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TOTAL DOMINATION POLYNOMIALS OF SOME SPLITTING GRAPHS

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Abstract

A hypergraph is an ordered pair $H = (V, E)$, where V is a finite nonempty set called vertices and E is a collection of subsets of V, called hyperedges or simply edges. A subset T of vertices in a hypergraph H is called a vertex cover if T has a nonempty intersection with every edge of H. The vertex covering number $\tau(H)$ of H is the minimum size of a vertex cover in H. Let $\mathcal{C}(H, i)$ be the family of vertex covering sets of H with cardinality i and let $C(H, i)$ be the cardinality of $\mathcal{C}(H, i)$. The polynomial $\sum_{i=\tau(H)}^{|V(H)|} C(H, i)x^i$ is

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defined as vertex cover polynomial of H. For a graph $G = (V, E)$, H_G denotes the hypergraph with vertex set V and edge set $\{N_G(x) | x \in V\}$. In this paper, we prove that the total domination polynomial of a connected graph G is the vertex cover polynomial of H_G. Using this result, we determine total domination polynomials of splitting graphs of order k of paths and cycles. Moreover, we introduce the terminology of iterated splitting graph $S^i(G)$ of a graph G and determine its total domination polynomials.

1. Introduction

All graphs considered in this paper are simple and connected unless otherwise stated. Notations and definitions not given here can be found in [1, 4, 8]. A *graph* is an ordered pair $G = (V(G), E(G))$, where $V(G)$ is a finite nonempty set and $E(G)$ is a collection of 2-point subsets of V. The sets $V(G)$ and $E(G)$ are the vertex set and edge set of G, respectively. The *open neighbourhood* of a vertex $v \in V(G)$ is $N_G(v) = \{u \in V \mid uv \in E(G)\}$. If the graph G is clear from the context, then we write $N(v)$ rather than $N_G(v)$. A *total dominating set*, abbreviated TD-set, of a graph $G = (V, E)$ with no isolated vertex is a set S of vertices of G such that every vertex is adjacent to a vertex in S. If no proper subset of S is a TD-set of G, then S is a *minimal TD-set* of G. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set of G. A TD-set of G of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$*-set*. Let $\mathcal{D}_t(G, i)$ be the family of total dominating sets of G with cardinality i and let $d_t(G, i) = |\mathcal{D}_t(G, i)|$. The polynomial $\mathcal{D}_t(G, x) = \sum_{i=\gamma_t(G)}^{|V(G)|} d_t(G, i)x^i$ is defined as *total domination polynomial* of G. A *hypergraph* $H = (V, E)$ is a finite nonempty set $V = V(H)$ of elements called *vertices*, together with a finite multiset $E = E(H)$ of subsets of V, called *hyperedges* or simply *edges*. The *order* and *size* of H are $|V|$

and $|E|$, respectively. A *k-edge* in *H* is an edge of size *k*. A subset *T* of vertices in a hypergraph *H* is a *transversal* (also called *vertex cover*) if *T* has a nonempty intersection with every edge of *H*. The *transversal number* $\tau(H)$ of *H* is the minimum size of a transversal in *H*. For further information on hypergraphs, refer [3]. Let $\mathcal{C}(H, i)$ be the family of vertex covering sets of *H* with cardinality *i* and let $C(H, i) = |\mathcal{C}(H, i)|$. The polynomial $\mathcal{C}(H, x) = \sum_{i=\tau(H)}^{|V(H)|} C(H, i)x^i$ is defined as *vertex cover polynomial* of *H*. For a graph $G = (V, E)$, the $ONH(G)$ or H_G is the *open neighbourhood hypergraph* of *G*; $H_G = (V, C)$ is the hypergraph with vertex set $V(H_G) = V$ and with edge set $E(H_G) = C = \{N_G(x) | x \in V\}$ consisting of the open neighbourhoods of vertices of *V* in *G*.

