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ENVELOPING ALGEBRA AND SKEW CALABI-YAU ALGEBRAS OVER SKEW POINCARÉ-BIRKHOFF-WITT EXTENSIONS

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Abstract

In this paper, we show that the tensor product of skew PBW extensions is a skew PBW extension. We also characterize the enveloping algebra of a skew PBW extension. Finally, we establish conditions for sufficiency to guarantee the property of being skew Calabi-Yau algebra over skew PBW extensions.

1. Introduction

Let k be a commutative ring and B be an associative k-algebra. By definition, the enveloping algebra of B is the tensor product $B^e = B \otimes_k B^{\mathrm{op}}$

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of B with its opposite algebra B^{op}. Bimodules over B are essentially the same as modules over the enveloping algebra of B, so, in particular, B can be considered as a B^e-module. Since that the opposite algebra of a Koszul algebra is also a Koszul algebra, and the tensor product of Koszul algebras is also a Koszul algebra, and having in mind that the authors have studied the property of being Koszul for skew Poincaré-Birkhoff-Witt (PBW for short) extensions (see [18-21]), in this paper, we are interested in the characterization of the enveloping algebra of skew PBW extensions. These non-commutative rings of polynomial type were introduced in [1], and they are defined by a ring and a set of variables with relations between them. Skew PBW extensions include rings and algebras coming from mathematical physics such PBW extensions, group rings of polycyclic-by-finite groups, Ore algebras, operator algebras, diffusion algebras, some quantum algebras, quadratic algebras in three variables, some 3-dimensional skew polynomial algebras, some quantum groups, some types of Auslander-Gorenstein rings, some Calabi-Yau algebras, some quantum universal enveloping algebras, and others. A detailed list of examples can be consulted in [5] and [3]. Several ring, module and homological properties of these extensions have been studied (see, for example, [5, 6, 3, 7, 8, 2, 9-13, 20, 18, 21] and others). Besides our interest, it is important to say that the concept of enveloping algebra is of great importance in the research of another concepts in physics and mathematics (for instance, Calabi-Yau algebras [22, 14, 23], see also Section 5).

The paper is organized as follows: In Section 2, we establish the necessary results about skew PBW extensions for the rest of the paper. Next, in Section 3, we establish some results about tensor product of skew PBW extensions. Section 4 contains the proof of the fact that the enveloping algebra of a bijective skew PBW extension is again a PBW extension. Finally, in Section 5, we study the skew Calabi-Yau algebras and show that graded quasi-commutative skew PBW extensions over connected Calabi-Yau algebras are skew Calabi-Yau. Throughout the paper, the word ring means a ring not necessarily commutative with unity. The symbols k and $\Bbbk$ will denote a commutative ring and a field, respectively.

2. Skew PBW Extensions

In this section, we recall the definition of a skew PBW extension and present some of their properties. The proofs of these properties can be found in [3].

Definition 2.1 [1, Definition 1]**.** Let R and A be rings. Then we say that A *is a skew PBW extension over R* (also called a σ-*PBW extension of R*), if the following conditions hold:

(i) $R \subseteq A$;

(ii) there exist elements $x_1, ..., x_n \in A$ such that A is a left free R-module, whose basis has the basic elements described in the set:

$$\mathrm{Mon}(A) := \{x^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, ..., \alpha_m) \in \mathbb{N}^n\};$$

(iii) for each $1 \leq i \leq n$ and any $r \in R \backslash \{0\}$, there exists an element $c_{i,r} \in R\backslash\{0\}$ such that $x_i r - c_{i,r}x_i \in R$;

(iv) for any elements $1 \leq i, j \leq n$, there exists $c_{i,j} \in R\backslash\{0\}$ such that $x_j x_i - c_{i,j}x_i x_j \in R + Rx_1 + \cdots + Rx_n$.

Under these conditions, we will write $A := \sigma(R)\langle x_1, ..., x_n\rangle$.

Remark 2.2 [1, Remark 2]**.** (i) Since $\mathrm{Mon}(A)$ is a left R-basis of A, the elements $c_{i,r}$ and $c_{i,j}$ in Definition 2.1 are unique.

(ii) In Definition 2.1(iv), $c_{i,i} = 1$. This follows from $x_i^2 - c_{i,i}x_i^2 = s_0 + s_1x_1 + \cdots + s_nx_n$, with $s_i \in R$, which implies $1 - c_{i,i} = 0 = s_i$.

(iii) Let $i < j$. By Definition 2.1(iv), there exist elements $c_{j,i}, c_{i,j} \in R$ such that $x_i x_j - c_{j,i}x_j x_i \in R + Rx_1 + \cdots + Rx_n$ and $x_j x_i - c_{i,j}x_i x_j \in$

$R + Rx_1 + \cdots + Rx_n$, and hence $1 = c_{j,i}c_{i,j}$, that is, for each $1 \le i < j \le n$, $c_{i,j}$ has a left inverse and $c_{j,i}$ has a right inverse. In general, the elements $c_{i,j}$ are not two sided invertible. For instance, $x_1x_2 = c_{2,1}x_2x_1 + p = c_{21}(c_{1,2}x_1x_2 + q) + p$, where $p, q \in R + Rx_1 + \cdots + Rx_n$, so $1 = c_{2,1}c_{1,2}$, since x_1x_2 is a basic element of $\mathrm{Mon}(A)$. Now, $x_2x_1 = c_{1,2}x_1x_2 + q = c_{1,2}(c_{2,1}x_2x_1 + p) + q$, but we cannot conclude that $c_{12}c_{21} = 1$ because x_2x_1 is not a basic element of $\mathrm{Mon}(A)$ (we recall that $\mathrm{Mon}(A)$ consists of the standard monomials).

(iv) Every element $f \in A\setminus\{0\}$ has a unique representation in the form $f = c_0 + c_1X_1 + \cdots + c_tX_t$, with $c_i \in R\setminus\{0\}$ and $X_i \in \mathrm{Mon}(A)$, for $1 \le i \le t$.

Proposition 2.3 [1, Proposition 3]**.** *Let A be a skew PBW extension over R. For each* $1 \le i \le n$, *there exist an injective endomorphism* $\sigma_i : R \to R$ *and a* σ_i*-derivation* $\delta_i : R \to R$ *such that* $x_ir = \sigma_i(r)x_i + \delta_i(r)$, *for every* $r \in R$.

Definition 2.4 [1, Definition 6]**.** Let A be a skew PBW extension over R with endomorphisms σ_i, $1 \le i \le n$, as in Proposition 2.3.

(i) For $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$, $\sigma^\alpha := \sigma_1^{\alpha_1} \cdots \sigma_n^{\alpha_n}$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$. If $\beta = (\beta_1, ..., \beta_n) \in \mathbb{N}^n$, then $\alpha + \beta := (\alpha_1 + \beta_1, ..., \alpha_n + \beta_n)$.

(ii) For $X = x^\alpha \in \mathrm{Mon}(A)$, $\exp(X) := \alpha$ and $\deg(X) := |\alpha|$. The symbol $\succeq$ will denote a total order defined on $\mathrm{Mon}(A)$ (a total order on $\mathbb{N}_0^n$). For an element $x^\alpha \in \mathrm{Mon}(A)$, $\exp(x^\alpha) := \alpha \in \mathbb{N}_0^n$. If $x^\alpha \succeq x^\beta$ but $x^\alpha \ne x^\beta$, then we write $x^\alpha \succ x^\beta$. If $f = c_1X_1 + \cdots + c_tX_t \in A$, $c_i \in R\setminus\{0\}$, with $X_1 \prec \cdots \prec X_t$, then $\mathrm{lm}(f) := X_t$ is the *leading monomial* of *f*, $\mathrm{lc}(f) := c_t$ is the *leading coefficient* of *f*, $\mathrm{lt}(f) := c_tX_t$ is the *leading term* of *f*, $\exp(f) := \exp(X_t)$ is the *order* of *f*, and $E(f) := \{\exp(X_i) \mid 1 \le i \le t\}$.

