Advances and Applications in Discrete Mathematics

http://www.pphmj.com
http://dx.doi.org/10.17654/DM018030317
Volume 18, Number 3, 2017, Pages 317-330 **ISSN: 0974-1658**

ON 4-EDGE COLORING OF CUBIC GRAPHS CONTAINING "SMALL" NON-PLANAR SUBGRAPHS

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Abstract

The problem we deal with is to find the minimum size of the four color classes among all proper 4-edge colorings of a cubic graph G. The bound for the number t of times that color 4 is required given in [8] is modified for the case of cubic graphs G having subgraphs that contain all the crossing points with l connected edges common to the rest of graph. The new bound becomes much better than the previous one if the fraction of the order of subgraphs to the order of whole G is "small" or if the number of connected edges l is "small".

1. Introduction

1.1. Motivation

We deal with the following problem: "among all proper 4-edge colorings of a cubic and bridgeless graph G with colors 1, 2, 3 and 4 what is the minimum number t of times that color 4 is required?" So, we ask for an upper bound for t. This number is related to the existence (or non-existence) of bicolor paths between the pairs of crossing edges in G and therefore is related to the distance d between the crossing points of G. Such an upper bound for t is given in [8].

Received: October 6, 2016; Revised: March 2, 2017; Accepted: April 20, 2017
2010 Mathematics Subject Classification: 05C15.
Keywords and phrases: edge coloring, max-3-edge coloring.

In the present note, we demonstrate in brief the concept of “bicolor connections” (a kind of *Kempe* chains) as it is described in [8] and study the case of cubic graphs G that have “small” subgraphs which contain all the crossing points.

The new bound for t we found is better than the previous one if the fraction of the order of subgraphs to the order of whole G is “small” or if the number of connected edges l is “small”. Moreover, our result gives the possibility to apply the previous upper bound only for subgraphs of G.

1.2. Definitions and related work

1.2.1. Definitions

The graphs we consider are without loops or multiple edges. We recall some definitions and results we need.

The *chromatic index* of a graph G is the minimum number of colors that color all edges, such that adjacent edges have different colors and it is denoted by $\chi'(G)$. A graph is called 3-*regular* or *cubic* if all the vertices have degree 3.

Vizing [12] proved that each graph belongs to one of two classes: In class 1, if $\chi'(G) = \Delta(G)$ or in class 2, if $\chi'(G) = \Delta(G) + 1$, where $\Delta = \Delta(G)$ denotes the maximum degree of the graph G. Recognizing these classes is a difficult problem, even when it is limited to cubic graphs [6] or in triangle free graphs with $\Delta = 3$ [7].

So, it is reasonable to search for approximating algorithms that decide whether a graph with $\Delta = k$ is k-edge colorable for some restricted classes of graphs as for example in [3], [1] and [10].

A new definition follows:

Let G be a non-planar graph with $\Delta = 3$. We construct from G another graph H by adding to every crossing point a vertex and the corresponding four edges. We define distance d between two crossing points in G, say p and q, to be the length of the shortest path in H between vertices p and q minus the number of the intermediate vertices of degree 4 in this path.

1.2.2. Related work

Related to the *k*-edge coloring problem is the max edge *k*-coloring problem: "Given a graph *G* and a number *k*, color as many edges as possible using *k* colors". In the max edge *k*-coloring problem, we try to find the best possible ratio of the number of edges that are colored with *k*-colors to the size of the given graph. Some known approximating ratios are: $1-\left(1-\frac{1}{k}\right)^k$ in [5], $\frac{k}{k+1}$ in [2]. In [9] and for $k = 3$, a ratio $\frac{7}{9}$ is achieved and in [11], is shown that a triangle free graph with maximum degree 3 has 3 matchings which cover at least $\frac{13}{15}$ of its edges.

Our approach to max 3-edge coloring problem is different and deals with the concept of *bicolor connection*.

We demonstrate the framework we need. In [8], an algorithm is used that gives a 4-edge assignment to *G*; where color 4 appears only on crossing edges and after that tries to reduce the number of edges with color 4, replacing it by one of the colors 1, 2 or 3.

