Advances and Applications in Statistics
© 2017 Pushpa Publishing House, Allahabad, India
http://www.pphmj.com
http://dx.doi.org/10.17654/AS051010059
Volume 51, Number 1, 2017, Pages 59-84 **ISSN: 0972-3617**

ON LARGE SAMPLE CONFIDENCE INTERVALS FOR THE COMMON INVERSE MEAN OF SEVERAL NORMAL POPULATIONS

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Abstract

The objective of this paper is to construct confidence intervals for the common inverse mean of several normal populations based on adjusted method of variance estimates recovery approach (adjusted MOVER approach) and to compare with generalized confidence interval approach and large sample approach. The coverage probability and average length of the confidence intervals are evaluated by a Monte Carlo simulation. The results showed that the generalized confidence interval approach provides the best confidence interval, but the coverage probabilities of the adjusted MOVER confidence intervals are close to the nominal confidence level of 0.95 when the sample size is large. Finally, the proposed approaches are illustrated by an example.

Received: May 9, 2017; Accepted: July 3, 2017

2010 Mathematics Subject Classification: 62F25.

Keywords and phrases: inverse mean, normal distribution, generalized confidence interval approach, adjusted MOVER approach.

1. Introduction

The inverse mean of normal distribution is defined as the ratio of one to the population mean. It has been statistical estimation in many fields such as experimental nuclear physics, econometrics, and biological sciences. In experimental nuclear physics, Lamanna et al. [6] studied charged particle momentum $\rho = 1/\mu$, where μ is the track curvature of a particle. Zaman [21] estimated the inverse mean in the one-dimensional special case of the single period control problem. Zaman [22] discussed an estimate of the inverse mean without moments. In econometrics, Zellner [23] studied the inverse of common mean of structural coefficient of linear structural econometric models. Voinov [15] presented the unbiased estimators of power for the inverse of mean. Niwitpong and Wongkhao [7] presented three new confidence intervals for the inverse mean of normal distribution. Niwitpong and Wongkhao [8] constructed three new confidence intervals for the difference between inverse means of normal distributions.

In many practical situations, there are common practices to collect different settings. Several researchers have been studied confidence intervals for the common parameter, for example, Krishnamoorthy and Lu [5] presented procedures for hypothesis testing and interval estimation of the common mean of several normal populations. Tian [12] dealt with the problem of making inference about the common populations with a common coefficient of variation. Tian and Wu [14] proposed the confidence interval estimation and hypothesis testing of the common mean of several log-normal populations using the concept of generalized variable. Ye et al. [20] presented procedures for hypothesis testing and interval estimation for the common mean of several inverse Gaussian populations. Thangjai et al. [9] proposed the generalized confidence interval approach and the large sample approach for confidence interval estimation about the common inverse mean based on several independent normal samples. Thangjai and Niwitpong [11] proposed new confidence intervals for the weighted coefficients of variation of two-parameter exponential distributions.

Therefore, confidence interval estimation for the common inverse mean based on several independent normal samples is of practical and theoretical importance. The goal of this paper is to extend the recent work of Thangjai et al. [9] to construct the confidence intervals for the common inverse mean of normal distributions. We propose a novel approach, the adjusted method of variance estimates recovery approach (adjusted MOVER approach), for confidence interval estimation for the common inverse mean of normal distributions. Then there are the concepts of generalized confidence interval, large sample confidence interval, and adjusted MOVER confidence interval. The first confidence interval was introduced by Weerahandi [17]. Many researchers have successfully used the generalized confidence interval approach to construct confidence interval for common parameter, i.e., see Krishnamoorthy and Lu [5], Tian [12], Tian and Wu [14], and Ye et al. [20]. Moreover, the concept of the generalized confidence interval has been applied to variety of practical settings where standard solutions do not exist for confidence intervals, i.e., see Weerahandi [18], Weerahadi and Berger [19], Krishnamoorthy and Lu [5], Tian and Cappelleri [13], Tian [12], Tian and Wu [14], and Thangjai et al. [9]. The second confidence interval was constructed based on the large sample approach which was constructed based on central limit theorem (CLT). The paper by Tian and Wu [14] presented the confidence interval for the common mean of several log-normal populations based on generalized confidence interval approach and compared it with a large sample approach. Thangjai et al. [9] proposed the confidence interval for the common inverse mean of several normal populations based on generalized confidence interval approach and compared with large sample approach. The third confidence interval was motivated based on the method of variance estimates recovery approach (MOVER approach), was introduced by Zou and Donner [24] and Zou et al. [25], is called *adjusted MOVER confidence interval*. The MOVER approach was inspired by the score interval approach which proposed by Bartlett [1]. Many researchers have successfully used the MOVER approach for constructing the confidence interval for parameter; for example, see Zou and Donner [24], Zou et al. [25], and Donner and Zou [3]. Moreover, several researchers have used the concept of the adjusted MOVER approach to construct the confidence interval for common parameters; for example, see Thangjai et al. [10] and Thangjai and Niwitpong [11].

This paper is organized as follows: In Section 2, the proposed approach and existing approaches are described. In Section 3, simulation results are presented to evaluate the coverage probabilities and average lengths of the proposed approach and existing approaches. Section 4 illustrates the proposed approach and existing approaches with real example. Finally, Section 5 summarizes this paper.

2. Confidence Intervals for the Common Inverse Mean of Several Normal Populations

Recall that a random variable X is distributed normally with mean μ and variance σ^2. The inverse mean of X is defined as $1/\mu$, where $\mu \neq 0$.

Let X_i, $1 = 2, \ldots, k$ be random samples from normal distributions and let $\theta_i = 1/\mu_i$ be the ith inverse mean population.

Let X_{ij}, $i = 1, 2, \ldots, k$; $j = 1, 2, \ldots, n_i$ be random samples from the X_i, i.e., $X_i = (X_{i1}, X_{i2}, \ldots, X_{in_i})$.

For the ith sample, let $\bar{X}_i$ and $\bar{x}_i$ be sample mean and observed sample mean of X_{ij}, respectively. And let S_i^2 and s_i^2 be sample variance and observed sample variance of X_{ij}, respectively.