Finally, if $f = 0$, then $\mathrm{lm}(0) := 0$, $\mathrm{lc}(0) := 0$ and $\mathrm{lt}(0) := 0$. We also consider $X \succ 0$ for any $X \in \mathrm{Mon}(A)$. For a detailed description of monomial orders in skew PBW extensions, see [1, Section 3].

(iii) If f is an element as in Remark 2.2(iv), then $\deg(f) := \max\{\deg(x_i)\}_{i=1}^{t}$.

Skew PBW extensions are characterized in the following way.

Proposition 2.5 [1, Theorem 7]**.** *Let A be a polynomial ring over R with respect to a set of indeterminates $\{x_1, ..., x_n\}$. A is a skew PBW extension over R if and only if the following conditions are satisfied*:

(i) *For each* $x^{\alpha} \in \mathrm{Mon}(A)$ *and every* $0 \neq r \in R$, *there exist unique elements* $r_{\alpha} := \sigma^{\alpha}(r) \in R \setminus \{0\}$, $p_{\alpha, r} \in A$ *such that* $x^{\alpha} r = r_{\alpha} x^{\alpha} + p_{\alpha, r}$, *where* $p_{\alpha, r} = 0$, *or* $\deg(p_{\alpha, r}) < |\alpha|$ *if* $p_{\alpha, r} \neq 0$. *If r is left invertible*, *then so is* r_{α}.

(ii) *For each* $x^{\alpha}, x^{\beta} \in \mathrm{Mon}(A)$, *there exist unique elements* $c_{\alpha, \beta} \in R$ *and* $p_{\alpha, \beta} \in A$ *such that* $x^{\alpha} x^{\beta} = c_{\alpha, \beta} x^{\alpha+\beta} + p_{\alpha, \beta}$, *where* $c_{\alpha, \beta}$ *is left invertible*, $p_{\alpha, \beta} = 0$, *or* $\deg(p_{\alpha, \beta}) < |\alpha + \beta|$ *if* $p_{\alpha, \beta} \neq 0$.

There are some examples of skew PBW extensions which are very important for several results in the paper (see Propositions 3.1, 4.1 and Theorem 5.5). This is the content of the following definition:

Definition 2.6. Let A be a skew PBW extension over R, $\Sigma := \{\sigma_1, ..., \sigma_n\}$ and $\Delta := \{\delta_1, ..., \delta_n\}$, where σ_i and δ_i $(1 \leq i \leq n)$ are as in Proposition 2.3.

(a) For any element r of R such that $\sigma_i(r) = r$ and $\delta_i(r) = 0$, for all $1 \leq i \leq n$, it is called a *constant*. A is called a *constant* if every element of R is a constant.

(b) A is called *quasi-commutative* if the conditions (iii) and (iv) of Definition 2.1 are replaced by the following conditions: (iii') for each $1 \leq i \leq n$ and every $r \in R \setminus \{0\}$, there exists $c_{i,r} \in R \setminus \{0\}$ such that $x_i r = c_{i,r} x_i$; and (iv') for any $1 \leq i, j \leq n$, there exists $c_{i,j} \in R \setminus \{0\}$ such that $x_j x_i = c_{i,j} x_i x_j$.

(c) A is called *bijective*, if σ_i is bijective for each $\sigma_i \in \Sigma$, and the elements $c_{i,j}$ are invertible for any $1 \leq i < j \leq n$.

The importance of the restriction on the injective endomorphisms σ_i is precisely due to the fact that under this condition, we have $Rx_i + R = x_i R + R$, and hence every element $f = c_0 + c_1 X_1 + \cdots + c_t X_t \in A$ (Remark 2.2(iv)) can be rewritten in the reverse form $f = c_0' + X_1 c_1' + \cdots + X_t c_t'$. In other words, if the functions σ_i are bijective, then A_R is a right free R-module with basis $\mathrm{Mon}(A)$ [3, Proposition 1.7]. In fact, a lot of properties (Noetherianess, regularity, Serre's theorem, global homological dimension, Gelfand-Kirillov dimension, Goldie dimension, semisimple Jacobson, prime ideals, Quillen's K-groups of higher algebraic K-theory, Baerness, quasi-Baerness, Armendariz, etc) of skew PBW extensions have been studied using this assumption of bijectivity (see [5, 6, 3, 7-10, 20, 18, 21] and others).

3. Tensor Product of Skew PBW Extensions

Proposition 3.1 (Change of scalars)**.** *If A is a skew PBW extension over k and B is a free commutative k-algebra, then $B \otimes_k A$ is a skew PBW extension over $B \otimes_k k$, that is, an extension over B.*

Proof. We show that the four conditions established in Definition 2.1 are satisfied. First of all, it is clear that $B \otimes_k k \subseteq B \otimes_k A$. Second of all, we know that $B \otimes_k A$ is a B-algebra under the product $b'(b \otimes a) := b'b \otimes a$, and $B \otimes_k A$ is left B-free with the same rank of A as k-module [17, Remark 18.27], and since $B \cong B \otimes_k k$, it follows that $B \otimes_k A$ is a left $B \otimes_k k$-free module. Note that

$$\mathrm{Mon}(B \otimes_k A) = \{(1 \otimes x_1)^{\alpha_1} \cdots (1 \otimes x_n)^{\alpha_n} \mid \alpha_i \in \mathbb{N}, 1 \leq i \leq n\},$$

which shows that $B \otimes_k A$ is left $B \otimes_k k$-free of the same rank of A as k-module. Now, if $r \otimes k'$ is a non-zero element of $B \otimes_k k$ (that is, the element rk' since $B \cong B \otimes_k k$), then we can see that for every element $1 \otimes x_i$ of the basis, there exists a non-zero element $1 \otimes c_{i,k'} \in B \otimes_k k$ such that

$$(1 \otimes x_i)(r \otimes k') - (1 \otimes c_{i,k'})(1 \otimes x_i) \in B \otimes_k k,$$

where $c_{i,r} \in k$ satisfies $x_i r - c_{i,r} x_i \in k$, since A is a skew PBW extension over k. Finally, the condition

$$(1 \otimes x_i)(1 \otimes x_j) - (1 \otimes c_{i,j})(1 \otimes x_j)(1 \otimes x_i)$$
$$\in B \otimes_k k + (B \otimes_k k)(1 \otimes x_1) + \cdots + (B \otimes_k k)(1 \otimes x_n)$$

follows from Definition 2.1(iv) applied to the extension A. □

Remark 3.2. The injective endomorphisms and the derivations for the skew PBW extension mentioned in Proposition 3.1 are given by $\sigma_i^{\otimes} : B \otimes_k k \to B \otimes_k k$, $\sigma_i^{\otimes}(b \otimes r) := b \otimes \sigma_i(r)$, and $\delta_i^{\otimes} : B \otimes_k k \to B \otimes_k k$, $\delta_i^{\otimes}(b \otimes r) := b \otimes \delta_i(r)$, respectively. It is straightforward to see that the functions $\sigma_i^{\otimes}$ are actually injective endomorphisms and that the functions $\delta_i^{\otimes}$ are $\sigma_i^{\otimes}$-derivations, for every $1 \leq i \leq n$.