Two crossing edges have a "bad" coloring if they get color 4 otherwise they have a "good" coloring.

The algorithm in [8] consists of two parts. The first part transforms graph *G* into another graph G' replacing each crossing point by a configuration, which is called "*basic configuration*" or *BASCON* for short, cf. Figure 1, and gives an assignment with colors 1, 2 and 3 to G'.

In the colored graph G', BASCONs have their own 3-edge coloring. A BASCON, as one can see in Figure 1, has four *outer* edges and each pair of *opposite outer* edges corresponds to a crossing edge. According to the coloring of the *outer edges* of a BASCON, we say that a BASCON has a "*bad*" or a "*good*" *coloring*, giving in this way, respectively, the "bad" or the "good" coloring to the pairs of crossing edges in graph *G*, cf. Figure 2.

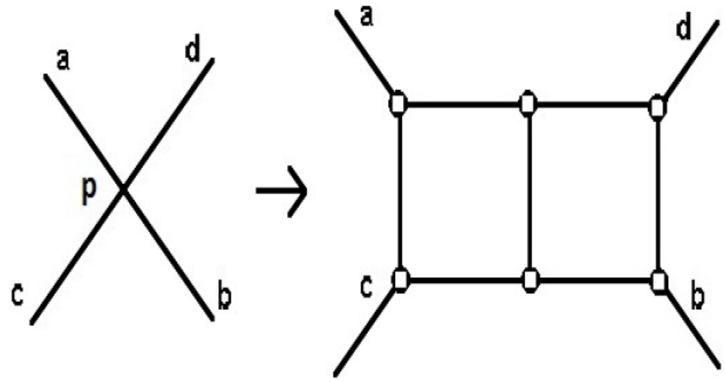

Figure 1. In the crossing point *p*, we put the configuration shown on the right. We called it *basic configuration* or *BASCON* for short. This BASCON has four *outer* edges: *a*, *b*, *c* and *d*, where the pairs *a*, *b* and *c*, *d* are *opposite* outer edges.

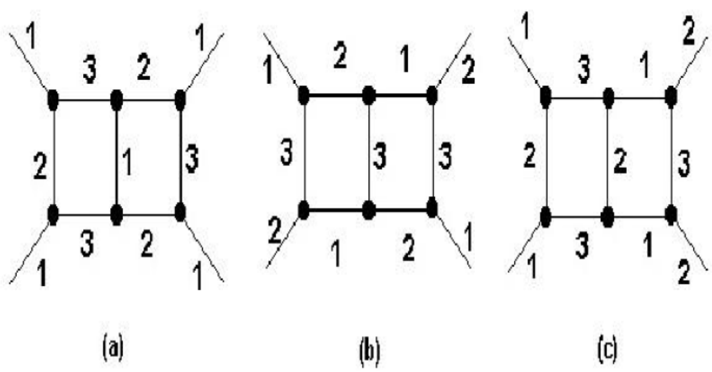

Figure 2. In (a) and in (b) is assigned a *good* coloring. In (c) is assigned a *bad* coloring. The two crossing edges that correspond in (a) get both color 1. The two crossing edges that correspond in (b) get a different color. The first gets color 1 and the second one color 2. The crossing edges that correspond in (c) both get color 4.

We say that a bicolor circle *passes* from a *BASCON* if it passes from two outer edges of the BASCON, which are not opposite edges. We say that two BASCONs are *connected* by a bicolor circle if this circle passes from both of them.

At this point starts the second part of the algorithm, which is called "*procedure recover*" or *prorec* for short. Procedure recover *tries* to reduce the number of BASCONs with a "bad" coloring based on the following facts:

Fact 1. *In every BASCON, exactly two bicolor circles pass. These circles are both colored with the same pair of colors,* 1-2, 1-3 *or* 2-3.

Fact 2. *The "bad" coloring of a BASCON can always change to a "good" coloring by interchanging the colors of two of its outer edges that are not opposite edges*.

So, *prorec* starts from a BASCON with a "bad" coloring and follows a bicolor circle, interchanging its colors, say colors 1-2. But, in this way, there is the possibility to "spoil" the "good" coloring of another BASCON turning it into a "bad" one. Trying to recover the "good" coloring in the second BASCON, it is possible to spoil the "good" coloring of a third BASCON and so on.

