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ON RAMSEY NUMBERS OF CYCLES WITH RESPECT EVEN WHEELS OF TWO HUBS

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Abstract

For given graphs G and H, the Ramsey number $R(G, H)$ is the smallest positive integer N such that for every graph F of order N the following holds: either F contains G as a subgraph or the complement of F contains H as a subgraph. In this paper, we determine the Ramsey numbers of cycles with respect to even wheels of two hubs: $R(C_n, W_{2,m}) = 3n - 2$ for even $m \geq 4$ and $n \geq \frac{9m}{2} + 1$.

1. Introduction

Throughout the paper, all graphs are finite and simple. Let G be such a graph. We write $V(G)$ or V for the vertex set of G and $E(G)$ or E for the edge set of G. For given graphs G and H, the *Ramsey number* $R(G, H)$ is the smallest positive integer N such that for every graph F of order N the following holds: either F contains G as a subgraph or the complement of F contains H as a subgraph. Since then the Ramsey numbers $R(G, H)$ for many combinations of graphs G and H have been extensively studied by various authors, see nice survey paper "small Ramsey numbers" in [8]. In particular, the Ramsey numbers for combination involving cycles and wheels have also been investigated.

Let C_n be a cycle of n vertices and $W_{1,m}$ be the join $K_1 + C_m$. It is called a *wheel* with m spokes. Burr and Erdös [3] showed that $R(C_3, W_{1,m}) = 2m + 1$ for each $m \geq 5$. Ten years later Radziszowski and Xia [9] gave a simple and unified method to establish the Ramsey number $R(C_3, G)$, where G is either a path, a cycle or a wheel. Surahmat et al. [12] showed $R(C_4, W_{1,m}) = 9, 10$ and 9 for $m = 4, 5$ and 6, respectively. Independently, Tse [14] showed $R(C_4, W_{1,m}) = 9,$ 10, 9, 11, 12, 13, 14, 15 and 17 for m = 4, 5, 6, 7, 8, 9, 10, 11 and 12, respectively. Recently, in [11], the Ramsey numbers of cycles versus small wheels were obtained, e.g., $R(C_n, W_{1,4}) = 2n - 1$ for $n \geq 5$ and $R(C_n, W_{1,5}) = 3n - 2$ for $n \geq 5$.

The aim of this paper is to determine the Ramsey number of cycles C_n with respect to wheels of two hubs $W_{2,m}$. The main result of this paper is the following.

Theorem. $R(C_n, W_{2,m}) = 3n - 2$ *for even* $m \geq 4$ *and* $n \geq \frac{9m}{2} + 1$.

Before proving the Theorem let us present some notation used in this note. For $x \in V$ and a subgraph B of G, define $N_B(x) = \{y \in V(B) : xy \in E\}$ and $N_B[x] = N_B(x) \cup \{x\}$. The degree $d_G(x)$ of a vertex x is $|N_G(x)|$; $\delta(G)$ denotes the minimum degree in G. For any nonempty subset $S \subset V$, the *subgraph induced* by S is the maximal subgraph of G with the vertex set S, it is denoted by $G[S]$. A *cycle* C_n of length $n \geq 3$ is a connected graph on n vertices in which every vertex has degree two. A *wheel* $W_{1,n} = K_1 + C_n$ is a graph on $n+1$ vertices obtained from a C_n by adding one vertex x, called the *hub* of the wheel, and making x adjacent to all vertices of C_n, called the *rim* of the wheel. A *wheel* of t-hubs $W_{t,n} = K_t + C_n$ is a graph on $n + t$ vertices obtained form a cycle C_m by adding a complete graph K_t and making vertices of K_t adjacent to all vertices of C_n.

If G contains cycles, let $c(G)$ be the *circumference* of G, that is, the length of a longest cycle, and $g(G)$ be the *girth* of G, that is, the length of a shortest cycle. A graph on n vertices is *pancyclic* if it contains cycles of every length l, $3 \leq l \leq n$. A graph is *weakly pancyclic* if it contains cycles of length from the girth to the circumference.

We will also use the short notations $H \subseteq F$, $F \supseteq H$, $H \nsubseteq F$, and $F \nsupseteq H$ to denote that H is (is not) a subgraph of F, with the obvious meanings.

For given graphs G and H, Chvátal and Harary [5] established the lower bound $R(G, H) \geq (C(G) - 1)(\chi(H) - 1) + 1$, where $C(G)$ is the number of vertices of the largest component of G and $\chi(H)$ is the chromatic number of

H. In particular, if $G = C_n$ and $H = W_{2,m}$ for even m, then we have $R(C_n, W_{2,m}) \geq 3n - 2$. In order to prove this Theorem, we need the following known results and lemmas.

2. Some Lemmas

Some lemmas in what follows will be used to prove the main result of this paper.