Example 3.3. If $A = \sigma(k)\langle x_1, ..., x_n\rangle$ and $B = k[y] = k[y_1, ..., y_m]$, then

$$k[y] \otimes_k A \cong \sigma(k[y] \otimes_k k)\langle 1 \otimes x_1, ..., 1 \otimes x_n\rangle \cong \sigma(k[y])\langle z_1, ..., z_n\rangle.$$

Let us see some examples of remarkable non-commutative rings which illustrated this isomorphism. A detailed reference of every example can be found in [5] or [3].

(a) *Additive analogue of the Weyl algebra*. This algebra is the $\Bbbk$-algebra $A_n(q_1, \dots, q_n)$ generated by the indeterminates $x_1, \dots, x_n, y_1, \dots, y_n$ subject to the relations:

$$x_j x_i = x_i x_j, \quad 1 \le i, j \le n,$$

$$y_j y_i = y_i y_j, \quad 1 \le i, j \le n,$$

$$y_i x_j = x_j y_i, \quad i \ne j,$$

$$y_i x_i = q_i x_i y_i + 1, \quad 1 \le i \le n,$$

where $q_i \in \Bbbk \backslash \{0\}$. From [3, Example 3.5(a)], we have the isomorphisms

$$A_n(q_1, \dots, q_n) \cong \sigma(\Bbbk)\langle x_1, \dots, x_n; y_1, \dots, y_n\rangle \cong \sigma(\Bbbk[x_1, \dots, x_n])\langle y_1, \dots, y_n\rangle,$$

that is, $A_n(q_1, \dots, q_n)$ is a skew PBW extension of the field $\Bbbk$ or the polynomial ring $\Bbbk[x_1, \dots, x_n]$. Now, by Proposition 3.1, we obtain the isomorphism

$$\Bbbk[x_1, \dots, x_n] \otimes_{\Bbbk} \sigma(\Bbbk)\langle y_1, \dots, y_n\rangle \cong \sigma(\Bbbk[x_1, \dots, x_n])\langle 1 \otimes y_1, \dots, y_n\rangle,$$

whence $A_n(q_1, \dots, q_n) \cong \Bbbk[x_1, \dots, x_n] \otimes_{\Bbbk} \sigma(\Bbbk)\langle y_1, \dots, y_n\rangle$.

(b) *Multiplicative analogue of the Weyl algebra*. By definition, this non-commutative ring is the $\Bbbk$-algebra $\mathcal{O}_n(\lambda_{ji})$ generated by $x_1, \dots, x_n$ satisfying $x_j x_i = \lambda_{ji} x_i x_j$, $1 \le i < j \le n$, $\lambda_{ji} \in \Bbbk \backslash \{0\}$. It can be proved that $\mathcal{O}_n(\lambda_{ji}) \cong \sigma(\Bbbk)\langle x_1, \dots, x_n\rangle \cong \sigma(\Bbbk[x_1])\langle x_2, \dots, x_n\rangle$ [3, Example 3.5(a)]. Now, Proposition 3.1 guarantees that

$$\Bbbk[x_1] \otimes_{\Bbbk} \sigma(\Bbbk)\langle x_2, \dots, x_n\rangle \cong \sigma(\Bbbk[x_1])\langle 1 \otimes x_2, \dots, 1 \otimes x_n\rangle,$$

and so $\mathcal{O}_n(\lambda_{ji}) \cong \Bbbk[x_1] \otimes_{\Bbbk} \sigma(\Bbbk)\langle x_2, \dots, x_n\rangle$.

The next proposition treats the construction of skew PBW extensions over the same ring of coefficients. Note that if A is a skew PBW extension over a ring R, then A is a right R-module under the multiplication in A, that is, $f \cdot r := fr$, $f \in A$, $r \in R$. However, A is not necessarily a right free R-module; in fact, if A is bijective, then A_R is free with basis the set $\mathrm{Mon}(A)$ established in Definition 2.1(ii) (see [3, Proposition 1.7] for a detailed proof of this fact).

Proposition 3.4. *If* $A = \sigma(R)\langle x_1, ..., x_n\rangle$ *and* $A' = \sigma(R)\langle y_1, ..., y_m\rangle$ *are two skew PBW extensions over* R, *then* $A \otimes_R A'$ *is also a skew PBW extension over* R, *and we have*

$$A \otimes_R A' = \sigma(R)\langle x_1 \otimes 1, ..., x_n \otimes 1, 1 \otimes y_1, ..., 1 \otimes y_m\rangle.$$

Proof. Again, let us illustrate the four conditions of Definition 2.1. It is clear that $R \subseteq A \otimes_R A'$. Now, since the product of left free R-modules is a left free R-module with R-basis $\{x^{\alpha} \otimes y^{\beta} \mid \alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^m\}$, and having in mind that

$$\begin{aligned}
&\{x^{\alpha} \otimes y^{\beta} \mid \alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^m\}\\
&= \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \otimes y_1^{\beta_1} \cdots y_m^{\beta_m} \mid \alpha_i, \beta_j \in \mathbb{N}\}\\
&= \{(x_1^{\alpha_1} \otimes 1) \cdots (x_n^{\alpha_n} \otimes 1)(1 \otimes y_1^{\beta_1}) \cdots (1 \otimes y_m^{\beta_m}) \mid \alpha_i, \beta_j \in \mathbb{N}\}\\
&= \{(x_1 \otimes 1)^{\alpha_1} \cdots (x_n \otimes 1)^{\alpha_n}(1 \otimes y_1)^{\beta_1} \cdots (1 \otimes y_m)^{\beta_m} \mid \alpha_i, \beta_j \in \mathbb{N}\},
\end{aligned}$$

the set

$$\mathrm{Mon}(A \otimes A') := \{(x_1 \otimes 1)^{\alpha_1} \cdots (x_n \otimes 1)^{\alpha_n}(1 \otimes y_1)^{\beta_1} \cdots$$
$$(1 \otimes y_m)^{\beta_m} \mid \alpha_i, \beta_j \in \mathbb{N}\}$$

is the R-basis for the left free R-module $A \otimes A'$.

For a non-zero element r of R, we have $(x_i \otimes 1)r - c_{i,r}(x_i \otimes 1) \in R \otimes 1$, $(1 \otimes y_j)r - c_{j,r}(1 \otimes y_j) \in 1 \otimes R$, for $(1 \le i \le n, 1 \le j \le m)$, because A and A' are skew PBW extensions of R.

Now, note that

$$(x_j \otimes 1)(x_i \otimes 1) - c_{i,j}(x_i \otimes 1)(x_j \otimes 1)$$

$$\in R \otimes 1 + \sum_{l=1}^{n} R(x_l \otimes 1) + \sum_{p=1}^{m} R(1 \otimes y_p), \quad 1 \leq i, \ j \leq n,$$

$$(1 \otimes y_j)(1 \otimes y_i) - c'_{i,j}(1 \otimes y_i)(1 \otimes y_j)$$

$$\in 1 \otimes R + \sum_{l=1}^{n} R(x_l \otimes 1) + \sum_{p=1}^{m} R(1 \otimes y_p), \quad 1 \leq i, \ j \leq m,$$

where the elements $c_{i,j}, c'_{i,j} \in R$ are considered from Definition 2.1(iv) for the extensions A and A', respectively. Finally, we impose the relations $(1 \otimes y_j)(x_i \otimes 1) - (x_i \otimes 1)(1 \otimes y_j) = 0 \quad (1 \leq i \leq n, 1 \leq j \leq m)$, with the aim of guarantee the condition (iv) of the definition of the skew PBW extension $A \otimes_R A'$ over R. □