Theorem 1.1 [7]**.** *The ONH of a connected bipartite graph consists of two components, while the ONH of a connected graph that is not bipartite is connected*.

Theorem 1.2 [8]**.** *If G is a graph with no isolated vertex and* H_G *is the ONH of G, then* $\gamma_t(G) = \tau(H_G)$.

Theorem 1.3 [9]**.** $D_t(C_n, x) = x[D_t(C_{n-1}, x) + D_t(C_{n-2}, x)]$.

The corona $G \circ K_1$ of a graph *G* is the graph obtained from *G* by adding a pendant edge to each vertex of *G*. The splitting graph of *G* is defined as, for each vertex *v* of *G*, take a new vertex v' and join v' to all vertices of *G* adjacent to *v*. The graph $spl(G)$ thus obtained is called the *splitting graph* of *G*. The splitting graph of order *k* of a graph *G*, denoted by $spl^k(G)$ is defined as for each vertex *v* of *G*, take *k* new vertices $v_1, v_2, ..., v_k$ and join each of these vertices to all vertices of *G* adjacent to *v*. The iterated splitting graph $S^i(G)$ of a graph *G* is defined as $S^i(G) = S(S^{i-1}(G))$, where $S^1(G)$ denotes the splitting graph $spl(G)$ of *G*.

2. Main Results

Theorem 2.1. *The total domination polynomial of a connected bipartite graph G is the product of the vertex cover polynomials of the two components of* H_G, *while the total domination polynomial of a connected graph that is not bipartite is the vertex cover polynomial of* H_G.

Proof. The proof follows immediately from the definitions of total dominating set of G and vertex cover polynomial of H_G. □

Using Theorem 2.1, we can easily prove Theorems 2.2 and 2.3 due to Chaluvaraju and Chaitra [2].

Theorem 2.2. $D_t(K_{m,n}, x) = [(1+x)^m - 1][(1+x)^n - 1]$.

Proof. Let (X, Y) be the bipartition and H_G be the open neighbourhood hypergraph of $K_{m,n}$. Then $E(H_G) = \{X, Y\}$ and the vertex cover polynomial of H_G is

$$\left[\binom{m}{1}x + \binom{m}{2}x^2 + \cdots + \binom{m}{m}x^m\right]\left[\binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n\right].$$

Thus the proof follows by Theorem 2.1. □

Theorem 2.3. *Let G be a connected graph with n vertices. Then*

$$D_t(G \circ K_1, x) = x^n(1+x)^n.$$

Proof. Let $V(G) = \{1, 2, 3, ..., n\}$ and $a_1, a_2, a_3, ..., a_n$ be the new vertices of $G \circ K_1$ such that $N(a_i) = \{i\}$ for $i = 1, 2, 3, ..., n$. So if S is a total dominating set of $G \circ K_1$, then $\{1, 2, 3, ..., n\} \subseteq S$. Therefore,

$$D_t(G \circ K_1, x) = x^n + \binom{n}{1}x^{n+1} + \binom{n}{2}x^{n+2} + \cdots + \binom{n}{n}x^{n+n} = x^n(1+x)^n.$$

This completes the proof. □

Theorem 2.4. *If* $B_{m,n}$ *is the bistar graph, then* $D_t(B_{m,n}, x) = x^2(1+x)^{m+n}$.

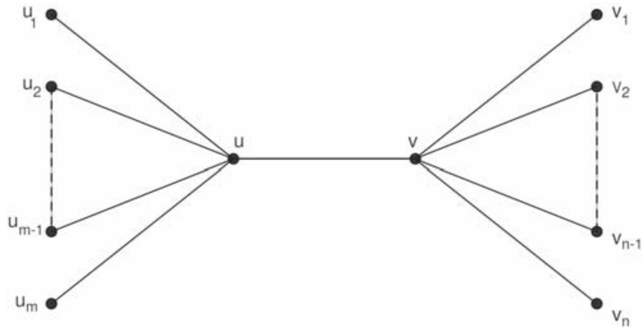

Figure 1. The graph $B_{m,n}$.