In other words, *prorec* fails to turn the "bad" colorings into "good" colorings for all the BASCONs if it is forced to return again and again to BASCONs that it has already tried to change the "bad" coloring into a "good" one. A necessary condition for this failure of *prorec* is given below:

If the recproc fails to assign a "good" coloring in two crossing edges in G*, then for every* 3*-edge coloring of* G'*, each one of the two bicolor circles that pass from the corresponding to these crossing edges BASCON always passes from another BASCON* (*one that corresponds to a different pair of crossing edges*), cf. Figure 3.

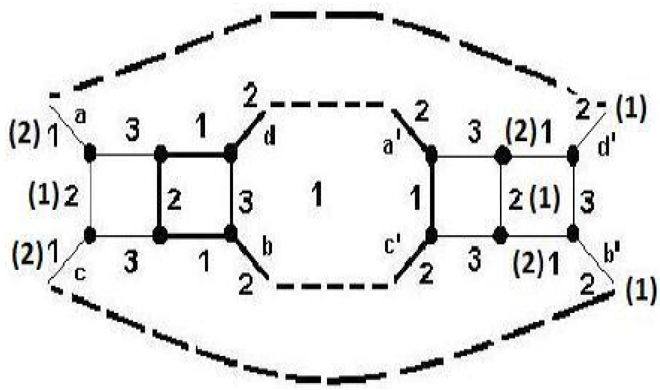

Figure 3. The BASCON on the left has a “bad” coloring. Interchanging colors 1-2, see the colors on the brackets, this BASCON gets a “good” coloring but the “good” coloring of the BASCON on the right turn a “bad” one. Trying to recover the previous “good” coloring, by interchanging colors 1-2 in the other bicolor circle, the current “good” coloring of the BASCON on the left “spoils” and returns to the previous “bad” coloring.

The main result in [8], that is Theorem 3, is proved using the previous condition. The key idea is that the bicolor circles that connect pairs of BASCONs cannot cover more vertices than the order of graph G' and therefore combining the minimum distance d with the order n, we can find an upper bound for t.

Notice that since G' is a cubic planar and bridgeless graph, in a 3-edge coloring all of its vertices can be covered by 1-2 circles and some of these 1-2 circles can connect pairs of BASCONs. Can by 1-2 and 1-3 circles that connect BASCONs be covered all the vertices of G'? The answer is “no”. This case is impossible and all the vertices of G' can be covered only with the existence of extra 1-2 circles that do not connect any pair of BASCONs. Take for example two pairs of BASCONs such as the first one is connected by 1-2 circles and the second one by 1-3 circles and suppose that these circles share m edges with the common color “1”. In that case, the sum of the distances between the corresponding pairs of crossing points is not $2d$ but $2d + m$ otherwise two crossing points in G are at distance less than the minimum distance d. Therefore, since nothing change in our calculations, we can assume that only 1-2 circles are involved in the connected pairs of BASCONs.

Now, if we consider the BASCONs in G' as crossing points, we return from graph G' to the graph G, we get the inequality in the following theorem:

Theorem 3 [8]**.** *Let G be a bridgeless non-planar cubic graph, of order n. Let d be the minimum distance between any pair of crossing points for a drawing of G. If G is in class* 2 *and in a* 4*-edge coloring, color* 4 *is required at most t times, then the following holds*:

(i) $t \leq \frac{n}{2(d-1)}$.

1.3. Comments on inequality (i)

We can use inequality (i) without the need to find a drawing of G with the minimum possible number of crossings. Actually, this is a NP-hard problem [4]. On the other hand, "many" crossings means "small" value for distance d. Also, "big" values of d usually means "big" values of n. The ratio of the number of crossing points to the order n balances around the value $\frac{1}{2(d-1)}$. We try to find conditions under which this ratio deviates to smaller values.

It is clear from the proof that having distance $d = 1$ does not make sense, since covering two vertices of degree 4 in H no vertex is covered in the corresponding graph G.