Remark 3.5. If A and A' are two skew PBW extensions over R as in Proposition 3.4, then the injective endomorphisms $\sigma_i^{\otimes}$ of R, and the $\sigma_i^{\otimes}$-derivations $\delta_i^{\otimes}$ of R for the extension $A \otimes_R A'$, are obtained using the injective endomorphisms σ_i and σ'_j, and the σ_i-derivations, and σ'_j-derivations of the extensions A and A', respectively. Therefore, Proposition 3.4 can be established in the following way: the tensor product of two skew PBW extensions $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$ and $A' = \sigma(R)\langle y_1, \ldots, y_m \rangle$ is given by the relations:

$$x_i r = c_{i,r} x_i + \delta_i(r), \quad 1 \leq i \leq n,$$

$$x_j x_i - c_{i,r} x_i x_j \in R + \sum_{l=1}^{n} R x_l, \quad 1 \leq i, \ j \leq n$$

and

$$y_j r = c'_{j,r} y_j + \delta'_j(r), \quad 1 \le i \le m,$$

$$y_j y_i - c'_{i,j} y_i y_j \in R + \sum_{p=1}^{m} R y_p, \quad 1 \le i, j \le m,$$

respectively, it is the left free algebra

$$A \otimes A' = R\langle x_1, \ldots, x_n \rangle / I,$$

where $R\langle x_1, \ldots, x_n \rangle$ is the free R-algebra, and I is the left ideal generated by the relations

$$(x_i \otimes 1) r - c_{i,r}(x_i \otimes 1) - \delta_i(r) \otimes 1, \quad 1 \le i \le n,$$

$$(1 \otimes y_j) r - c_{j,r}(1 \otimes y_j) - 1 \otimes \delta'_j(r), \quad 1 \le j \le m,$$

$$(x_j \otimes 1)(x_i \otimes 1) - c_{i,j}(x_i \otimes 1)(x_j \otimes 1) + R \otimes 1 + \sum_{l=1}^{n} R(x_l \otimes 1),$$

$$1 \le i, j \le n,$$

$$(1 \otimes y_j)(1 \otimes y_i) - c'_{i,j}(1 \otimes y_i)(1 \otimes y_j) + 1 \otimes R + \sum_{p=1}^{m} R(1 \otimes y_p),$$

$$1 \le i, j \le m,$$

$$(1 \otimes y_j)(x_i \otimes 1) - (x_i \otimes 1)(1 \otimes y_j), \quad 1 \le i \le n, 1 \le j \le m.$$

From Proposition 3.4, it follows the next result.

Corollary 3.6. *If* $\{A_i\}_{i \in I}$ *is a family of skew PBW extensions over the ring* R, *then* $\bigotimes_{i \in I} A_i$ *is also a skew PBW extension of* R.

Example 3.7. The Weyl algebra $A_{n+m}(k)$ is the skew PBW extension $A_n(k) \otimes_k A_m(k)$, which can be obtained by using Proposition 3.4. More precisely, since the Weyl algebra $A_n(k)$ is the left free k-algebra $k\langle x_1, ..., x_n, y_1, ..., y_n\rangle$ with ideal of relations generated by $x_j x_i - x_i x_j$, $y_j x_i - x_i y_j - \delta_{ij}$, $y_j y_i - y_i y_j$, for $1 \leq i < j \leq n$, and similarly $A_m(\Bbbk)$,

$$A_n(k) \otimes_k A_m(k) = k\langle (x_1 \otimes 1), ..., (x_n \otimes 1), (y_1 \otimes 1), ..., (y_n \otimes 1), (1 \otimes x'_1), ..., (1 \otimes x'_m), (1 \otimes y'_1), ..., (1 \otimes y'_m)\rangle / I,$$

where I is the left ideal generated by the relations

$$\begin{aligned} I = \langle &(x_j \otimes 1)(x_i \otimes 1) - (x_i \otimes 1)(x_j \otimes 1), \quad 1 \leq i < j \leq n, \\ &(y_j \otimes 1)(x_i \otimes 1) - (x_i \otimes 1)(y_j \otimes 1) - \delta_{ij} \otimes 1, \quad 1 \leq i < j \leq n, \\ &(y_j \otimes 1)(y_i \otimes 1) - (y_i \otimes 1)(y_j \otimes 1), \quad 1 \leq i < j \leq n, \\ &(1 \otimes x'_j)(x_i \otimes 1) - (x_i \otimes 1)(1 \otimes x'_j), \quad 1 \leq i \leq n, 1 \leq j \leq m, \\ &(1 \otimes y'_j)(x_i \otimes 1) - (x_i \otimes 1)(1 \otimes y'_j), \quad 1 \leq i \leq n, 1 \leq j \leq m, \\ &(1 \otimes x'_j)(y_i \otimes 1) - (y_i \otimes 1)(1 \otimes x'_j), \quad 1 \leq i \leq n, 1 \leq j \leq m, \\ &(1 \otimes y'_j)(y_i \otimes 1) - (y_i \otimes 1)(1 \otimes y'_j), \quad 1 \leq i \leq n, 1 \leq j \leq m, \\ &(1 \otimes x'_j)(1 \otimes x'_i) - (1 \otimes x'_i)(1 \otimes x'_j), \quad 1 \leq i < j \leq m, \\ &(1 \otimes y'_j)(1 \otimes x'_i) - (1 \otimes x'_i)(1 \otimes y'_j) - 1 \otimes \delta_{ij}, \quad 1 \leq i < j \leq m, \\ &(1 \otimes y'_j)(1 \otimes y'_i) - (1 \otimes y'_i)(1 \otimes y'_j), \quad 1 \leq i < j \leq m\rangle. \end{aligned}$$

If we identify $p_i := x_i \otimes 1\ (1 \leq i \leq n)$, $p_{n+i} := 1 \otimes x'_i\ (1 \leq i \leq m)$, $q_i := y_i \otimes 1\ (1 \leq i \leq n)$ and $q_{n+i} := 1 \otimes y'_i\ (1 \leq i \leq m)$, then we can see that the algebra $A_n(k) \otimes_k A_m(k)$ is precisely the Weyl algebra $A_{n+m}(k)$.

Next, we study the tensor product of skew PBW extensions whose coefficient rings are not necessarily the same. In this way, we generalize Proposition 3.4.

Proposition 3.8. *If* $A = \sigma(R)\langle x_1, ..., x_n\rangle$ *and* $A' = \sigma(R')\langle x_1, ..., x_n\rangle$ *are two skew PBW extensions over the k-algebras R and* R', *respectively*, *then* $A \otimes_k A'$ *is a skew PBW extension over* $R \otimes_k R'$.

Proof. Let A and A' be skew PBW extensions over the k-algebras R and R', respectively. From the definition, we know that

$$x_i r = \sigma_i(r)x_i + \delta_i(r), \quad 1 \leq i \leq n,$$

$$x_j x_i - c_{i,j}x_i x_j \in R + \sum_{l=1}^{n} Rx_l, \quad 1 \leq i, j \leq n,$$

$$y_j s = \sigma'_j(s)y_j + \delta'_j(s), \quad 1 \leq j \leq m,$$

$$y_j y_i - d_{i,j}y_i y_j \in R' + \sum_{l=1}^{m} R'y_l, \quad 1 \leq i, j \leq m,$$

where $\sigma_i, \delta_i : R \to R$ and $\sigma'_j, \delta'_j : R' \to R'$ are as in Proposition 2.3. We assume that the elements of k commute with every element of A and each element of A', so A and A' are k-algebras (this assumption, for example, was used in the computation of Gelfand-Kirillov dimension for these non-commutative rings, see [6]). Note that $A \otimes_k A'$ is a k-algebra with the product given by $(a \otimes a')(b \otimes b') = (ab) \otimes (a'b')$ [16, Proposition 2.60]. Moreover, $R \cong R \otimes_k k$, $R' \cong k \otimes_k R'$, $A \cong A \otimes_k k$ and $A' \cong k \otimes_k A'$. We endow $A \otimes_k A'$ with the natural structure of left $R \otimes_k R'$-module, i.e., $(r \otimes s) \cdot (a \otimes a') := (ra) \otimes (sa')$.