Proof. Let us label the vertices of $B_{m,n}$ as shown in Figure 1. Since $N(u) = \{v, u_1, u_2, ..., u_m\}$, $N(v) = \{u, v_1, v_2, ..., v_n\}$, $N(u_i) = \{u\}$ and $N(v_i) = \{v\}$, a set S is a TD-set of $B_{m,n}$ if and only if $\{u, v\} \subseteq S$. So the TD-polynomial is

$$D_t(B_{m,n}, x) = x^2 + \binom{m+n}{1}x^3 + \binom{m+n}{2}x^4 + \cdots + \binom{m+n}{m+n}x^{m+n}$$

$$= x^2(1+x)^{m+n}.$$

This completes the proof. □

Theorem 2.5. *Let* G *and* H *be graphs of order* m *and* n, *respectively. Then* $D_t(G \vee H, x) = [(1+x)^m - 1][(1+x)^n - 1] + D_t(G, x) + D_t(H, x)$.

Proof. If $S \subseteq V(G) \cup V(H)$, such that $S \cap V(G) \neq \varnothing$ and $S \cap V(H) \neq \varnothing$, then S is a TD-set of $G \vee H$. Moreover, if S is a TD-set of G or H, then S is a TD-set of $G \vee H$. Therefore, $D_t(G \vee H, x) = [(1+x)^m - 1][(1+x)^n - 1] + D_t(G, x) + D_t(H, x)$. □

Theorem 2.6. $D_t(C_{2n}, x) = [\mathcal{C}(C_n, x)]^2$.

Proof. The ONH of C_{2n} consists of two components isomorphic to C_n. Therefore, the proof follows from Theorem 2.1. $\square$

Lemma 2.7. *If G is bipartite, then* $spl^k(G)$ *and* $S^k(G)$ *are bipartite.*

Proof. Let (X, Y) be the bipartition of G and X', Y' be collections of new vertices of $spl^k(G)$ corresponding to the vertices of X and Y, respectively. Then $X \cup X'$ and $Y \cup Y'$ are the partite sets of $spl^k(G)$. Similarly, we can show that $S^k(G)$ is bipartite. $\square$

Theorem 2.8. $\mathcal{C}(C_n, x) = x[\mathcal{C}(C_{n-1}, x) + \mathcal{C}(C_{n-2}, x)]$.

Proof. The proof follows from Theorem 1.3 and from the definition of vertex cover polynomial. $\square$

Theorem 2.9. *If*

$$\mathcal{C}(C_n, x) = b_s x^s + b_{s+1}x^{s+1} + b_{s+2}x^{s+2} + \cdots + b_{n-1}x^{n-1} + b_n x^n,$$

then $C(G_1, s+j) = b_s\binom{nk}{j} + b_{s+1}\binom{nk}{j-1} + b_{s+2}\binom{nk}{j-2} + \cdots + b_{s+j}$, *where* G_1 *is a component of ONH of* $spl^k(C_{2n})$ *and* $\binom{n}{r} = 0 = b_r$ *if* $r > n$.

Proof. Let $X = \{1, 3, 5, ..., 2n-1\}$ and $Y = \{2, 4, 6, ..., 2n\}$ be the bipartitions of C_{2n}. Let G_1 be the component of $ONH(spl^k(C_{2n}))$ corresponding to the partite set $X \cup X'$. For $i = 1, 2, 3, ..., k$, let v_i denote the new vertex in $spl^i(C_{2n})$, corresponding to the vertex v in C_{2n}. Then $N(1_i) = \{2\}$ and $N(2n_i) = \{2n-1\}$ and for v in $\{2, 3, 4, ..., 2n-1\}$, $N(v_i)$ $= \{v-1, v+1\}$. So if S is a vertex covering set of C_n with $V(C_n) =$ $\{2, 4, 6, ..., 2n\}$, then S is a vertex covering set of G_1. Also, $V(spl^k(C_{2n}))$