The maximum likelihood estimator and unbiased estimator of parameter θ_i are defined by

$$\hat{\theta}_i = \frac{1}{\hat{\mu}_i} = \frac{1}{\bar{X}_i}; \quad i = 2, \ldots, k.$$

Theorem 1. *Let* $X = (X_1, X_2, \ldots, X_n)$ *be a random sample from the* *normal population with mean* μ *and variance* σ^2. *Let* $\hat{\theta}$ *be the unbiased estimator of* θ. *The variance of* $\hat{\theta}$ *is*

$$Var(\hat{\theta}) = \frac{\sigma^2}{n\mu^4}. \tag{1}$$

Proof. Let $X_1, X_2, \ldots, X_n$ be an independent and identically distributed random variables with mean μ and variance σ^2. Then the estimators $\hat{\mu}$ and $\hat{\sigma}^2$ have the following normal distribution in large samples due to the central limit theorem:

$$\sqrt{n}(\hat{\mu} - \mu) \sim N(0, \sigma^2) \quad \text{and} \quad \sqrt{n}(\hat{\sigma}^2 - \sigma^2) \sim N(0, 2\sigma^4),$$

where $\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{\mu})^2$.

Denote $\hat{\theta} = (\hat{\mu}, \hat{\sigma}^2)'$ and $\theta = (\mu, \sigma^2)'$. Then

$$\sqrt{n}(\hat{\theta} - \theta) \sim N(0, V_\theta),$$

where $V_\theta \equiv \begin{bmatrix} \sigma^2 & 0 \\ 0 & 2\sigma^4 \end{bmatrix}$, Casella and Berger [2].

The inverse mean estimator $1/\hat{\mu}$ can be written as function of $\hat{\theta}$, i.e., $f(\hat{\theta})$. The delta method is applied to derive asymptotic distribution,

$$\sqrt{n}(f(\hat{\theta}) - f(\theta)) \sim N(0, V_f),$$

where $V_f \equiv \frac{\partial f(\theta)}{\partial \theta} V_\theta \frac{\partial f(\theta)}{\partial \theta'}$.

The function of θ is denoted by

$$f(\theta) = \frac{1}{\mu}.$$

The partial derivative of $f(\theta)$ with respect to μ and σ^2 are, respectively,

$$\frac{\partial f(\theta)}{\partial \mu} = -\frac{1}{\mu^2} \quad \text{and} \quad \frac{\partial f(\theta)}{\partial \sigma^2} = 0.$$

Thus

$$\frac{\partial f(\theta)}{\partial \theta'} = \begin{bmatrix} -1/\mu^2 \\ 0 \end{bmatrix} \quad \text{and} \quad V_\theta \equiv \begin{bmatrix} \sigma^2 & 0 \\ 0 & 2\sigma^4 \end{bmatrix}.$$

The asymptotic distribution of estimator $1/\hat{\mu}$ is

$$\sqrt{n}\left(\frac{1}{\hat{\mu}} - \frac{1}{\mu}\right) \sim N(0, V_{iid}),$$

where $V_{iid} = \left(\frac{\partial f}{\partial \mu}\right)^2 \sigma^2 + \left(\frac{\partial f}{\partial \sigma^2}\right)^2 2\sigma^4 = \left(-\frac{1}{\mu^2}\right)^2 \sigma^2 + 0 = \frac{\sigma^2}{\mu^4}.$

Thus

$$Var(\hat{\theta}) = Var\left(\frac{1}{\hat{\mu}}\right) = Var\left(\frac{1}{\overline{X}}\right) = \frac{\sigma^2/\mu^4}{n} = \frac{\sigma^2}{n\mu^4}.$$

Hence, Theorem 1 is proved.

2.1. Generalized confidence interval approach (GCI approach)

Definition. Let $X = (X_1, X_2, \ldots, X_n)$ be a random sample from a distribution $F(x|\delta)$, where $x = (x_1, x_2, \ldots, x_n)$ be an observed sample, $\delta = (\theta, v)$ is a vector of unknown parameters, θ is a parameter of interest, v is a vector of nuisance parameters. Let $R = R(X; x, \delta)$ be a function of X, x and δ. The random quantity R is called a *generalized pivotal quantity* if it has the following two properties; see Weerahandi [17]:

(i) The distribution of R is free of all unknown parameters.

(ii) The observed value of R, $X = x$, is the parameter of interest θ.

Let $R(\alpha)$ be the $100(\alpha)$th percentile of R. Then $R(\alpha)$ is a $100(1-\alpha)\%$ lower bound of one-sided generalized confidence interval for θ and $(R(\alpha/2),\ R(1-\alpha/2))$ is the $100(1-\alpha)\%$ two-sided generalized confidence interval for θ.

In the following, the above definition is used to develop a generalized pivotal quantity for the common inverse mean of several normal populations.

Consider k independent normal populations with a common inverse mean θ. Let $X_{i1},\ X_{i2},\ \ldots X_{in_i}$ be a random sample from the ith normal population as follows:

$$X_{ij} \sim N(\mu_i,\ \sigma_i^2).$$

The inverse mean is defined by

$$\theta_i = \frac{1}{\mu_i}; \quad i = 1,\ 2,\ \ldots,\ k.$$

It is well known that

$$\frac{(n_i - 1)S_i^2}{\sigma_i^2} = V_i \sim \chi^2_{n_i - 1}, \tag{2}$$

where V_i is chi-square distribution with degrees of freedom $n_i - 1$. Then

$$\sigma_i^2 = \frac{(n_i - 1)S_i^2}{V_i}.$$

The generalized pivotal quantity for σ_i^2 is defined by

$$R_{\sigma_i^2} = \frac{(n_i - 1)s_i^2}{V_i} \sim \frac{(n_i - 1)s_i^2}{\chi^2_{n_i - 1}}. \tag{3}$$

According to Niwitpong and Wongkhao [7], the generalized pivotal quantity for μ_i is defined as

$$R_{\mu_i} = \bar{x}_i - \frac{Z_i s_i}{\sqrt{U_i}}, \tag{4}$$

where Z_i and U_i denote standard normal distribution and chi-square distribution with degrees of freedom $n_i - 1$, respectively.