With the aim of showing that $A \otimes_k A'$ is a skew PBW extension of $R \otimes_k R'$, we consider the free algebra $(R \otimes_k R')/I$, where I is the left ideal generated by the relations

$$(x_i \otimes 1)(r \otimes 1) - (\sigma_i(r) \otimes 1)(x_i \otimes 1) - \delta_i(r) \otimes 1, \quad 1 \leq i \leq n,$$

$$(x_j \otimes 1)(x_i \otimes 1) - c_{i,j}(x_i \otimes 1)(x_j \otimes 1) - R \otimes 1 + \sum_{l=1}^{n} R(x_l \otimes 1),$$

$1 \leq i,\ j \leq n,$

$$(1 \otimes y_j)(1 \otimes s) - (1 \otimes \sigma'_j(s))(1 \otimes y_j) - 1 \otimes \delta'_i(s), \quad 1 \leq j \leq m,$$

$$(1 \otimes y_j)(1 \otimes y_i) - d_{i,j}(1 \otimes y_i)(1 \otimes y_j) - 1 \otimes S + \sum_{l=1}^{m} S(1 \otimes y_l),$$

$1 \leq i,\ j \leq m,$

$$(x_i \otimes 1)(1 \otimes s) - (1 \otimes s)(x_i \otimes 1), \quad 1 \leq i \leq n,$$

$$(1 \otimes y_j)(r \otimes 1) - (r \otimes 1)(1 \otimes y_j), \quad 1 \leq j \leq s,$$

$$(x_i \otimes 1)(1 \otimes y_j) - (1 \otimes y_j)(x_i \otimes 1), \quad 1 \leq i \leq n, 1 \leq j \leq m.$$

It is clear that $R \otimes_k R' \subseteq A \otimes_k A'$. Since $A \otimes_k A'$ is a left $R \otimes_k R'$-module, following the notation established in Definition 2.4, and using Remark 2.2(iv), we can see that $A \otimes_k A'$ is left free over $R \otimes_k R'$ (the proof is similar to that established in [15, Theorem 14.5]) and uses some arguments about the union of sets of basic monomials (see [7, Lemma 4.3] for details about this procedure) with basis

$$\mathrm{Mon}(A \otimes A') := \{(x_1 \otimes 1)^{\alpha_1} \cdots (x_n \otimes 1)^{\alpha_n} (1 \otimes y_1)^{\alpha_{n+1}} \cdots (1 \otimes y_m)^{\alpha_{n+m}}\}.$$

The injective endomorphims and the derivations of the skew PBW extension $A \otimes_k A$ are given by

$$\overline{\sigma_i} : R \otimes S \to R \otimes S, \quad \overline{\sigma_i}(r \otimes s) = \begin{cases} \sigma_i(r) \otimes s, & 1 \leq i \leq n, \\ r \otimes \sigma'_i(r), & n+1 \leq i \leq n+m \end{cases}$$

and

$$\overline{\delta_i} : R \otimes S \to R \otimes S, \quad \overline{\delta_i}(r \otimes s) = \begin{cases} \delta_i(r) \otimes s, & 1 \leq i \leq n, \\ r \otimes \delta'_i(s), & n+1 \leq i \leq n+m, \end{cases}$$

respectively. Note that the functions $\overline{\sigma_i}$ are injective endomorphisms because σ_i and σ'_j so are. Next, we show that $\overline{\delta_i}$ is a $\overline{\sigma_i}$-derivation for $1 \leq i \leq n$:

$$\begin{aligned}
\overline{\delta_i}((r \otimes s)(r' \otimes s')) &= \overline{\delta_i}(rr' \otimes ss') \\
&= \delta_i(rr') \otimes ss' \\
&= (\sigma_i(r)\delta_i(r') + \delta_i(r)r') \otimes ss' \\
&= \sigma_i(r)\delta_i(r') \otimes ss' + \delta_i(r)r' \otimes ss' \\
&= (\sigma_i(r) \otimes s)(\delta_i(r') \otimes s) + (\delta_i(r) \otimes s)(r' \otimes s') \\
&= \overline{\sigma_i}(r \otimes s)\overline{\delta_i}(r' \otimes s') + \overline{\delta_i}(r \otimes s)(r' \otimes s').
\end{aligned}$$

Similarly, we can see that $\overline{\delta_i}$ is a $\overline{\sigma_i}$-derivation, for $n + 1 \leq i \leq n + m$. □

Remark 3.9. Note that if f is a non-zero element of A and g is a non-zero element of A', then $\exp(f \otimes g) = (\exp(f), \exp(g)) \in \mathbb{N}^{n+m}$, where $\exp(f \otimes g)$ is obtained using an order of elimination, either A or A'.

4. Enveloping Algebra

The *opposite* of a ring is the ring with the same elements and addition operation, but with the multiplication performed in the reverse order. More precisely, the opposite of a ring $(B, +, \cdot)$ is the ring $(B, +, *)$, whose multiplication $*$ is defined by $a * b = b \cdot a$. In this section, we show that the enveloping algebra of a bijective skew PBW extension A is again a PBW extension. We recall that if B is a k-algebra, then the *enveloping algebra* of B is $B^e := B \otimes_k B^{\mathrm{op}}$, where B^{op} is the opposite algebra of B.

Proposition 4.1. *If A is a bijective skew PBW extension over R, then A^{op} is a bijective skew PBW extension over R^{op}. In fact, for A^{op}, we have* *the automorphisms* $\delta_i^{\text{op}} : R^{\text{op}} \to R^{\text{op}}$ *given by* $\sigma_i^{\text{op}} := \sigma_i^{-1}(r)$, *and the* σ_i^{op}*-derivations* $\delta_i^{\text{op}} : R^{\text{op}} \to R^{\text{op}}$ *defined by* $\delta_i^{\text{op}}(r) := -\delta_i(\sigma_i^{-1}(r))$, *for every element* $r \in R^{\text{op}}$.

Proof. Let $A = \sigma(R)\langle x_1, \dots, x_n \rangle$ be a bijective skew PBW extension of R. We will verify the four conditions of Definition 2.1 for the rings R^{op} and A^{op}:

(i) It is clear that $R^{\text{op}} \subseteq A^{\text{op}}$.

(ii) Since A is a left free R-module with basis

$$\text{Mon}(A) := \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\},$$

by the definition of the product in A^{op}, we have that A^{op} is a free right R-module whose basis has the basic elements described in the set:

$$\text{Mon}(A^{\text{op}}) := \{x^{\alpha^{\text{op}}} = x_n^{\alpha_n} \cdots x_1^{\alpha_1} \mid \alpha^{\text{op}} = (\alpha_n, \dots, \alpha_1) \in \mathbb{N}^n\}.$$

Hence, A^{op} is a left free R^{op}-module.

(iii) We will see that for each $1 \le i \le n$, and for every $r \in R^{\text{op}} \setminus \{0\}$, there exists $c'_{i,r} \in R^{\text{op}} \setminus \{0\}$ such that $rx_i - x_i c'_{i,r} \in R^{\text{op}}$. Put $c'_{i,r} := \sigma_i^{-1}(r)$. Given that

$$x_i c'_{i,r} = x_i \sigma_i^{-1}(r) = \sigma_i(\sigma_i^{-1}(r))x_i + \delta_i(\sigma_i^{-1}(r)) = rx_i + \delta_i(\sigma_i^{-1}(r)),$$

we have that $rx_i - x_i c'_{i,r} = -\delta_i(\sigma_i^{-1}(r)) \in R^{\text{op}}$.