We also note that the inequality (i) works very well either in the case of replacement of minimum distance d with an "average" distance d or with the use of other proper techniques.

For example, suppose that two pairs of crossing points are at distance 3 apart and all the other pairs are at distance 5. In that case, for the first two pairs, it is possible that color "4" be required twice and $4(3-1)=8$ vertices be covered by four bicolor paths. That means that inequality (i) takes the form:

$$t-2 \leq \frac{n-8}{2(5-1)} = t \leq \frac{n}{8}+1.$$

Sometimes inequality (i) gives as an upper bound of t an odd integer, for example $t \leq 3$. The times that color "4" is required is always less than or equal to the number of crossing points when this number is even but less than or equal to the number of crossing points minus 1 when this number is odd. So, in the case where color "4" is required, at most 3 times two pairs of BASCONs must be connected by *bcircles* (so inequality (i) takes the form $t \leq 4$) and at least one of these pairs shares a common *outer* edge.

It is obvious that the knowledge of the minimum length of bicolor paths that are realizable in a 3-edge coloring of G' and connect BASCONs would improve inequality (i). Distance d is used instead, since it is easy to get computed and is always less than the minimum length of bicolor paths that connect BASCONs.

2. The Main Result

2.1. How to use the upper bound in [8] in subgraphs of G

The basic idea is to find structures of G which either "prevent" the connection of BASCONs by bicolor circles or "impose" these connections. We study the case of cubic graphs having each crossing point "close" to the other ones.

The reason for that choice is based on the fact that a bicolor connected pair of BASCONs needs more edges than a pair of BASCONs without this connection. So, if bicolor paths are very "close" to each other, some bicolor connections cannot be completed unless there exist "long" bicolor paths that first go away from the other ones and then return back to them. In other words, we study cubic and bridgeless graphs G having "small" subgraphs which contain all the crossing points of *G*. The existence of "long" bicolor paths that go out of the subgraphs and return to them depends on the number of connected edges of these subgraphs with the rest graph and therefore this number involves upper bound.

For short by *bpath*, *bpaths* or *bcircles*, we mean *bicolor path*, *bicolor paths* or *bicolor circles*, respectively. Now, we state the following theorem:

Theorem 1. *Let G be a cubic graph with n vertices. Suppose that G belongs to class* 2 *and t is the minimum number of times that need the fourth color* (*over all possible* 4-*edge colorings*). *Let H be a subgraph of G having l connecting edges with the rest graph* $G - H$. *Suppose that all crossing edges of G belong to H and the order of H is* n'. *Let d be the minimum distance between any pair of crossing points inside H for a drawing of G. Then the followings hold*:

(i) $t \leq \dfrac{n}{2(d-1)}$;

(ii) $t \leq \dfrac{n'}{2(d-1)}$ *for* $d \leq 5$;

(iii) $t \leq \dfrac{n'}{2(d-1)} + \dfrac{l}{4} - \dfrac{l}{(d-1)}$ *for* $d \geq 5$.

Proof. Due to the assumption that color "4" is required to *t* edges, there exist at least 2*t* crossing edges in *G*. We can transform *G* to another cubic graph G' by setting BASCONs into the place of the crossing points. So, subgraph *H* is transformed to another subgraph H' and has all the BASCONs. We know from [8] that in any 3-edge coloring of G', there exist *t* pairs of BASCONs each one of which consists from a BASCON with a "good" coloring and a BASCON with a "bad" coloring. We also know that these two BASCONs are connected by two *bcircles*, so there are 4 *bpaths* that connect them. Some of these *bpaths* can pass from the connected edges leaving subgraph H' and return to it passing again from the connecting edges, cf. Figure 4.

