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A NOTE ON SEMICENTRAL IDEMPOTENTS AND SEMICOMMUTATIVE NEAR-RINGS

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Abstract

In this paper, we investigate some properties of semicentral idempotents related to the left regularity, right regularity and zero-symmetric reducibility. We obtain that every zero-symmetric near-ring with reversibility is left semicentral idempotent, and that a right regular near-ring is semicommutative.

1. Introduction

Throughout this paper, our near-ring R is an associative left near-ring.

We say that a near-ring R is *reduced* if R has no nonzero nilpotent elements, that is, for each a in R, $a^n = 0$, for some positive integer n implies that $a = 0$. McCoy [4] proved that R is reduced if and only if for each a in R, $a^2 = 0$ implies $a = 0$. A near-ring R is called *reversible* if for any $a, b \in R$, $ab = 0$ implies $ba = 0$.

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On the other hand, R has the *insertion of factors property* (briefly, IFP) provided that for all a, b, x in R with $ab = 0$ implies $axb = 0$, [5].

For other notations and basic results, we refer to Pilz [5].

2. Properties of Right Regular Near-rings

For a near-ring R, an *idempotent* of R is an element $e \in R$ such that $e^2 = e$.

A near-ring R is called (*von Neumann*) *regular* if for any element $a \in R$, there exists an element x in R such that $a = axa$. Such an element a is called *regular*.

A near-ring R is called *right regular* if, for each $a \in R$, there exists $x \in R$ such that $a = a^2x$. Left regularity is defined in a similar way.

The following concepts are introduced in [3].

An idempotent $e \in R$ is *left semicentral* in R if $Re = eRe$. An idempotent $e \in R$ is *right semicentral* in R if $eR = eRe$ and an idempotent $e \in R$ is *central* in R if $er = re$ for all $r \in R$.

An element $a \in R$ is called *nilpotent* if $a^n = 0$ for some integer n.

A near-ring in which every idempotent is left semicentral (resp. right semicentral) is called *left semicentral* (resp. *right semicentral*). Also, a near-ring in which every idempotent is central is called *central*. See [1].

Now, we define a notion which is a generalization of commutativity.

Let R be a near-ring. If for $a \in R$, there exists an element x in R such that $ax = xa$, then R is called *semicommutative*. Such an element a is called *semicommutative*.

There are lots of examples of semicommutative near-rings as can be obtained from Proposition 12.

First, we provide some basic properties of right or left regularity in near-rings, and check some errata in [1] and [2].

Lemma 1 [1]**.** *Let R be a right or left regular near-ring. If for any a, b* $\in R$ *with* $ab = 0$, *then* $(ba)^n = 0a$, *for all positive integer n. In particular,* $ba = 0a$. *Thus if R is zero-symmetric, then R is reversible.*

Lemma 2 [1]**.** *Let R be a right or left regular near-ring. If for any* $a \in R$ *with* $a^2 = 0a$, *then* $a = 0a$. *Thus if R is zero-symmetric, then R is reduced.*

Consequently, we obtain the following:

Proposition 3 [2]**.** *Let R be a zero-symmetric right or left regular near-ring. Then R is reversible and reduced.*

Proposition 4 [2]**.** *Let R be a zero-symmetric and reduced near-ring. Then R is reversible.*

Proof. Suppose that a, b in R such that $ab = 0$. Then, since R is zero-symmetric, we have $(ba)^2 = baba = b0a = b0 = 0$. Reducibility of R implies that $ba = 0$. □

Now, we give easy characterizations of left semicentral and right semicentral conditions in a near-ring R.

Proposition 5 [1]**.** *For an idempotent* $e \in R$, *the following conditions hold*:

(1) *e is left semicentral,* $\Leftrightarrow ae = eae$, *for all* $a \in R$.

(2) *e is right semicentral,* $\Leftrightarrow ea = eae$, *for all* $a \in R$.