(iv) Let $c'_{i,j} := \sigma_i^{-1}(\sigma_j^{-1}(c_{i,j}^{-1}))$. Then

$$\begin{aligned} x_i x_j - x_j x_i c'_{i,j} &= x_i x_j - x_j x_i \sigma_i^{-1}(\sigma_j^{-1}(c_{i,j}^{-1})) \\ &= x_i x_j - x_j[\sigma_i(\sigma_i^{-1}(\sigma_j^{-1}(c_{i,j}^{-1})))x_i + \delta_i(\sigma_i^{-1}(\sigma_j^{-1}(c_{i,j}^{-1})))] \end{aligned}$$

$$= x_i x_j - x_j \sigma_j^{-1}(c_{i,j}^{-1}) x_i - x_j \delta_i(\sigma_i^{-1}(\sigma_j^{-1}(c_{i,j}^{-1})))$$

$$= x_i x_j - [\sigma_j(\sigma_j^{-1}(c_{i,j}^{-1})) x_j + \delta_j(\sigma_j^{-1}(c_{i,j}^{-1}))] x_i$$

$$- x_j \delta_i(\sigma_i^{-1}(\sigma_j^{-1}(c_{i,j}^{-1})))$$

$$= x_i x_j - c_{i,j}^{-1} x_j x_i - \delta_j(\sigma_j^{-1}(c_{i,j}^{-1})) x_i$$

$$- x_j \delta_i(\sigma_i^{-1}(\sigma_j^{-1}(c_{i,j}^{-1}))). \tag{4.1}$$

From Definition 2.1(iv), we have that $x_j x_i - c_{i,j} x_i x_j = r^{(i,j)} + \sum_{l=1}^{n} r_l^{(i,j)} x_l$, whence $c_{i,j}^{-1} x_j x_i = x_i x_j + c_{i,j}^{-1} r^{(i,j)} + \sum_{l=1}^{n} c_{i,j}^{-1} r_l^{(i,j)} x_l$. So, by replacing the term $c_{i,j}^{-1} x_j x_i$ in the above expression (4.1), we have that

$$x_i x_j - x_j x_i c'_{i,j} = -c_{i,j}^{-1} r^{(i,j)} - \left(\sum_{l=1}^{n} c_{i,j}^{-1} r_l^{(i,j)} x_l \right) - \delta_j(\sigma_j^{-1}(c_{i,j}^{-1})) x_i$$

$$- x_j \delta_i(\sigma_i^{-1}(\sigma_j^{-1}(c_{i,j}^{-1})))$$

$$= -c_{i,j}^{-1} r^{(i,j)} - \left(\sum_{l=1}^{n} x_l \sigma_l^{-1}(c_{i,j}^{-1} r_l^{(i,j)}) - \delta_l(\sigma_l^{-1}(c_{i,j}^{-1} r_l^{(i,j)})) \right)$$

$$- [x_i \sigma_i^{-1}(\delta_j(\sigma_j^{-1}(c_{i,j}^{-1}))) - \delta_i(\sigma_i^{-1}(\delta_j(\sigma_j^{-1}(c_{i,j}^{-1}))))]$$

$$- x_j \delta_i(\sigma_i^{-1}(\sigma_j^{-1}(c_{i,j}^{-1})))$$

$$= -c_{i,j}^{-1} r^{(i,j)} + \left(\sum_{l=1}^{n} \delta_l(\sigma_l^{-1}(c_{i,j}^{-1} r_l^{(i,j)})) \right)$$

$$+ \delta_i(\sigma_i^{-1}(\delta_j(\sigma_j^{-1}(c_{i,j}^{-1}))))$$

$$- \sum_{l=1, l \neq i, j} x_l \sigma_l^{-1}(c_{i,j}^{-1} r_l^{(i,j)}) - x_i [\sigma_i^{-1}(c_{i,j}^{-1} r_i^{(i,j)})$$

$$+\sigma_i^{-1}(\delta_j(\sigma_j^{-1}(c_{i,j}^{-1})))]$$

$$-x_j[\sigma_j^{-1}(c_{i,j}^{-1}r_j^{(i,j)}) + \delta_i(\sigma_i^{-1}(\sigma_j^{-1}(c_{i,j}^{-1})))],$$

which shows that $x_ix_j - x_jx_ic'_{i,j} \in R + x_1R + \cdots + x_nR.$

Finally, let r and r' be elements of R^{op}. Then we have:

$$\sigma_i^{\text{op}}(r + r') = \sigma_i^{-1}(r + r') = \sigma_i^{-1}(r) + \sigma_i^{-1}(r') = \sigma_i^{\text{op}}(r) + \sigma_i^{\text{op}}(r'),$$

$$\sigma_i^{\text{op}}(1_{R^{\text{op}}}) = \sigma_i^{\text{op}}(1_R) = \sigma_i^{-1}(1_R) = 1_R = 1_{R^{\text{op}}},$$

$$\sigma_i^{\text{op}}(rr') = \sigma_i^{-1}(r'r) = \sigma_i^{-1}(r')\sigma_i^{-1}(r) = \sigma_i^{-1}(r)\sigma_i^{-1}(r') = \sigma_i^{\text{op}}(r)\sigma_i^{\text{op}}(r').$$

Given that σ_i is injective and surjective, so it is σ_i^{op}, for every $1 \le i \le n$. With respect to the functions δ_i^{op}, we have

$$\begin{aligned}\delta_i^{\text{op}}(r + r') &= -\delta_i(\sigma_i^{-1}(r + r')) = -\delta_i(\sigma_i^{-1}(r) + \sigma_i^{-1}(r'))\\ &= -\delta_i(\sigma_i^{-1}(r)) - \delta_i(\sigma_i^{-1}(r'))\\ &= \delta_i^{\text{op}}(r) + \delta_i^{\text{op}}(r'),\end{aligned}$$

and using the product on R^{op},

$$\begin{aligned}\delta_i^{\text{op}}(rr') &= -\delta_i(\sigma_i^{-1}(r'r)) = -\delta_i(\sigma_i^{-1}(r')\sigma_i^{-1}(r))\\ &= -[\sigma_i(\sigma_i^{-1}(r'))\delta_i(\sigma_i^{-1}(r')) + \delta_i(\sigma_i^{-1}(r'))\sigma_i^{-1}(r)]\\ &= -r'\delta_i(\sigma_i^{-1}(r')) - \delta_i(\sigma_i^{-1}(r'))\sigma_i^{-1}(r)\\ &= \sigma_i^{-1}(r)\,(-\delta_i(\sigma_i^{-1}(r'))) + (-\delta_i(\sigma_i^{-1}(r)))r'\\ &= \sigma_i^{\text{op}}(r)\delta_i^{\text{op}}(r') + \delta_i^{\text{op}}(r)r',\end{aligned}$$

which concludes the proof. □

Remark 4.2. We note also that A^{op} is a skew PBW extension over R, where the elements of Definition 2.1 are written in reverse order, and $\preceq^{\mathrm{op}}$ is the order given by $\alpha \preceq^{\mathrm{op}} \beta$ if and only if $\alpha^{\mathrm{op}} \leq \beta^{\mathrm{op}}$. So, we can see that the set $\mathrm{Mon}(A^{\mathrm{op}}) = \{x_n^{\alpha_n} \cdots x_1^{\alpha_1} \mid \alpha^{\mathrm{op}} = (\alpha_n, \ldots, \alpha_1) \in \mathbb{N}^n\}$ is a free R-basis of A^{op}.