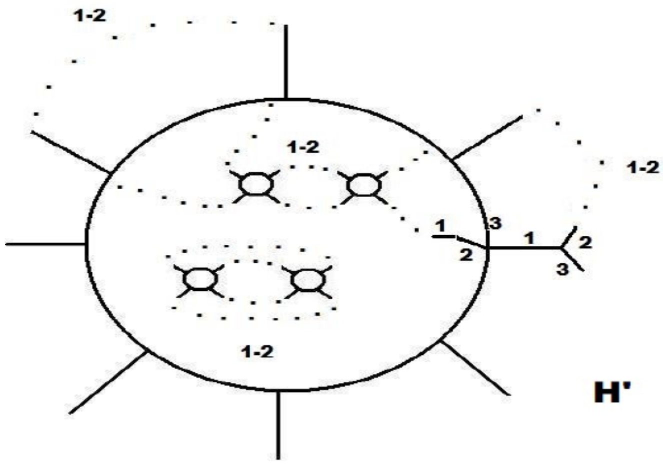

Figure 4. Subgraph H is connected with the rest graph by 8 connected edges. H contains all the crossing points of G. In H', some of *bpaths* that connect pairs of BASCONs pass from the connected edges into subgraph G'-H' and return to H'. The rest of the connecting *bpaths* cover vertices that belong only to H'.

We shall try to use inequality (i) for subgraph H. It is clear that as many vertices are covered by the *bpaths* inside H' a better upper bound for t is achieved. So, we try to find the worst case that is the case with the minimum covering by the *bpaths*.

Assume that *bpaths* pass from a fraction of the l connecting edges, so from $l' = kl$ connected edges, where $0 \leq k \leq 1$. We choose 1-2 paths be the *bpaths* that will cover the vertices of H' and we notice that in every one 3-edge coloring of subgraph H' at most one 1-2 path can pass from one connected edge. Therefore, at most $\frac{l'}{2}$ 1-2 paths go out from H' and return to it. We also suppose that these $\frac{l'}{2}$ 1-2 paths have "almost" all of their edges outside of graph H'. It means that we can have 2t 1-2 paths in total and $2t - \frac{l'}{2}$ 1-2 paths having all of their edges inside the subgraph H'.

Considering BASCONs as crossing points, we return to graph G and to subgraph H. Notice that the 1-2 paths in H' cover vertices in H and the number of these vertices does not exceed the order n' of H minus $2l'$. Indeed, if a 1-2 path passes from one of the l' connected edges, it covers at least two vertices of H. The first one is the endpoint of the connected edge that belongs to H. The 1-2 path that we consider goes to a crossing point and therefore passes from one crossing edge. So, the second vertex that is covered is the endpoint of a crossing edge that also belongs to H. Finally, we get: $\left(2t - \frac{l'}{2}\right)(d-1) \leq n' - 2l'$ or $t \leq \frac{n'}{2(d-1)} + \frac{kl}{4} - \frac{kl}{d-1}$, since $l' = kl$. For $d \leq 5$, the right side of the last inequality takes a maximum when $k = 0$ which gives the upper bound in (ii). In the case where $d \geq 5$, the right side of the inequality takes a maximum when $k = 1$ and this gives the upper bound in (iii). □

Corollary 1. *Let G be a bridgeless non-planar cubic graph. Suppose that instead of a subgraph H in the previous Theorem* 1, *there are k disjoint subgraphs* H_i, $i = 1, 2, \ldots, k$ *with corresponding minimum distances between their crossing points* d_i, $i = 1, 2, \ldots, k$. *If all the crossing points of G are contained in these subgraphs, then the inequalities* (ii) *and* (iii) *of Theorem* 1 *can be applied for each one of these subgraphs and get an inequality for graph G using the sum of these inequalities*.

Proof. In the case where $d_i = d$ for all $i = 1, 2, \ldots, k$, we can simply consider the subgraph H of Theorem 1 as a disjoint union of the H_i subgraphs. In the case where we have different distances d_i, we have only to notice that the structure of graph $G - H$ does not involve in the proof of Theorem 1 and the minimum distance d is used only in subgraph H. So, each one of the H_i subgraphs can be considered as the subgraph H in Theorem 1 and therefore the inequalities we get from subgraphs H_i are independent of each other. □