The generalized pivotal quantity for θ_i is defined by

$$R_{\theta_i} = \frac{1}{R_{\mu_i}}. \tag{5}$$

Following Ye et al. [20], the generalized pivotal quantity for the common inverse mean θ is a weighted average of the generalized pivotal quantity R_{θ_i} based on k individual samples given by

$$R_\theta = \sum_{i=1}^{k} \frac{R_{\theta_i}}{R_{Var(\hat{\theta}_i)}} \Bigg/ \sum_{i=1}^{k} \frac{1}{R_{Var(\hat{\theta}_i)}}, \tag{6}$$

where (from equation (1))

$$R_{Var(\hat{\theta}_i)} = \frac{R_{\sigma_i^2}}{n_i (R_{\mu_i})^4}. \tag{7}$$

Therefore, the $100(1-\alpha)\%$ two-sided confidence interval for the common inverse mean θ based on the generalized confidence interval approach is

$$CI_{GCI} = (L_{GCI}, U_{GCI}) = (R_\theta(\alpha/2),\ R_\theta(1-\alpha/2)), \tag{8}$$

where $R_\theta(\alpha/2)$ and $R_\theta(1-\alpha/2)$ denote the $100(\alpha/2)$th and $100(1-\alpha/2)$th percentiles of R_θ, respectively.

The following algorithm is used to construct the generalized confidence interval:

Algorithm 1

For a given $\bar{x}_i$ and s_i^2, $i = 1, 2, \ldots k$:

For $g = 1$ to m:

Generate V_i from chi-square distribution with degrees of freedom $n_1 - 1$.

Compute $R_{\sigma_i^2}$ from equation (3).

Generate Z_i from standard normal distribution.

Generate U_i from chi-square distribution with degrees of freedom $n_i - 1$.

Compute R_{μ_i} from equation (4).

Compute R_{θ_i} from equation (5).

Compute $R_{Var(\hat{\theta}_i)}$ from equation (7).

Compute R_θ from equation (6).

(end *g* loop)

Compute the $100(\alpha/2)$th percentile of R_θ defined by $R_\theta(\alpha/2)$.

Compute the $100(1-\alpha/2)$th percentile of R_θ defined by $R_\theta(1-\alpha/2)$.

2.2. Large sample approach

According to Graybill and Deal [4], the large sample estimate of inverse mean is a pooled estimated unbiased estimator of the inverse mean defined as

$$\hat{\theta} = \sum_{i=1}^{k} \frac{\hat{\theta}_i}{Var(\hat{\theta}_i)} \Bigg/ \sum_{i=1}^{k} \frac{1}{Var(\hat{\theta}_i)}, \tag{9}$$

where $\hat{\theta}_i = 1/\bar{X}_i$ and $Var(\hat{\theta}_i)$ is an estimate of $Var(\hat{\theta}_i)$ in equation (1) with μ_i and σ_i^2 replaced by $\bar{x}_i$ and s_i^2, respectively.

For large sample size, the distribution of $\hat{\theta}$ is approximately normal distribution. Then the quantile of the normal distribution is used to construct confidence interval for θ. Therefore, the $100(1-\alpha)\%$ two-sided confidence interval for the common inverse mean θ based on the large sample approach is

$$CI_{LS} = (L_{LS}, U_{LS})$$

$$= \left(\hat{\theta} - z_{1-\alpha/2}\sqrt{1 \Big/ \sum_{i=1}^{k} 1 \Big/ Var(\hat{\theta}_i)},\ \hat{\theta} + z_{1-\alpha/2}\sqrt{1 \Big/ \sum_{i=1}^{k} 1 \Big/ Var(\hat{\theta}_i)} \right), \tag{10}$$

where $z_{1-\alpha/2}$ denotes the $1-\alpha/2$th quantile of the standard normal distribution.

2.3. Adjusted method of variance estimates recovery approach (adjusted MOVER approach)

In two parameters case, Zou and Donner [24], and Zou et al. [25] introduced the method of variance estimates recovery approach (MOVER approach). Let θ_1 and θ_2 be the parameters of interest and let L and U be the lower limit and upper limit of $100(1-\alpha)\%$ two-sided confidence interval for the parameter $\theta_1 + \theta_2$. Using the central limit theorem and the assumption of independence between the point estimates $\hat{\theta}_1$ and $\hat{\theta}_2$, the lower limit L is defined as

$$L = \hat{\theta}_1 + \hat{\theta}_2 - z_{\alpha/2}\sqrt{V\hat{a}r(\hat{\theta}_1) + V\hat{a}r(\hat{\theta}_2)}, \tag{11}$$

where $z_{\alpha/2}$ is the upper $\alpha/2$ th percentile of the standard normal distribution.

For $i = 1, 2,$ let (l_i, u_i) be a $100(1 - \alpha)\%$ two-sided confidence interval for θ_i. The lower limit L must be closer to $l_1 + l_2$ than to $\hat{\theta}_1 + \hat{\theta}_2$. The variance estimate for $\hat{\theta}_i$ at $\theta_i = l_i$ is

$$V\hat{a}r(\hat{\theta}_i) = \frac{(\hat{\theta}_i - l_i)^2}{z_{\alpha/2}^2}.$$

Substituting back into equation (11) as follows:

$$L = \hat{\theta}_1 + \hat{\theta}_2 - \sqrt{(\hat{\theta}_1 - l_1)^2 + (\hat{\theta}_2 - l_2)^2} \tag{12}$$

and similarly

$$U = \hat{\theta}_1 + \hat{\theta}_2 + \sqrt{(u_1 - \hat{\theta}_1)^2 + (u_2 - \hat{\theta}_2)^2}. \tag{13}$$