consists of $(k+1)2n$ vertices. So if $\tau(C_n) = s$ and C_n has b_{s+j} vertex covering subsets of order $s + j$, then G_1 has $b_s\binom{nk}{j} + b_{s+1}\binom{nk}{j-1} + b_{s+2}\binom{nk}{j-2} + \cdots + b_{s+j}$ vertex covering sets of order $s + j$. This completes the proof. □

Theorem 2.10. *If G_1 is a component of ONH of $spl^k(C_{2n})$, then*

$$D_t(spl^k(C_{2n}, x)) = [\mathcal{C}(G_1, x)]^2.$$

Proof. The proof follows from Theorems 2.1 and 2.9. □

Theorem 2.11. *If*

$$\mathcal{C}(C_n, x) = b_s x^s + b_{s+1}x^{s+1} + b_{s+2}x^{s+2} + \cdots + b_{n-1}x^{n-1} + b_n x^n,$$

then $C(G_1, s+j) = b_s\binom{n(2^k-1)}{j} + b_{s+1}\binom{n(2^k-1)}{j-1} + b_{s+2}\binom{n(2^k-1)}{j-2} + \cdots + b_{s+j}$, *where G_1 is a component of ONH of $S^k(C_{2n})$, where $\binom{n}{r} = 0 = b_r$ if $r > n$.*

Proof. Let $X = \{1, 3, 5, ..., 2n-1\}$ and $Y = \{2, 4, 6, ..., 2n\}$ be the bipartitions of C_{2n}. Let $a'_1, a'_3, a'_5, ..., a'_{2n-1}$ be the vertices of a component G_1 of *ONH* of $S^k(C_{2n})$ of degree 2. Let $N(a'_1) = \{2n, 2\}$, $N(a'_3) = \{2, 4\}$, $N(a'_5) = \{4, 6\}, ..., N(a'_{2n-1}) = \{2n-2, 2n\}$. If $v \in V(G_1)$, then there is a vertex a'_i such that $N(a'_i) \subseteq N(v)$. So if S is a vertex covering set of C_n with $V(C_n) = \{2, 4, 6, ..., 2n\}$, then S is a vertex covering set of G_1. Also, if $|V(G)| = n$, then $|V(S^k(G))| = 2^k n$. So if $\tau(C_n) = s$ and C_n has b_{s+j} vertex covering subsets of order $s + j$, then G_1 has $b_s\binom{n(2^k-1)}{j} +$

$b_{s+1}\binom{n(2^k-1)}{j-1} + b_{s+2}\binom{n(2^k-1)}{j-2} + \cdots + b_{s+j}$ vertex covering sets of order $s+j$. This completes the proof. □

Theorem 2.12. *If G_1 is a component of ONH of $S^k(C_{2n})$, then*

$$D_t(S^k(C_{2n}, x)) = [\mathcal{C}(G_1, x)]^2.$$

Proof. The proof follows from Theorems 2.1 and 2.11. □

Theorem 2.13. *If n is odd and $D_t(C_n, x) = b_s x^s + b_{s+1}x^{s+1} + b_{s+2}x^{s+2} + \cdots + b_{n-1}x^{n-1} + b_n x^n$, then $d_t(spl^k(C_n), s+j) = b_s\binom{nk}{j} + b_{s+1}\binom{nk}{j-1} + b_{s+2}\binom{nk}{j-2} + \cdots + b_{s+j}$ and $\binom{n}{r} = 0 = b_r$ if $r > n$.*

Proof. Since n is odd, $ONH(C_n)$ is isomorphic to C_n and $\mathcal{C}(C_n, x) = D_t(C_n, x)$. The rest of the proof is exactly similar to Theorem 2.9. □