The following statements are alternative versions of Proposition 2.6 in [1], using the conditions of Proposition 5.

Proposition 6 [2]**.** *Let R be a zero-symmetric right or left regular near-ring. Then R is left semicentral idempotent.*

Proof. Let e be an idempotent element in R, and let $a \in R$. Then

$e(ae - eae) = 0,$ $ae(ae - eae) = 0$ and $eae(ae - eae) = 0.$ By Proposition 3, $(ae - eae)ae = 0$ and $(ae - eae)eae = 0.$ Hence $(ae - eae)^2 = (ae - eae)ae$ $-(ae - exe)eae = 0 - 0 = 0.$ Since R is reduced, by Proposition 3, $ae - eae$ $= 0,$ that is, $ae = eae.$ Consequently, R is left semicentral. □

From Lemmas 1, 2 and Proposition 4, we get the following:

Corollary 7. *Let R be a zero-symmetric reduced near-ring. Then R is left semicentral idempotent.*

Proposition 8 [2]**.** *Let R be a reversible and reduced near-ring. Then R is a left semicentral idempotent near-ring.*

Example 9. Let $R = Z_6 = \{0, 1, 2, 3, 4, 5\}.$ This is a near-ring with the following multiplication table [5, p. 410]:

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

This near-ring R is a zero-symmetric near-ring with identity 1. Moreover, R is reduced, because R is right regular, for example, $2 = 2^2 \cdot 2, 3 = 3^2 \cdot 3,$ etc. Thus, from Proposition 6 or Corollary 7, R is a left semi-central idempotent near-ring.

Lemma 10. *Let R be a right or left regular near-ring. If for any* $a, b \in R$ *with* $ab = 0$ *and* $b^2 = 0b,$ *then* $b = 0.$ *Moreover, R has the IFP.*

Proof. In case R is a right regular near-ring, let $a, b \in R$ with $ab = 0$ and $b^2 = 0b.$ Then $b = b^2x,$ for some $x \in R.$ From this equality, $b = b^2x$ $= 0bx = a0bx = ab = 0,$ by assumption.

Analogously, we can prove for the case of left regular near-ring.

Next, assume the given condition. To prove the IFP, first we must prove that $ba = 0$. Indeed, $(ba)^2 = baba = 0a$, which implies $ba = 0$. From this new condition, $(axb)^2 = axbaxb = 0xb$. Our assumption implies $axb = 0$. Hence, R has the IFP. □

Also, using the conditions of Proposition 5, we prove the following:

Proposition 11. *Every right regular near-ring R is regular and left semicentral idempotent.*

Proof. Let R be a right regular near-ring. Then clearly, R is regular. Now, let $a \in R$ and $e^2 = e \in R$. This shows that $e(ae - eae) = 0$, $ae(ae - eae) = 0$ and $eae(ae - eae) = 0$. By Lemma 1, $(ae - eae)ae = 0ae$ and $(ae - eae)eae = 0eae.$ It follows that $(ae - eae)^2 = (ae - eae)ae - (ae - exe)eae = 0ae - 0eae = 0(ae - eae)$. From Lemma 10, $ae = eae.$ Consequently, R is a left semicentral near-ring. □

Proposition 12. *Every right regular near-ring R is semicommutative.*

Proof. Suppose that R is a right regular near-ring, and let $a \in R$. Then there exists $x \in R$ such that $a = a^2x$. Also, we have that $a(ax - xa) = aax - axa = a - a = 0$, by Proposition 11. From Lemma 1 we have that $(ax - xa)a = 0a$ and also, similarly we have $(ax - xa)xa = 0xa$. Hence $(ax - xa)^2 = (ax - xa)ax - (ax - xa)xa = 0ax - 0xa = 0(ax - xa)$. From Lemma 10, we see that $ax - xa = 0$, that is, $ax = xa$. Hence R is a semicommutative near-ring. □

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