightarrow{a.s.} \psi(t) \tag{2.8}$$

as $n \to \infty$. *Hence,* $\hat{\psi}_{n,\beta}(t)$ *is a weakly and strongly consistent estimator of* $\psi(t)$.

Theorem 4 (Consistency of $\hat{V}_{n,\beta}(t)$). *Suppose that the intensity function* λ *satisfies* (1.1) *and is locally integrable*. *If* $Z(t)$ *satisfies* (1.2), *then*

$$\hat{V}_{n,\beta}(t) \xrightarrow{P} V(t) \tag{2.9}$$

and

$$\hat{V}_{n,\beta}(t) \xrightarrow{a.s.} V(t) \tag{2.10}$$

as $n \to \infty$. *Hence,* $\hat{V}_{n,\beta}(t)$ *is a weakly and strongly consistent estimator of* $V(t)$.

3. Proofs of Theorems 1-4

Note that, by (2.1), (2.3) and (2.4), we obtain

$$\begin{aligned}\hat{\psi}_{n,\beta}(t) &= \frac{\hat{\gamma}_{n,\beta}}{\beta}(\exp(\beta t) - 1)\hat{\mu}_{1,n} \\ &= \frac{1}{\beta}\frac{\beta(\exp(\beta t) - 1)}{(\exp(\beta n) - 1)} N([0, n]) \frac{1}{N([0, n])}\sum_{i=1}^{N([0,n])} X_i \\ &= \frac{(\exp(\beta t) - 1)}{(\exp(\beta n) - 1)}\sum_{i=1}^{N([0,n])} X_i. \end{aligned} \tag{3.1}$$

Similar to (3.1), by (2.2), (2.3) and (2.5), we obtain

$$\hat{V}_{n,\beta}(t) = \frac{(\exp(\beta t) - 1)}{(\exp(\beta n) - 1)}\sum_{i=1}^{N([0,n])} X_i^2. \tag{3.2}$$

We note that, since $\{X_i, i \geq 1\}$ is a sequence of independent and identically distributed non-negative random variables with $\mu_4 = E[X_1^4] < \infty$, we also have that $\{X_i^2, i \geq 1\}$ is a sequence of independent and identically distributed random variables with mean $\mu_2 < \infty$ and variance $\sigma_2^2 < \infty$.

Proof of Theorem 1. By (1.3), (1.6), (2.6) and (3.1), we obtain

$$E(\hat{\psi}_{n,\beta}(t)) = \frac{(\exp(\beta t) - 1)}{(\exp(\beta n) - 1)} E(N[0, n])\mu_1$$

$$= \frac{(\exp(\beta t) - 1)}{(\exp(\beta n) - 1)} \frac{\gamma}{\beta} (\exp(\beta n) - 1)\mu_1 = \psi(t). \tag{3.3}$$

Similar to (3.3), by (1.3), (1.7), (2.6) and (3.2), we obtain

$$E(\hat{V}_{n,\beta}(t)) = V(t). \tag{3.4}$$

This completes the proof of Theorem 1.

Proof of Theorem 2. By (1.4), (2.6) and (3.1), we obtain

$$Var(\hat{\psi}_{n,\beta}(t)) = \left(\frac{(\exp(\beta t) - 1)}{(\exp(\beta n) - 1)}\right)^2 Var\left(\sum_{i=1}^{N([0,n])} X_i\right)$$

$$= \left(\frac{(\exp(\beta t) - 1)}{(\exp(\beta n) - 1)}\right)^2 E(N([0, n]))\mu_2$$

$$= \left(\frac{(\exp(\beta t) - 1)}{(\exp(\beta n) - 1)}\right)^2 \frac{\gamma}{\beta} (\exp(\beta n) - 1)\mu_2$$

$$= \frac{1}{(\exp(\beta n) - 1)} \frac{\gamma\mu_2(\exp(\beta t) - 1)^2}{\beta}. \tag{3.5}$$

Similar to (3.5), by (1.4), (2.6) and (3.2), we obtain

$$Var(\hat{V}_{n,\beta}(t)) = \frac{1}{(\exp(\beta n) - 1)} \frac{\gamma\mu_4(\exp(\beta t) - 1)^2}{\beta}. \tag{3.6}$$

This completes the proof of Theorem 2.

In this paper, for any random variables X_n and X on a probability space $(\Omega, \mathcal{F}, P)$, $X_n \xrightarrow{c} X$ denotes that X_n converges completely to X, as $n \to \infty$. The random variable X_n is called *converges* completely to X if, for each $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(|X_n - X| > \varepsilon) < \infty. \tag{3.7}$$

Proof of Theorem 3. Since (2.8) implies (2.7), to prove this theorem, it suffices to check (2.8). First, we prove that

$$\hat{\psi}_{n,\beta}(t) \xrightarrow{c} \psi(t) \tag{3.8}$$

as $n \to \infty$. By (3.3), (3.5), (3.7) and Chebyshev's inequality, we obtain

$$\sum_{n=1}^{\infty} P(|\hat{\psi}_{n,\beta}(t) - \psi(t)| > \varepsilon)$$

$$= \sum_{n=1}^{\infty} P(|\hat{\psi}_{n,\beta}(t) - E(\hat{\psi}_{n,\beta}(t))| > \varepsilon)$$

$$\leq \sum_{n=1}^{\infty} \frac{Var(\hat{\psi}_{n,\beta}(t))}{\varepsilon^2}$$

$$= \frac{\gamma\mu_2(\exp(\beta t) - 1)^2}{\beta\varepsilon^2} \sum_{n=1}^{\infty} \frac{1}{(\exp(\beta n) - 1)} < \infty.$$

Hence, we have (3.8). By (3.8) and the Borel-Cantelli lemma, we obtain (2.8). This completes the proof of Theorem 3.

Proof of Theorem 4. Since (2.10) implies (2.9), to prove this theorem, it suffices to check (2.10). First, we verify

$$\hat{V}_{n,\beta}(t) \xrightarrow{c} V(t) \tag{3.9}$$

as $n \to \infty$. By (3.4), (3.6), (3.7) and Chebyshev's inequality, we obtain

$$\sum_{n=1}^{\infty} P(|\hat{V}_{n,\beta}(t) - E(\hat{V}_{n,\beta}(t))| > \varepsilon)$$

$$\leq \frac{\gamma\mu_4(\exp(\beta t) - 1)^2}{\beta\varepsilon^2} \sum_{n=1}^{\infty} \frac{1}{(\exp(\beta n) - 1)} < \infty.$$

Hence, we have (3.9). By (3.9) and the Borel-Cantelli lemma, we obtain (2.10). This completes the proof of Theorem 4.

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