Theorem 2.14. *If n is odd and $D_t(C_n, x) = b_s x^s + b_{s+1}x^{s+1} + b_{s+2}x^{s+2} + \cdots + b_{n-1}x^{n-1} + b_n x^n$, then*

$$d_t(S^k(C_n), s+j) = b_s\binom{n(2^k-1)}{j} + b_{s+1}\binom{n(2^k-1)}{j-1} + b_{s+2}\binom{n(2^k-1)}{j-2} + \cdots + b_{s+j}$$

and $\binom{n}{r} = 0 = b_r$ if $r > n$.

Proof. Since n is odd, $ONH(C_n)$ is isomorphic to C_n and $\mathcal{C}(C_n, x) = D_t(C_n, x)$. The rest of the proof is exactly similar to Theorem 2.11. □

Next, we determine the total domination polynomials of $spl^k(P_n)$ and $S^k(P_n)$.

Let P_n' be the graph shown in Figure 2.

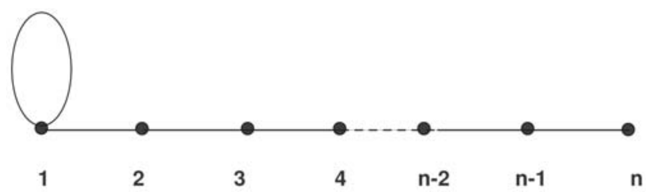

Figure 2. The graph P_n'.

Theorem 2.15. $C(P_n', i+1) = C(P_{n-1}', i) + C(P_{n-2}', i)$.

Proof. Let $\mathcal{C}(P_{n-1}', i) = A_{n-1} \bigcup A$, where $A_{n-1} = \{S \mid S \in \mathcal{C}(P_{n-1}', i)$ and $n-1 \in S\}$ and $A = \mathcal{C}(P_{n-1}', i) \backslash A_{n-1}$. Then $A \subseteq \mathcal{C}(P_{n-2}', i)$. Also, let $B = \mathcal{C}(P_{n-2}', i) \backslash A$. If $S \in A_{n-1}$, then $S \bigcup \{n\} \in \mathcal{C}(P_n', i+1)$. If $S \in A$, then $S \bigcup \{n-1\}$ and $S \bigcup \{n\} \in \mathcal{C}(P_n', i+1)$. If $S \in B$, then $S \bigcup \{n-1\} \in \mathcal{C}(P_n', i+1)$.

Conversely, let $S \in \mathcal{C}(P_n', i+1)$. Then either $n-1 \in S$ or $n \in S$ or both.

Case 1. $n-1 \in S$ and $n \notin S$. In this case, $S \backslash \{n-1\} \in A \bigcup B$.

Case 2. $n-1 \notin S$ and $n \in S$. In this case, $S \backslash \{n\} \in A$.

Case 3. $n-1 \in S$ and $n \in S$. In this case, $S \backslash \{n\} \in A_{n-1}$. Therefore,

$$C(P_n', i+1) = |A_{n-1}| + 2|A| + |B| = C(P_{n-1}', i) + C(P_{n-2}', i). \qquad \square$$

Theorem 2.16. $\mathcal{C}(P_n', x) = x[\mathcal{C}(P_{n-1}', x) + \mathcal{C}(P_{n-2}', x)]$, *with initial values* $\mathcal{C}(P_2', x) = x + x^2$, $\mathcal{C}(P_3', x) = 2x^2 + x^3$.

Proof. The proof follows immediately from Theorem 2.15. $\square$

Theorem 2.17. $D_t(P_{2n}, x) = [\mathcal{C}(P'_n, x)]^2$.