Let $\theta_1, \theta_2, \ldots, \theta_k$ be the parameters of interest. Then we determine the confidence interval for the common inverse mean based on Graybill and Deal [4] defined by

$$\hat{\theta} = \sum_{i=1}^{k} \frac{\hat{\theta}_i}{Var(\hat{\theta}_i)} \Bigg/ \sum_{i=1}^{k} \frac{1}{Var(\hat{\theta}_i)}.$$

The MOVER approach is motivated with confidence intervals for $\theta_1, \theta_2, \ldots, \theta_k$. Let $(l_1, u_1), (l_2, u_2), \ldots, (l_k, u_k)$ be the confidence intervals for $\theta_1, \theta_2, \ldots, \theta_k$, respectively, and let L and U be the lower limit and upper limit of $100(1 - \alpha)\%$ two-sided confidence interval for the parameter $\theta_1 + \theta_2 + \cdots + \theta_k$. Using the central limit theorem and the assumption of independence between the point estimates $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_k$, the lower limit L is defined as

$$L = \hat{\theta}_1 + \cdots + \hat{\theta}_k - z_{\alpha/2}\sqrt{V\hat{a}r(\hat{\theta}_1) + \cdots + V\hat{a}r(\hat{\theta}_k)},$$

where $z_{\alpha/2}$ is the upper $\alpha/2$ th percentile of the standard normal distribution.

For $i = 1, 2, \ldots, k,$ let (l_i, u_i) be a $100(1-\alpha)\%$ two-sided confidence interval for θ_i. The lower limit L must be closer to $l_1 + l_2 + \ldots + l_k$ than to $\hat{\theta}_1 + \hat{\theta}_2 + \ldots + \hat{\theta}_k$. The lower limit L for $\theta_1 + \theta_2 + \ldots + \theta_k$ is

$$L = \hat{\theta}_1 + \cdots + \hat{\theta}_k - \sqrt{(\hat{\theta}_1 - l_1)^2 + \cdots + (\hat{\theta}_k - l_k)^2} \tag{14}$$

and similarly, the upper limit

$$U = \hat{\theta}_1 + \cdots + \hat{\theta}_k + \sqrt{(u_1 - \hat{\theta}_1)^2 + \cdots + (u_k - \hat{\theta}_k)^2}. \tag{15}$$

Using the concepts of large sample approach and MOVER approach defined in equations (9)-(15), it is called the *adjusted MOVER approach*. According to Graybill and Deal [4], the common inverse mean θ is weighted average of the inverse mean $\hat{\theta}_i$ based on k individual samples defined as

$$\hat{\theta} = \sum_{i=1}^{k} \frac{\hat{\theta}_i}{Var(\hat{\theta}_i)} \Bigg/ \sum_{i=1}^{k} \frac{1}{Var(\hat{\theta}_i)}, \tag{16}$$

where the variance estimate for $\hat{\theta}_i$ at $\theta_i = l_i$ and $\theta_i = u_i$ is the average variance between these two variances and given by

$$Var(\hat{\theta}_i) = \frac{1}{2}\left(\frac{(\hat{\theta}_i - l_i)^2}{z_{\alpha/2}^2} + \frac{(u_i - \hat{\theta}_i)^2}{z_{\alpha/2}^2}\right). \tag{17}$$

Therefore, the lower limit L and upper limit U for the common inverse mean θ are

$$L = \hat{\theta} - z_{1-\alpha/2}\sqrt{1 \Bigg/ \sum_{i=1}^{k} \frac{z_{\alpha/2}^2}{(\hat{\theta}_i - l_i)^2}} \tag{18}$$

and

$$U = \hat{\theta} + z_{1-\alpha/2}\sqrt{1 \Bigg/ \sum_{i=1}^{k} \frac{z_{\alpha/2}^2}{(u_i - \hat{\theta}_i)^2}}. \tag{19}$$

Therefore, the $100(1-\alpha)\%$ two-sided confidence interval for the common inverse mean θ based on adjusted MOVER approach is

$$CI_{AM} = (L_{AM}, U_{AM})$$

$$= \left(\hat{\theta} - z_{1-\alpha/2}\sqrt{1\Big/\sum_{i=1}^{k}\frac{z_{\alpha/2}^2}{(\hat{\theta}_i - l_i)^2}},\ \hat{\theta} + z_{1-\alpha/2}\sqrt{1\Big/\sum_{i=1}^{k}\frac{z_{\alpha/2}^2}{(u_i - \hat{\theta}_i)^2}}\right). \tag{20}$$

According to Niwitpong and Wongkhao [7], three confidence intervals for inverse mean $1/\mu$ are defined by

$$(l_{1i}, u_{1i}) = \left(\frac{\sqrt{n_i}}{d_i S_i + \sqrt{n_i}\bar{X}_i},\ \frac{\sqrt{n_i}}{-d_i S_i + \sqrt{n_i}\bar{X}_i}\right), \tag{21}$$

where d_i is an upper $1-\alpha/2$ th quantile of the *t*-distribution with degrees of freedom $n_i - 1$,

$$(l_{2i}, u_{2i}) = \left(\frac{1}{\bar{X}_i} - c n_i^{-1/2}\left(\frac{1}{\bar{X}_i}\right)^2 S_i,\ \frac{1}{\bar{X}_i} + c n_i^{-1/2}\left(\frac{1}{\bar{X}_i}\right)^2 S_i\right), \tag{22}$$

where *c* is an upper $1-\alpha/2$ th quantile of the standard normal distribution,

$$(l_{3i}, u_{3i}) = \left(\frac{1}{\bar{X}_i} - b_i n_i^{-1/2}\left(\frac{1}{\bar{X}_i}\right)^2 S_i,\ \frac{1}{\bar{X}_i} + b_i n_i^{-1/2}\left(\frac{1}{\bar{X}_i}\right)^2 S_i\right), \tag{23}$$

where b_i is an upper $1-\alpha/2$ th quantile of the *t*-distribution with degrees of freedom $n_i - 1$.