Proposition 1 (Faudree and Schelp [7], Rosta [10]).

$$R(C_n, C_m)$$

$$= \begin{cases} 2n - 1 \text{ for } 3 \leq m \leq n,\ m \text{ odd},\ (n, m) \neq (3, 3). \\ n + \frac{m}{2} - 1 \text{ for } 4 \leq m \leq n,\ m \text{ even and } n \text{ even},\ (n, m) \neq (4, 4). \\ \max\left\{n + \frac{m}{2} - 1,\ 2m - 1\right\} \text{ for } 4 \leq m < n,\ m \text{ even and } n \text{ odd}. \end{cases}$$

Theorem 1 (Surahmat et al. [13]). *$R(C_n, W_{1,m}) = 2n - 1$ for even $m \geq 4$ and $n \geq \frac{5m}{2} - 1$.*

Lemma 1 (Bondy [1]). *Let G be a graph of order n. If $\delta(G) \geq \frac{n}{2}$, then either G is pancyclic or n is even and $G \cong K_{\frac{n}{2}, \frac{n}{2}}$.*

Lemma 2 (Brandt et al. [2]). *Every non-bipartite graph G of order n with $\delta(G) \geq \frac{n+2}{3}$ is weakly pancyclic and has girth 3 or 4.*

Lemma 3 (Dirac [6]). *Let G be a 2-connected graph of order $n \geq 3$ with $\delta(G) = \delta$. Then $c(G) \geq \min\{2\delta, n\}$.*

Lemma 4. *Let F be a graph with $|V(F)| \geq R(C_n, W_{1,m}) + 1$. If there is a vertex $x \in V(F)$ such that $|N_F[x]| \leq |V(F)| - R(C_n, W_{1,m})$ and $F \not\supseteq C_n$, then $\overline{F} \supseteq W_{2,m}$.*

Proof. Let $A = V(F) \backslash N_F[x]$ and so $|A| \geq R(C_n, W_{1,m})$. Since the subgraph $F[A]$ of F induced by A contains no C_n, by the definition of $R(C_n, W_{1,m})$ we get $\overline{F}[A] \supseteq W_{1,m}$ and hence $\overline{F}$ contains a $W_{2,m}$. □

Lemma 5 (Chvátal and Erdös [4], Zhou [15])**.** *If $H = C_s \subseteq F$ for a graph F, while $F \not\supseteq C_{s+1}$ and $\overline{F} \not\supseteq K_r$, then $|N_H(x)| \leq r - 2$ for each* $x \in V(F) \setminus V(H)$.

3. Proof of Theorem

Proof of Theorem. Let G be a graph of order $3n - 2$, where $n \geq \frac{9m}{2} + 1$ for even $m \geq 4$ and containing no C_n. We shall show that $\overline{G}$ contains $W_{2,m}$. By contradiction, suppose $\overline{G}$ contains no $W_{2,m}$. By Lemma 4, we have $\delta(G) \geq n - 1$ since $|N_G[x]| > V(F) - R(C_n, W_{1,m}) = (3n - 2) - (2n - 1) = n - 1$ for any $x \in V(G)$. Now we shall distinguish two cases below.

Case 1. $\delta(G) \geq n$.

Subcase 1.1. G is non-bipartite.

Since $\delta(G) \geq n = \frac{(3n - 2) + 2}{3}$, by Lemma 2, we get that G is weakly pancyclic with girth 3 or 4.

If $\kappa(G) \geq 2$, then G is a 2-connected graph. By Lemma 3, we have $c(G) \geq \min\{2n, 3n - 2\}$. This implies that G contains C_n, a contradiction.

Let $\kappa(G) = 1$. There exists a cut-vertex $v \in V(G)$ such that $G - v$ is disconnected. Let $G_1, ..., G_r$ be the components of $G - v$. Since $\delta(G) \geq n$ we deduce $\delta(G_i) \geq n - 1$, hence $|V(G_i)| \geq n$ for every $i = 1, 2, ..., r$. This implies $r = 2$ and $G - v$ has two components G_1 and G_2, such that $|V(G_1)| + |V(G_2)| = 3n - 3$. This implies that, we have at least one component, say G_1, such that $|V(G_1)| \leq \frac{3n-3}{2}$. So, we find $\delta(G_1) \geq n - 1$ $\geq \frac{\frac{3n-3}{2}}{2} \geq \frac{|V(G_1)|}{2}$. Now Lemma 1 applies to G_1, hence G_1 is pancyclic and G_1 contains C_n, a contradiction, or $G_1 \cong K_{\frac{p}{2}, \frac{p}{2}}$, where $p = |V(G_1)|$ $\geq n$ is even. Since $n \geq \frac{9m}{2} + 1$ and $m \geq 4$, $\frac{p}{2} \geq \frac{9m+2}{4} \geq m + 2$, so we deduce that $\overline{G} \supseteq W_{2,m}$, a contradiction.