Proof. The open neighbourhood hypergraph of P_{2n} consists of two components isomorphic to P'_n. Then, by Theorem 2.1, $D_t(P_{2n}, x) = [\mathcal{C}(P'_n, x)]^2$. □

Theorem 2.18. *If* $\mathcal{C}(P'_n, x) = b_s x^s + b_{s+1}x^{s+1} + b_{s+2}x^{s+2} + \cdots + b_{n-1}x^{n-1} + b_n x^n$, *then*

$$C(G_1, s+j) = b_s\binom{nk}{j} + b_{s+1}\binom{nk}{j-1} + b_{s+2}\binom{nk}{j-2} + \cdots + b_{s+j},$$

where G_1 *is a component of ONH of* $spl^k(P_{2n})$ *and* $\binom{n}{r} = 0 = b_r$ *if* $r > n$.

Proof. For $i = 1, 2, \ldots, k$, let v_i be the new vertex corresponding to the vertex in $spl^i(P_{2n})$. Then for all i, $N_{spl^i(P_{2n})} = N_{P_{2n}}(v)$. If S is a vertex covering subset of P'_n, then S is a vertex covering subset of G_1. So, if $\mathcal{C}(P'_n, x) = b_s x^s + b_{s+1}x^{s+1} + b_{s+2}x^{s+2} + \cdots + b_{n-1}x^{n-1} + b_n x^n$, then

$$C(G_1, s+j) = b_s\binom{nk}{j} + b_{s+1}\binom{nk}{j-1} + b_{s+2}\binom{nk}{j-2} + \cdots + b_{s+j}.$$

This completes the proof. □

Theorem 2.19. *If* G_1 *is a component of ONH of* $spl^k(P_{2n})$, *then*

$$D_t(spl^k(P_{2n}, x)) = [\mathcal{C}(G_1, x)]^2.$$

Proof. The proof follows immediately from Theorems 2.1 and 2.18. □

Theorem 2.20. *If the vertex cover polynomial of* P'_n *is* $\mathcal{C}(P'_n, x) = b_s x^s + b_{s+1}x^{s+1} + b_{s+2}x^{s+2} + \cdots + b_{n-1}x^{n-1} + b_n x^n$ *and if* G_1 *is a component of* $ONH(S^k(P_{2n}))$, *then* $C(G_1, s+j) = b_s\binom{n(2^k-1)}{j} + b_{s+1}\binom{n(2^k-1)}{j-1} + b_{s+2}\binom{n(2^k-1)}{j-2} + \cdots + b_{s+j}$ *and* $\binom{n}{r} = 0 = b_r$ *if* $r > n$.

Proof. Observe that $|V(S^k(P_{2n}))| = 2^{k+1}n$, $|V(G_1)| = 2^k n$ and $|V(P'_n)| = n$. The remaining part can be proved as in Theorem 2.18. □

Corollary 2.21. *If* G_1 *is a component of* $ONH(S^k(P_{2n}))$, *then*

$$D_t(S^k(P_{2n}, x)) = [\mathcal{C}(G_1, x)]^2.$$

Proof. The proof is obvious. □

Let P''_n be the graph shown in Figure 3.

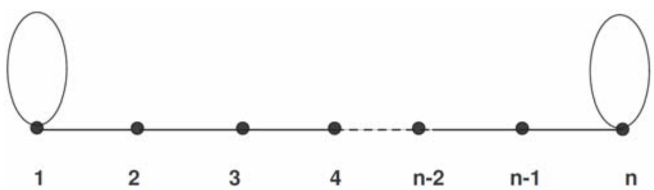

Figure 3. P''_n.

Theorem 2.22. $\mathcal{C}(P''_n, x) = x[\mathcal{C}(P''_{n-1}, x) + \mathcal{C}(P''_{n-2}, x)]$ *with initial values* $\mathcal{C}(P''_2, x) = x^2$ *and* $\mathcal{C}(P''_3, x) = x^2 + x^3$.