Therefore, the $100(1-\alpha)\%$ two-sided confidence intervals for the common inverse mean θ based on adjusted MOVER approach are

$$CI_{AM1} = (L_{AM1}, U_{AM1})$$

$$= \left(\hat{\theta} - z_{1-\alpha/2}\sqrt{1\Big/\sum_{i=1}^{k}\frac{z_{\alpha/2}^2}{(\hat{\theta}_i - l_{1i})^2}},\ \hat{\theta} + z_{1-\alpha/2}\sqrt{1\Big/\sum_{i=1}^{k}\frac{z_{\alpha/2}^2}{(u_{1i} - \hat{\theta}_i)^2}} \right), \tag{24}$$

$$CI_{AM2} = (L_{AM2}, U_{AM2})$$

$$= \left(\hat{\theta} - z_{1-\alpha/2}\sqrt{1\Big/\sum_{i=1}^{k}\frac{z_{\alpha/2}^2}{(\hat{\theta}_i - l_{2i})^2}},\ \hat{\theta} + z_{1-\alpha/2}\sqrt{1\Big/\sum_{i=1}^{k}\frac{z_{\alpha/2}^2}{(u_{2i} - \hat{\theta}_i)^2}} \right) \tag{25}$$

and

$$CI_{AM3} = (L_{AM3}, U_{AM3})$$

$$= \left(\hat{\theta} - z_{1-\alpha/2}\sqrt{1\Big/\sum_{i=1}^{k}\frac{z_{\alpha/2}^2}{(\hat{\theta}_i - l_{3i})^2}},\ \hat{\theta} + z_{1-\alpha/2}\sqrt{1\Big/\sum_{i=1}^{k}\frac{z_{\alpha/2}^2}{(u_{3i} - \hat{\theta}_i)^2}} \right). \tag{26}$$

3. Simulation Studies

In this section, simulation studies are performed to evaluate the coverage probabilities and the average lengths of each confidence interval via Monte Carlo simulation. The confidence interval is satisfactory when the coverage probability is greater than or close to the nominal confidence level $1 - \alpha$ and the shortest average length.

The following algorithm is used to estimate the coverage probability and average length:

Algorithm 2

For a given $(n_1, n_2, \ldots, n_k), (\mu_1, \mu_2, \ldots, \mu_k), (\sigma_1, \sigma_2, \ldots, \sigma_k)$ and θ:

For $h = 1$ to M:

Generate x_{ij} from $N(\mu_i, \sigma_i^2)$; $i = 1, 2, \ldots, k,\ j = 1, 2, \ldots, n_i$.

Compute $\bar{x}_i$ and s_i^2.

Use Algorithm 1 to construct generalized confidence interval $(L_{GCI(h)}, U_{GCI(h)})$.

Use equation (10) to construct large sample confidence interval $(L_{LS(h)}, U_{LS(h)})$.

Use equation (24) to construct adjusted MOVER confidence interval $(L_{AM1(h)}, U_{AM1(h)})$.

Use equation (25) to construct adjusted MOVER confidence interval $(L_{AM2(h)}, U_{AM2(h)})$.

Use equation (26) to construct adjusted MOVER confidence interval $(L_{AM3(h)}, U_{AM3(h)})$.

If $(L_{(h)} \le \theta \le U_{(h)})$, set $p_{(h)} = 1$; else set $p_{(h)} = 0$.

Compute $U_{(h)} - L_{(h)}$

(end *h* loop)

Compute mean of $p_{(h)}$ defined by the coverage probability.

Compute mean of $U_{(h)} - L_{(h)}$ defined by the average length.

In this simulation, there are five confidence intervals, i.e., the generalized confidence interval is defined as CI_{GCI}, the large sample confidence interval is defined as CI_{LS}, and three adjusted MOVER confidence intervals are defined as CI_{AM1}, CI_{AM2} and CI_{AM3}. Each confidence interval is evaluated at the nominal confidence level $1 - \alpha = 0.95$. The number of populations k is set to be 2 and 6. The sample sizes within each population $n_1 = n_2 = \cdots = n_k = n = 10, 20, 50, 100$ and 200. The population mean of normal data within each population $\mu_1 = \mu_2 = \cdots = \mu_k = \mu = 1$, and the population standard deviation $\sigma_1 = \sigma_2 = \cdots = \sigma_{k/2} = 0.10$ and $\sigma_{(k/2)+1} = \sigma_{(k/2)+2} = \cdots = \sigma_k = 0.01, 0.03, 0.05, 0.07, 0.09, 0.10, 0.30, 0.50$ and 0.70. For each parameter and sample size setting, 5000 random samples are generated. For the generalized confidence interval approach, for each of the 5000 random samples, 2500 R_θ 's are obtained by Algorithm 1.

For 2 sample cases, the data are generated from normal distribution with the population mean $\mu_1 = \mu_2 = 1$ and the population standard deviation $\sigma_1 = 0.10$, $\sigma_2 = 0.01, 0.03, 0.05, 0.07, 0.09, 0.10, 0.30, 0.50$ and 0.70. The coverage probabilities and average lengths are presented in Tables 1 and 2. The results show that the coverage probabilities of generalized confidence interval are greater than the nominal confidence level of 0.95 for all cases. The coverage probabilities of large sample confidence interval and adjusted MOVER confidence interval are close to nominal confidence level of 0.95 when sample size is large, i.e., $n \geq 50$. The average lengths of large sample confidence interval and adjusted MOVER confidence interval are a bit shorter than that of generalized confidence interval.

For 6 sample cases, the data are generated from normal distribution with the population mean $\mu_1 = \mu_2 = \cdots = \mu_6 = 1$ and the population standard deviation $\sigma_1 = \sigma_2 = \sigma_3 = 0.10$, $\sigma_4 = \sigma_5 = \sigma_6 =$ 0.01, 0.03, 0.05, 0.07, 0.09, 0.10, 0.30, 0.50 and 0.70. The coverage probabilities and average lengths are presented in Tables 3 and 4. The coverage probabilities of generalized confidence interval close to the nominal confidence level of 0.95 in all cases for every sample.