Theorem 4.3. *If A is a bijective skew PBW extension over R, then A^e is a bijective skew PBW extension over R^e.*

Proof. First of all, it is clear that $R^e \subseteq A^e$. Second, given that A is a left free R-module, we have that

$$A \cong R^{|\mathrm{Mon}(A)|} \cong (R \otimes_R R)^{|\mathrm{Mon}(A)|} \cong (R \otimes_R R^{\mathrm{op}})^{|\mathrm{Mon}(A)|}.$$

Similarly, since that A^{op} is right R^{op}-free,

$$A^{\mathrm{op}} \cong (R^{\mathrm{op}})^{|\mathrm{Mon}(A^{\mathrm{op}})|} \cong (R^{\mathrm{op}} \otimes_{R^{\mathrm{op}}} R^{\mathrm{op}})^{|\mathrm{Mon}(A^{\mathrm{op}})|}$$

$$\cong (R \otimes_R R^{\mathrm{op}})^{|\mathrm{Mon}(A^{\mathrm{op}})|},$$

which shows that A and A^{op} are left free $R \otimes_R R^{\mathrm{op}}$-modules, so $A^e = A \otimes A^{\mathrm{op}}$ is also a left free $R \otimes R^{\mathrm{op}}$-module with basis $\mathrm{Mon}(A) \otimes \mathrm{Mon}(A^{\mathrm{op}})$. Hence,

$$\mathrm{Mon}(A^{\mathrm{e}}) = \{(x_1 \otimes 1)^{\alpha_1} \cdots (x_n \otimes 1)^{\alpha_n} (1 \otimes x_n)^{\alpha_{n+1}}$$

$$\cdots (1 \otimes x_1)^{\alpha_{2n}} \mid (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^{2n}\}.$$

Note that the automorphisms $\overline{\sigma_i}$ and the $\overline{\sigma_i}$-derivations of A^e, for $1 \leq i \leq n$, are given by

$$\overline{\sigma_i} : R \otimes R^{\mathrm{op}} \to R \otimes R^{\mathrm{op}}, \quad \overline{\sigma_i}(r \otimes r') = \begin{cases} \sigma_i(r) \otimes r', & 1 \leq i \leq n, \\ r \otimes \sigma_i'(r), & n+1 \leq i \leq 2n \end{cases}$$

and

$$\overline{\delta_i} : R \otimes R^{\mathrm{op}} \to R \otimes R^{\mathrm{op}}, \quad \overline{\delta_i}(r \otimes r') = \begin{cases} \delta_i(r) \otimes r', & 1 \leq i \leq n, \\ r \otimes \delta_i'(r'), & n+1 \leq i \leq 2n. \end{cases}$$

In this way, the conditions (iii) and (iv) of Definition 2.1 follow from Propositions 3.4 and 4.1, and Remark 3.5. □

5. Skew Calabi-Yau Algebras

Suppose that M and N are both B^e-modules. Then there are two B^e-module structures on $M \otimes N$. One of them is called the *outer structure* defined by $(a \otimes b) \cdot (m \otimes n) \overset{\text{out}}{=} am \otimes nb$, and the other is called the *inner structure* defined by $(a \otimes b) \cdot (m \otimes n) \overset{\text{int}}{=} ma \otimes bn$, for any $a, b \in B$, $m \in M$, $n \in N$. Since B^e is identified with $B \otimes B$ as a $\Bbbk$-module $({}_{\Bbbk}B^e = {}_{\Bbbk}(B \otimes B^{\mathrm{op}}) = {}_{\Bbbk}(B \otimes B))$, $B \otimes B$ endowed with the outer structure is nothing but the left regular B^e-module B^e. ${}_{B^e}(B \otimes B) \overset{\text{out}}{=} {}_{B^e} B^e$: for ${}_{B^e}(B \otimes B)$, $(a \otimes b) \cdot (x \otimes y) = a \cdot (x \otimes y) \cdot b \overset{\text{out}}{=} ax \otimes yb$, whereas that in ${}_{B^e} B^e (a \otimes b) \cdot (x \otimes y) = ax \otimes b \circ y = ax \otimes yb$. $B \otimes B$ endowed with the inner structure is nothing but the right regular B^e-module B^e. ${}_{B^e}(B \otimes B)$ $\overset{\text{int}}{=} B^e_{B^e}$: for ${}_{B^e}(B \otimes B)$, $(a \otimes b) \cdot (x \otimes y) = a \cdot (x \otimes y) \cdot b \overset{\text{int}}{=} xa \otimes by$, whereas that in $B^e_{B^e}$, $(x \otimes y) \cdot (a \otimes b) = xa \otimes y \circ b = xa \otimes by$. Hence, we often say that B^e has the *outer* (*left*) and *inner* (*right*) B^e*-module structures*.

An algebra B is said to be *homologically smooth* if as an B^e-module, B has a finitely generated projective resolution of finite length. The length of this resolution is known as the *Hochschild dimension* of B (in [12], the authors considered this dimension to compute the cyclic homology of skew PBW extensions). In the next definition, the outer structure on B^e is used when computing the homology $\operatorname{Ext}^*_{B^e}(B, B^e)$. Thus, $\operatorname{Ext}^*_{B^e}(B, B^e)$ admits a B^e-module structure induced by the inner one on B^e.

Definition 5.1. An algebra B is called *skew Calabi-Yau of dimension d*, if the following conditions hold:

(i) B is homologically smooth.

(ii) There exists an algebra automorphism ν of B such that $\operatorname{Ext}^i_{B^e}(B, B^e)$ $\cong \begin{cases} 0, & i \neq d \\ B^{\nu}, & i = d \end{cases}$ as B^e-modules.

If ν is the identity, then B is said to be *Calabi-Yau*.

The automorphism ν is called the *Nakayama automorphism* of B, and it is unique up to inner automorphisms of B. Note that a skew Calabi-Yau algebra is Calabi-Yau if and only if its Nakayama automorphism is inner.

Definition 5.2. Let $B = \Bbbk \oplus B_1 \oplus B_2 \oplus \cdots$ be a finitely presented graded algebra over a field $\Bbbk$. The algebra B will be called *AS-regular*, if it has the following properties:

(i) B has finite global dimension d, i.e., every graded B-module has a projective dimension less than or equal to d;

(ii) B has finite Gelfand-Kirillov dimension;

(iii) B is Gorenstein, meaning that $\operatorname{Ext}^i_B(\Bbbk, B) = 0$ if $i \neq d$, and $\operatorname{Ext}^d_B(\Bbbk, B) \cong \Bbbk$.

The dimension of Gelfand-Kirillov and the notion of Gorenstein for skew PBW extensions were studied in [6] and [5], respectively. Now, from Definition 5.2, we can see that we need to consider graded algebras, and since, in general, skew PBW extensions are not graded rings, in the next definition, we impose three conditions to guarantee a notion of grade in these extensions. More exactly,

Definition 5.3 [18, Definition 2.6]**.** Let $A = \sigma(R)\langle x_1, ..., x_n\rangle$ be a bijective skew PBW extension over a $\mathbb{N}$-graded algebra R. We said that A is a *graded skew PBW extension*, if the following conditions hold:

(i) the indeterminates $x_1, ..., x_n$ have degree 1 in A;

(ii) σ_i is a graded ring homomorphism and $\delta_i : R(-1) \to R$ is a graded σ_i-derivation, for all $1 \leq i \leq n$, where σ_i and δ_i are established in Proposition 2.3;

(iii) $x_j x_i - c_{i,j} x_i x_j \in R_2 + R_1 x_1 + \cdots + R_1 x_n$, as in Definition 2.1(iv), and $c_{i,j} \in R_0$.

For the next proposition, consider the notation established in Definition 2.4.