2.2. Some examples

In Figure 5 is shown the Petersen graph with 5 crossing points, say *a*, *b*, *c*, *d* and *e*, and 10 vertices: 0, 1, 2, ..., 9. Let us forget for the moment that Petersen graph needs color "4" only twice and it can be drawn with only two crossing points. We can see that four possible *bpaths* that can connect *a* and *d* are: *a*-9-1-2-3-5-*d*, *a*-8-0-4-6-*d*, *a*-*c*-*e*-*d* and *a*-*b*-*d*. That means that the corresponding distances between crossing points *a* and *d* are: 6, 5, 0 and 0, so we get an average distance $d = \frac{11}{4}$ and the times t that color "4" is required is less than: $\frac{10}{2\left(\frac{11}{4}-1\right)} = \frac{20}{7}$ and since t is an integer, color "4" is needed at most twice. If another pair of crossing points has been chosen, say for example, *a* and *b*, we can have as possible *bpaths* connecting *a* and *b* to be: *a*-8-0-4-6-*b*, *a*-9-1-2-3-5-*d*-*b*, *a*-*b* and *a*-*c*-*e*-*d*-*b*. The corresponding distances are: 5, 6, 0 and 0 and we get the previous result. Due to the symmetry between these crossing points, it is no need to check any other pair.

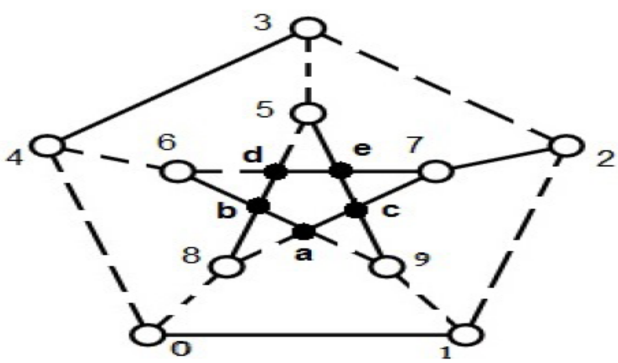

Figure 5. The Petersen graph in a drawing with 5 crossings is shown. Crossing points are denoted by letters and vertices by numbers.

In Figure 6, a subdivision of the Petersen graph is shown. We assume that it is a subgraph of a cubic and bridgeless graph G and as in Figure 5 crossing points are denoted by letters and vertices by numbers. The connected edges with the rest of graph are four. Suppose that from graph G, we construct graph G'. We can see that it is possible to exist *bcircles* that connect the BASCONs corresponding to crossing points e and b. The first one can be: e-5-3-10- ... -12-4-6-b-8-0-13-...-11-2-7-e and the second one: e-d-b-a-c-e. In that case, there is no vertex left inside the subgraph to complete a second bicolor connection between another pair of BASCONs. There is no connected edge left and therefore it is not possible to use outside of the subgraph. So, color "4" is needed at most twice. Using inequality (ii), we get the same conclusion, since $n' = 14,\ \ l = 4$ and as in the previous example, we get an average distance $d = \frac{13}{4}$. Indeed, $t \leq \frac{14}{2\left(\frac{13}{4} - 1\right)}$ or $t \leq \frac{28}{9}$ and finally $t \leq 3$. Since 3 is an odd integer, color "4" is required at most twice, as already noticed.

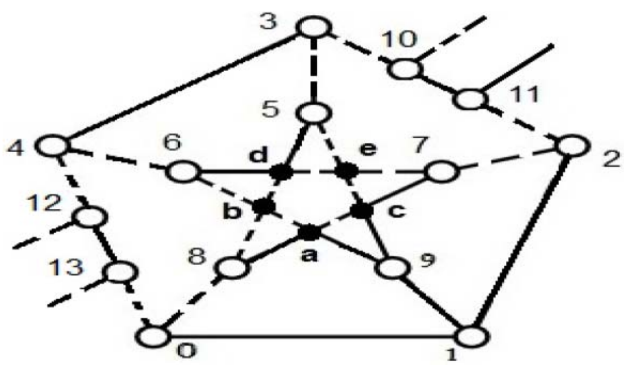

Figure 6. Here we have a subdivision of the Petersen graph as a subgraph of a graph G. It is connected with G by four edges. Crossing points are denoted by letters and vertices by numbers.

Acknowledgement

The author thanks the anonymous referees for their valuable suggestions and constructive criticism which improved the presentation of the paper.

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