Proof. Let $S \in \mathcal{C}(P''_{n-2}, i)$. Then $S \cup \{n\} \in \mathcal{C}(P''_n, i+1)$. If $S \in \mathcal{C}(P''_{n-1}, i)$, then $S \cup \{n\} \in \mathcal{C}(P''_n, i+1)$. Conversely, if $S \in \mathcal{C}(P''_n, i+1)$, then either $S \in \mathcal{C}(P''_{n-1}, i)$ or $S \in \mathcal{C}(P''_{n-2}, i)$. Hence, $C(P''_n, i+1) = C(P''_{n-1}, i) + C(P''_{n-2}, i)$. Therefore, $\mathcal{C}(P''_n, x) = x[\mathcal{C}(P''_{n-1}, x) + \mathcal{C}(P''_{n-2}, x)]$. □

Observation 2.23. $\mathcal{C}(P_n + 2, x) = x[\mathcal{C}(P_{n+1}, x) + \mathcal{C}(P_n, x)]$ with initial values $\mathcal{C}(P_2, x) = 2x + x^2$ and $\mathcal{C}(P_3, x) = x + 3x^2 + x^3$.

Theorem 2.24. $D_t(P_{2n+1}, x) = [\mathcal{C}(P_n'', x)][\mathcal{C}(P_{n+1}, x)]$.

Proof. Let $X = \{1, 3, 5, ..., 2n - 1\}$ and $Y = \{2, 4, 6, ..., 2n\}$ be the partite sets of P_{2n+1}. Let G_1 and G_2 be the components of $ONH(P_{2n+1})$ corresponding to the open neighbourhoods of vertices in X and Y, respectively. Then G_1 is isomorphic to P_n'' and G_2 is isomorphic to P_{2n+1}. Therefore, the result follows from Theorem 2.1. □

Theorem 2.25. *Let* $C(P_n', x) = b_s x^s + b_{s+1}x^{s+1} + b_{s+2}x^{s+2} + \cdots + b_{n-1}x^{n-1} + b_n x^n$ *and* $C(P_{n+1}, x) = c_l x^l + c_{l+1}x^{l+1} + c_{l+2}x^{l+2} + \cdots + c_n x^n + c_{n+1}x^{n+1}$. *If* G_1 *and* G_2 *are the components of* $ONH(spl^k(P_{2n+1}))$, *then the coefficients of* x^{s+j} *in* $\mathcal{C}(G_1, x)$ *and* $\mathcal{C}(G_2, x)$ *are* $C(G_1, s + j) = b_s\binom{nk}{j} + b_{s+1}\binom{nk}{j-1} + b_{s+2}\binom{nk}{j-2} + \cdots + b_{s+j}$, $C(G_2, s + j) = c_l\binom{(n+1)k}{j} + b_{s+1}\binom{(n+1)k}{j-1} + b_{s+2}\binom{(n+1)k}{j-2} + \cdots + c_{l+j}$, *where* $\binom{n}{r} = b_r = 0$ *if* $r > n$ *and* $c_r = 0$ *if* $r > n + 1$.

Proof. For $i = 1, 2, ..., k$, let v_i be the new vertex corresponding to the vertex v in $spl^k(P_{2n+1})$ and $X = \{1_i, 3_i, ..., (2n + 1)_i\}$ and $Y = \{2_i, 4_i, ..., (2n)_i\}$ be the partite sets. Let G_1 and G_2 be the components of $ONH(spl^k(P_{2n+1}))$ corresponding to X and Y, respectively. Then as in the previous results, we can prove the result immediately. □

Corollary 2.26. *If* G_1 *and* G_2 *are the components of* $ONH(spl^k(P_{2n+1}))$, *then*

$$D_t(spl^k(P_{2n+1}, x)) = \mathcal{C}(G_1, x)\mathcal{C}(G_1, x).$$

Proof. The proof is obvious. □

Observation 2.27. Adopting the procedure in Theorem 2.25, we can easily derive the total domination polynomial of $S^k(P_{2n+1})$ also.

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