Overall, the generalized confidence intervals provide the best coverage probabilities for all cases. The large sample confidence intervals and adjusted MOVER confidence intervals have coverage probabilities close to nominal confidence level of 0.95 when the sample size is large. The average lengths of large sample confidence intervals and adjusted MOVER confidence intervals are a bit shorter than that of generalized confidence intervals.

Table 1. The coverage probabilities of 95% two-sided confidence intervals for the common inverse mean of normal distributions: 2 sample cases

n	σ_1	σ_2	CI_{GCI}	CI_{LS}	CI_{AM1}	CI_{AM2}	CI_{AM3}
10	0.10	0.01	0.9594	0.9224	0.9514	0.9224	0.9276
		0.03	0.9604	0.9112	0.9458	0.9112	0.9198
		0.05	0.9614	0.9048	0.9404	0.9048	0.9100
		0.07	0.9648	0.9000	0.9402	0.9000	0.9086
		0.09	0.9646	0.8890	0.9366	0.8890	0.8988
		0.10	0.9670	0.9006	0.9396	0.9006	0.9092
		0.30	0.9622	0.9020	0.9396	0.9020	0.9096
		0.50	0.9642	0.9068	0.9490	0.9068	0.9130
		0.70	0.9602	0.9126	0.9494	0.9126	0.9184
20	0.10	0.01	0.9542	0.9342	0.9490	0.9342	0.9384
		0.03	0.9500	0.9272	0.9422	0.9272	0.9334
		0.05	0.9572	0.9262	0.9420	0.9262	0.9330
		0.07	0.9590	0.9308	0.9448	0.9308	0.9366
		0.09	0.9604	0.9324	0.9464	0.9324	0.9376
		0.10	0.9594	0.9268	0.9450	0.9268	0.9334
		0.30	0.9572	0.9334	0.9490	0.9334	0.9384
		0.50	0.9580	0.9294	0.9490	0.9294	0.9364
		0.70	0.9570	0.9324	0.9492	0.9326	0.9396
50	0.10	0.01	0.9516	0.9432	0.9506	0.9432	0.9504
		0.03	0.9536	0.9462	0.9528	0.9462	0.9520
		0.05	0.9552	0.9446	0.9506	0.9446	0.9498
		0.07	0.9532	0.9402	0.9462	0.9402	0.9458
		0.09	0.9584	0.9464	0.9528	0.9464	0.9520
		0.10	0.9504	0.9402	0.9468	0.9402	0.9442
		0.30	0.9484	0.9386	0.9466	0.9386	0.9456
		0.50	0.9482	0.9402	0.9452	0.9402	0.9448
		0.70	0.9546	0.9470	0.9502	0.9470	0.9510

100	0.10	0.01	0.9526	0.9500	0.9528	0.9500	0.9552
		0.03	0.9490	0.9428	0.9454	0.9428	0.9490
		0.05	0.9518	0.9466	0.9496	0.9468	0.9520
		0.07	0.9576	0.9512	0.9542	0.9512	0.9582
		0.09	0.9498	0.9440	0.9476	0.9440	0.9500
		0.10	0.9538	0.9464	0.9496	0.9464	0.9528
		0.30	0.9544	0.9486	0.9510	0.9486	0.9540
		0.50	0.9498	0.9452	0.9504	0.9452	0.9508
		0.70	0.9482	0.9436	0.9478	0.9436	0.9494
200	0.10	0.01	0.9498	0.9472	0.9488	0.9472	0.9536
		0.03	0.9466	0.9458	0.9470	0.9460	0.9514
		0.05	0.9542	0.9520	0.9542	0.9520	0.9578
		0.07	0.9502	0.9496	0.9490	0.9496	0.9532
		0.09	0.9540	0.9508	0.9528	0.9508	0.9582
		0.10	0.9488	0.9458	0.9482	0.9458	0.9528
		0.30	0.9520	0.9496	0.9506	0.9496	0.9552
		0.50	0.9506	0.9498	0.9500	0.9498	0.9558
		0.70	0.9512	0.9478	0.9506	0.9478	0.9540

Table 2. The average lengths of 95% two-sided confidence intervals for the common inverse mean of normal distributions: 2 sample cases

n	σ_1	σ_2	CI_{GCI}	CI_{LS}	CI_{AM1}	CI_{AM2}	CI_{AM3}
10	0.10	0.01	0.0148	0.0120	0.0139	0.0120	0.0123
		0.03	0.0439	0.0342	0.0395	0.0342	0.0351
		0.05	0.0679	0.0522	0.0603	0.0522	0.0535
		0.07	0.0871	0.0666	0.0771	0.0666	0.0683
		0.09	0.1011	0.0772	0.0895	0.0772	0.0792
		0.10	0.1072	0.0820	0.0952	0.0821	0.0841
		0.30	0.1464	0.1117	0.1299	0.1117	0.1146
		0.50	0.1551	0.1172	0.1361	0.1172	0.1202
		0.70	0.1597	0.1186	0.1377	0.1186	0.1216
20	0.10	0.01	0.0094	0.0086	0.0092	0.0086	0.0088
		0.03	0.0277	0.0248	0.0264	0.0248	0.0254
		0.05	0.0433	0.0382	0.0409	0.0382	0.0392
		0.07	0.0555	0.0487	0.0521	0.0487	0.0499
		0.09	0.0647	0.0567	0.0606	0.0567	0.0581
		0.10	0.0685	0.0600	0.0642	0.0600	0.0615
		0.30	0.0926	0.0817	0.0875	0.0817	0.0838