Proposition 5.4 [18, Proposition 2.7]**.** *Let A be a graded skew PBW extension over R, and let A_p be the $\mathbb{k}$-space generated by the set $\{r_i x^{\alpha} \mid t + |\alpha| = p,\, r_t \in R_t$ and $x^{\alpha} \in \text{Mon}(A)\}$, for $p \geq 0$. Then A is a graded algebra with graduation given by $A = \bigoplus_{p \geq 0} A_p$.*

Next theorem is one of the most important results of this paper. This theorem establishes that quasi-commutative skew PBW extensions over connected skew Calabi-Yau algebras are skew Calabi-Yau.

Theorem 5.5. *If A is a graded quasi-commutative skew PBW extension over a connected skew Calabi-Yau $\mathbb{k}$-algebra R, then A is skew Calabi-Yau.*

Proof. Note that A is isomorphic to an iterated Ore extension of endomorphism type $R[z_1, \theta_1]\cdots[z_n, \theta_n]$, where θ_i is bijective; $\theta_1 = \sigma_1$;

$$\theta_j : R[z_1; \theta_1]\cdots[z_{j-1}; \theta_{j-1}] \to R[z_1; \theta_1]\cdots[z_{j-1}; \theta_{j-1}]$$

is such that $\theta_j(z_1) = c_{i,j} z_i$ ($c_{i,j} \in R$ as in Definition 2.1), $1 \leq i < j \leq n$

and $\theta_i(r) = \sigma_i(r)$, for $r \in R$ [3, Theorem 2.3]. Since A is graded, σ_i is graded and $c_{i,j} \in R_0$. Now, using that $\theta_i(r) = \sigma_i(r)$ and $\theta_j(z_i) = c_{i,j}z_i$, we have that θ_i is a graded automorphism, for every i. Without loss of generality, we can assume that $z_i = x_i$, for every $1 \leq i \leq n$. Therefore, A is isomorphic to a graded iterated Ore extension, and using that R is AS-regular [14, Lemma 1.2], then A is AS-regular [4]. Since R is connected, A is also connected [18, Remark 2.10]. From [14, Lemma 1.2], we conclude that A is skew Calabi-Yau. □

Example 5.6. Theorem 5.5 allows us to obtain the following examples of skew PBW extensions which are skew Calabi-Yau algebras:

(1) For a fixed $q \in \Bbbk\backslash\{0\}$, the $\Bbbk$-*algebra of linear partial q-dilation operators* with the polynomial coefficients is $\Bbbk[t_1, \ldots, t_n][H_1^{(q)}, \ldots, H_m^{(q)}]$, $n \geq m$, subject to the relations: $t_jt_i = t_it_j$, $1 \leq i < j \leq n$; $H_i^{(q)}t_i = qt_iH_i^{(q)}$, $1 \leq i \leq m$; $H_j^{(q)}t_i = t_iH_j^{(q)}$, $i \neq j$; $H_j^{(q)}H_i^{(q)} = H_i^{(q)}H_j^{(q)}$, $1 \leq i < j \leq m$ (see [3, Subsection 3.3]). This algebra is a graded quasi-commutative skew PBW extension of $\Bbbk[t_1, \ldots, t_n]$, where $\Bbbk[t_1, \ldots, t_n]$ is endowed with usual graduation.

(2) The *quantum polynomial ring* $\mathcal{O}_n(\lambda_{ji})$ (also known as the *multiplicative analogue of the Weyl algebra*) is the algebra generated by the indeterminates $x_1, \ldots, x_n$ subject to the relations $x_jx_i = \lambda_{ji}x_ix_j$, $1 \leq i < j \leq n$, $\lambda_{ji} \in \Bbbk\backslash\{0\}$. In [3, Subsection 3.5], it was proved that $\mathcal{O}_n(\lambda_{ji}) \cong \sigma(\Bbbk)\langle x_1, \ldots, x_n\rangle \cong \sigma(\Bbbk[x_1])\langle x_2, \ldots, x_n\rangle$.

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References

[1] C. Gallego and O. Lezama, Gröbner bases for ideals of σ-PBW extensions, Comm. Algebra 39(1) (2011), 50-75.

[2] O. Lezama, J. P. Acosta and A. Reyes, Prime ideals of skew PBW extensions, Rev. Un. Mat. Argentina 56(2) (2015), 39-55.

[3] O. Lezama and A. Reyes, Some homological properties of skew PBW extensions, Comm. Algebra 42(3) (2014), 1200-1230.

[4] L.-Y. Liu, S. Wang and Q.-S. Wu, Twisted Calabi-Yau property of Ore extensions, J. Noncommut. Geom. 8(2) (2014), 587-609.

[5] A. Reyes, Ring and module theoretical properties of skew PBW extensions, Ph.D. Thesis, Universidad Nacional de Colombia, Bogotá, 2013, 142 pp.

[6] A. Reyes, Gelfand-Kirillov dimension of skew PBW extensions, Rev. Colombiana Mat. 47(1) (2013), 95-111.

[7] A. Reyes, Uniform dimension over skew PBW extensions, Rev. Colombiana Mat. 48(1) (2014), 79-96.

[8] A. Reyes, Jacobson's conjecture and skew PBW extensions, Rev. Integr. Temas Mat. 32(2) (2014), 139-152.

[9] A. Reyes, Skew PBW extensions of Baer, quasi-Baer, p.p. and p.q.-rings, Rev. Integr. Temas Mat. 33(2) (2015), 173-189.

[10] A. Reyes and H. Suárez, A note on zip and reversible skew PBW extensions, Bol. Mat. (N.S.) 23(1) (2016), 71-79.

[11] A. Reyes and H. Suárez, Armendariz property for skew PBW extensions and their classical ring of quotients, Rev. Integr. Temas Mat. 34(2) (2016), 147-168.

[12] A. Reyes and H. Suárez, Some remarks about cyclic homology of skew PBW extensions, Ciencia en Desarrollo 7(2) (2016), 99-107.

[13] A. Reyes and H. Suárez, PBW bases for some 3-dimensional skew polynomial algebras, Far East J. Math. Sci. (FJMS) 101(6) (2017), 1207-1228.

[14] M. Reyes, D. Rogalski and J. Zang, Skew Calabi-Yau algebras and homological identities, Adv. in Math. 264 (2014), 308-354.

[15] S. Roman, Advanced linear algebra, 3rd ed., Graduate Texts in Mathematics, Springer, 2008.

[16] J. Rotman, An Introduction to Homological Algebra, 2nd ed., Universitext, Springer, 2009.

[17] L. H. Rowen, Graduate algebra: noncommutative view, Graduate Studies in Mathematics, Vol. 91, American Mathematical Society, Providence, RI, 2008.

[18] H. Suárez, Koszulity for graded skew PBW extensions, Comm. Algebra 45(10) (2017), 4569-4580.

[19] H. Suárez, O. Lezama and A. Reyes, Some relations between N-Koszul, Artin-Schelter regular and Calabi-Yau algebras with skew PBW extensions, Ciencia en Desarrollo 6(2) (2015), 205-213.

[20] H. Suárez and A. Reyes, A generalized Koszul property for skew PBW extensions, Far East J. Math. Sci. (FJMS) 101(2) (2017), 301-320.

[21] H. Suárez and A. Reyes, Koszulity for skew PBW extensions over fields, JP J. Algebra Number Theory Appl. 39(2) (2017), 181-203.

[22] Q.-S. Wu and C. Zhu, Skew group algebras of Calabi-Yau algebras, J. Algebra 340 (2011), 53-76.

[23] C. Zhu, F. Van Oystaeyen and Y. Zhang, Nakayama automorphism of double Ore extensions of Koszul regular algebras, Manuscripta Math. 152(3) (2017), 555-584.