		0.50	0.0956	0.0846	0.0905	0.0846	0.0867
		0.70	0.0968	0.0854	0.0914	0.0854	0.0875
50	0.10	0.01	0.0057	0.0055	0.0056	0.0055	0.0056
		0.03	0.0165	0.0158	0.0162	0.0158	0.0162
		0.05	0.0258	0.0245	0.0252	0.0245	0.0252
		0.07	0.0331	0.0314	0.0322	0.0314	0.0322
		0.09	0.0386	0.0366	0.0375	0.0366	0.0375
		0.10	0.0408	0.0387	0.0397	0.0387	0.0397
		0.30	0.0546	0.0521	0.0535	0.0521	0.0535
		0.50	0.0563	0.0539	0.0553	0.0539	0.0553
		0.70	0.0571	0.0546	0.0560	0.0546	0.0560
100	0.10	0.01	0.0040	0.0039	0.0039	0.0039	0.0040
		0.03	0.0115	0.0112	0.0114	0.0112	0.0115
		0.05	0.0179	0.0174	0.0177	0.0174	0.0179
		0.07	0.0229	0.0223	0.0226	0.0223	0.0229
		0.09	0.0268	0.0260	0.0264	0.0260	0.0267
		0.10	0.0283	0.0275	0.0279	0.0275	0.0282
		0.30	0.0379	0.0370	0.0375	0.0370	0.0380
		0.50	0.0392	0.0384	0.0388	0.0384	0.0393
		0.70	0.0395	0.0387	0.0392	0.0387	0.0397
200	0.10	0.01	0.0028	0.0028	0.0028	0.0028	0.0028
		0.03	0.0080	0.0079	0.0080	0.0079	0.0082
		0.05	0.0125	0.0124	0.0124	0.0124	0.0127
		0.07	0.0161	0.0158	0.0159	0.0158	0.0162
		0.09	0.0187	0.0185	0.0186	0.0185	0.0189
		0.10	0.0198	0.0196	0.0197	0.0196	0.0201
		0.30	0.0266	0.0263	0.0264	0.0263	0.0269
		0.50	0.0274	0.0271	0.0273	0.0271	0.0278
		0.70	0.0277	0.0274	0.0276	0.0274	0.0281

Table 3. The coverage probabilities of 95% two-sided confidence intervals for the common inverse mean of normal distributions: 6 sample cases

n	σ_1	σ_4	CI_{GCI}	CI_{LS}	CI_{AM1}	CI_{AM2}	CI_{AM3}
10	0.10	0.01	0.9686	0.8872	0.9324	0.8872	0.8946
		0.03	0.9658	0.8928	0.9292	0.8928	0.8996
		0.05	0.9664	0.8782	0.9248	0.8782	0.8878
		0.07	0.9630	0.8850	0.9270	0.8850	0.8926
		0.09	0.9616	0.8744	0.9234	0.8744	0.8812

		0.10	0.9638	0.8778	0.9290	0.8778	0.8864
		0.30	0.9506	0.8776	0.9274	0.8776	0.8866
		0.50	0.9484	0.8788	0.9266	0.8788	0.8858
		0.70	0.9478	0.8706	0.9260	0.8708	0.8804
20	0.10	0.01	0.9572	0.9164	0.9368	0.9164	0.9242
		0.03	0.9582	0.9222	0.9386	0.9222	0.9302
		0.05	0.9602	0.9284	0.9454	0.9284	0.9350
		0.07	0.9576	0.9186	0.9376	0.9186	0.9258
		0.09	0.9554	0.9158	0.9364	0.9158	0.9228
		0.10	0.9558	0.9160	0.9360	0.9160	0.9230
		0.30	0.9530	0.9146	0.9380	0.9146	0.9224
		0.50	0.9506	0.9214	0.9404	0.9214	0.9276
		0.70	0.9518	0.9206	0.9400	0.9206	0.9266
50	0.10	0.01	0.9532	0.9376	0.9462	0.9376	0.9466
		0.03	0.9548	0.9434	0.9486	0.9434	0.9486
		0.05	0.9514	0.9364	0.9420	0.9364	0.9424
		0.07	0.9520	0.9386	0.9464	0.9386	0.9448
		0.09	0.9576	0.9446	0.9522	0.9446	0.9504
		0.10	0.9486	0.9342	0.9428	0.9342	0.9386
		0.30	0.9496	0.9394	0.9450	0.9394	0.9438
		0.50	0.9506	0.9396	0.9474	0.9396	0.9450
		0.70	0.9458	0.9348	0.9434	0.9348	0.9426
100	0.10	0.01	0.9508	0.9448	0.9472	0.9448	0.9502
		0.03	0.9536	0.9462	0.9490	0.9462	0.9524
		0.05	0.9514	0.9416	0.9464	0.9416	0.9492
		0.07	0.9512	0.9458	0.9478	0.9458	0.9516
		0.09	0.9540	0.9480	0.9510	0.9480	0.9536
		0.10	0.9576	0.9506	0.9534	0.9506	0.9552
		0.30	0.9512	0.9462	0.9492	0.9462	0.9510
		0.50	0.9498	0.9456	0.9478	0.9456	0.9500
		0.70	0.9488	0.9444	0.9468	0.9444	0.9482
200	0.10	0.01	0.9520	0.9470	0.9492	0.9470	0.9546
		0.03	0.9506	0.9472	0.9480	0.9472	0.9524
		0.05	0.9500	0.9458	0.9480	0.9458	0.9524
		0.07	0.9510	0.9464	0.9472	0.9464	0.9530
		0.09	0.9476	0.9456	0.9474	0.9456	0.9514
		0.10	0.9502	0.9462	0.9484	0.9462	0.9526

		0.30	0.9528	0.9528	0.9558	0.9528	0.9580
		0.50	0.9518	0.9484	0.9500	0.9484	0.9546
		0.70	0.9492	0.9474	0.9486	0.9474	0.9522

Table 4. The average lengths of 95% two-sided confidence intervals for the common inverse mean of normal distributions: 6 sample cases

n	σ_1	σ_4	CI_{GCI}	CI_{LS}	CI_{AM1}	CI_{AM2}	CI_{AM3}
10	0.10	0.01	0.0088	0.0065	0.0075	0.0065	0.0067
		0.03	0.0257	0.0187	0.0216	0.0187	0.0192
		0.05	0.0401	0.0290	0.0335	0.0290	0.0298
		0.07	0.0515	0.0369	0.0427	0.0369	0.0379
		0.09	0.0602	0.0430	0.0498	0.0430	0.0441
		0.10	0.0637	0.0456	0.0528	0.0456	0.0467
		0.30	0.0871	0.0619	0.0718	0.0619	0.0635
		0.50	0.0911	0.0639	0.0741	0.0639	0.0655
		0.70	0.0937	0.0644	0.0746	0.0644	0.0660
20	0.10	0.01	0.0056	0.0048	0.0052	0.0048	0.0050
		0.03	0.0162	0.0139	0.0149	0.0139	0.0143
		0.05	0.0253	0.0216	0.0231	0.0216	0.0222
		0.07	0.0325	0.0276	0.0296	0.0276	0.0283
		0.09	0.0380	0.0322	0.0345	0.0322	0.0331
		0.10	0.0403	0.0341	0.0365	0.0341	0.0350
		0.30	0.0542	0.0460	0.0492	0.0460	0.0472
		0.50	0.0561	0.0475	0.0508	0.0475	0.0487
		0.70	0.0569	0.0480	0.0514	0.0480	0.0493
50	0.10	0.01	0.0033	0.0031	0.0032	0.0031	0.0032
		0.03	0.0096	0.0091	0.0093	0.0091	0.0093
		0.05	0.0150	0.0141	0.0144	0.0141	0.0144
		0.07	0.0192	0.0180	0.0185	0.0180	0.0185
		0.09	0.0225	0.0210	0.0216	0.0210	0.0216
		0.10	0.0238	0.0222	0.0228	0.0222	0.0228
		0.30	0.0319	0.0298	0.0306	0.0298	0.0306
		0.50	0.0330	0.0309	0.0317	0.0309	0.0317
		0.70	0.0333	0.0312	0.0320	0.0312	0.0320
100	0.10	0.01	0.0023	0.0022	0.0023	0.0022	0.0023
		0.03	0.0067	0.0065	0.0065	0.0065	0.0066

		0.05	0.0104	0.0100	0.0102	0.0100	0.0103
		0.07	0.0133	0.0129	0.0130	0.0129	0.0132
		0.09	0.0155	0.0150	0.0152	0.0150	0.0154
		0.10	0.0164	0.0159	0.0161	0.0159	0.0163
		0.30	0.0220	0.0213	0.0216	0.0213	0.0219
		0.50	0.0227	0.0220	0.0223	0.0220	0.0226
		0.70	0.0230	0.0222	0.0225	0.0222	0.0228
200	0.10	0.01	0.0016	0.0016	0.0016	0.0016	0.0016
		0.03	0.0046	0.0046	0.0046	0.0046	0.0047
		0.05	0.0072	0.0071	0.0072	0.0071	0.0073
		0.07	0.0093	0.0091	0.0092	0.0091	0.0094
		0.09	0.0108	0.0107	0.0107	0.0107	0.0109
		0.10	0.0115	0.0113	0.0113	0.0113	0.0115
		0.30	0.0154	0.0151	0.0152	0.0151	0.0155
		0.50	0.0159	0.0156	0.0157	0.0156	0.0160
		0.70	0.0160	0.0158	0.0159	0.0158	0.0162

4. An Empirical Application

An example, given in Walpole et al. [16], was exhibited to illustrate the generalized confidence interval approach, the large sample approach and the adjusted MOVER approach. Four different levels of financial leverages given in Table 5 were used to estimate the inverse mean in rates of return on equity. The Shapiro-Wilk normality tests indicated that the four data sets come from normal populations with *p*-values 0.4770, 0.7172, 0.6736 and 0.4477. The sample means (sample variances) are 4.3833 (4.8257), 5.1000 (3.8840), 8.4167 (5.9937), and 8.3333 (5.4707) for control level, low level, medium level, and high level, respectively. The sample inverse means are 0.2281, 0.1961, 0.1188, and 0.1200 for control level, low level, medium level, and high level, respectively. The 95% two-sided confidence intervals for the common inverse mean were evaluated. The confidence interval based on the generalized confidence interval approach, CI_{GCI}, was (0.0948, 0.1631) with the length of interval 0.0683. The confidence interval based on the large sample approach, CI_{LS}, was (0.1124, 0.1485) with the length of interval 0.0361. Finally, the confidence intervals based on the adjusted

MOVER approach, CI_{AM1} was (0.1094, 0.1619) with the length of interval 0.0525, CI_{AM2} was (0.1124, 0.1485) with the length of interval 0.0361, and CI_{AM3} was (0.1119, 0.1489) with the length of interval 0.0370. These results support the simulation results in the previous section.

Table 5. Rates of return on equity for 24 randomly selected firms

Financial leverage			
Control	Low	Medium	High
2.1	6.2	9.6	10.3
5.6	4.0	8.0	6.9
3.0	8.4	5.5	7.8
7.8	2.8	12.6	5.8
5.2	4.2	7.0	7.2
2.6	5.0	7.8	12.0

5. Discussion and Conclusions

The study was conducted to investigate the performance of confidence intervals based on the generalized confidence interval approach (CI_{GCI}), the large sample approach (CI_{LS}), and adjusted MOVER approach $(CI_{AM1}, CI_{AM2}, CI_{AM3})$. The simulation studies showed that the generalized confidence intervals provide the best coverage probabilities for all cases. The large sample confidence intervals and adjusted MOVER confidence intervals have coverage probabilities close to nominal confidence level of 0.95 when the sample size is large, i.e., $n \geq 100$. All confidence intervals perform similarly for large sample size in terms of maintaining the coverage probability and the length of all confidence intervals by adjusted MOVER approach is slightly narrower than that of the generalized confidence interval approach. Hence, for sample sizes, i.e., $n = 10$ and 20, we chose the generalized confidence interval approach for the confidence interval for the common inverse mean based on several independent normal samples. However, the adjusted MOVER approach should be chosen to estimate the common inverse mean of several normal populations for large sample sizes because it is based on the formulas (24)-(26) and is more easy to use than that of the generalized confidence interval approach which is based on a computational approach.

Acknowledgments

The first author gratefully acknowledges the financial support from Science Achievement Scholarship of Thailand. The second author is grateful to grant number KMUTNB-60-ART-093 from King Mongkut's University of Technology North Bangkok.

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