Let $\kappa(G) = 0$. Then G is disconnected and we deduce as above that G has exactly two components, G_1 and G_2. Since $\delta(G) \geq n$, we deduce $|V(G_i)| \geq n + 1$ for each $i \in \{1, 2\}$. Suppose $|V(G_1)| \leq |V(G_2)|$, which implies $|V(G_1)| \leq \frac{3n-2}{2}$. We find that $\delta(G_1) \geq n > \frac{\frac{3n-2}{2}}{2} \geq \frac{|V(G_1)|}{2}$. By Lemma 1, we get that G_1 is either pancyclic and so $G_1 \supseteq C_n$, a contradiction, or $G_1 \cong K_{\frac{p}{2}, \frac{p}{2}}$, where $p = |V(G_1)| \geq n + 1$ is even. Since $n \geq \frac{9m}{2} + 1$ and $m \geq 4$, $\frac{p}{2} \geq \frac{n+1}{2} \geq \frac{9m+4}{4} \geq m + 2$, so we deduce that $\overline{G} \supseteq W_{2,m}$, a contradiction.

Subcase 1.2. G is bipartite.

Since G is bipartite and $\delta(G) \geq n$, we deduce that G is a spanning subgraph of $K_{j,t}$ for $j \geq n$ and $t \geq n$. This implies $\overline{G} \supseteq W_{2,m}$, a contradiction, since $E(\overline{G}) \supseteq E(K_j) \cup E(K_t)$ and $n \geq \frac{9m}{2} + 1 > m + 2$.

Case 2. $\delta(G) = n - 1$.

Let $x \in V(G)$ such that $|N_G(x)| = \delta(G) = n - 1$. Let H be the subgraph of G induced by $N_G(x)$. Let $A = V(G) \backslash N_G[x]$. So, we have $|A| = 2n - 2$.

Let T be the subgraph of G induced by A. Now, we shall consider in what follows two subcases.

Subcase 2.1. $\delta(T) < n - \frac{m}{2} - 3.$

Let $y \in V(T)$ such that $|N_T(y)| = \delta(T) < n - \frac{m}{2} - 3.$ Let $B = V(T) \setminus N_T[y].$ So, we have $|B| \geq (2n-2) - (n - \frac{m}{2} - 2) = n + \frac{m}{2} > n + \frac{m}{2} - 1 > 2m - 1.$ Since by Proposition 1, we have $R(C_n, C_m) = n + \frac{m}{2} - 1$ the complement of the subgraph $T[B]$ of T induced by B contains C_m, which implies $\overline{T} \supseteq W_{1,m}$, hence $\overline{G} \supseteq W_{2,m}$, a contradiction.

Subcase 2.2. $\delta(T) \geq n - \frac{m}{2} - 3.$

In this situation, we also consider two subcases: (a) T is non-bipartite and (b) T is bipartite.

(a) In the first case, let T be non-bipartite. Since $\delta(T) \geq n - \frac{m}{2} - 3 \geq \frac{2n}{3} = \frac{|V(T)| + 2}{3}$, by Lemma 2, we get that T is weakly pancyclic with girth 3 or 4.

If $\kappa(T) \geq 2$, then T is a 2-connected graph. By Lemma 3, we have $c(T) \geq \min\{2\delta(T), 2n-2\}$. This implies that T contains C_n, a contradiction.

Let $\kappa(T) = 1$. There exists a cut-vertex $v_0 \in V(T)$ such that $T - v_0$ is disconnected. Let $T_1, \ldots, T_r$ be the components of $T - v_0$. Since $\delta(T) \geq n - \frac{m}{2} - 3$ we deduce $\delta(T_i) \geq n - \frac{m}{2} - 4$, hence $|V(T_i)| \geq n - \frac{m}{2} - 3$ for every $i = 1, 2, \ldots, r$. This implies $r = 2$ and $T - v_0$ has two components T_1 and T_2, such that $|V(T_1)| + |V(T_2)| = 2n - 3$. Suppose that $\overline{T - v_0}$ contains

$W_{1,m}$. Since in $\overline{G}$, x is adjacent to all vertices in T, it follows that $\overline{G}$ contains $W_{2,m}$ and the proof is complete in this case. Otherwise, $\overline{T-v_0}$ contains no $W_{1,m}$. Since $T-v_0$ has $2n-3$ vertices, its complement contains no $W_{1,m}$ and by Theorem 1 $R(C_{n-1}, W_{1,m}) = 2n-3$, we obtain that $T-v_0$ contains C_{n-1}. This implies that C_{n-1} will be contained in one of the components of $T-v_0$, say T_1, such that $T_1 \supseteq C_{n-1}$. Thus, we have $|V(T_1)| \geq n-1$ and $|V(T_2)| \leq n-2$. Let $X = V(C_{n-1})$. If $\overline{T}$ contains $W_{1,m}$ we deduce as above that $\overline{G}$ contains $W_{2,m}$ and we are done. Otherwise, $\overline{T}$ contains no $W_{1,m}$. Since $\overline{T}$ contains no $W_{1,m}$, it contains also no K_{m+1} and by Lemma 5, we have:

$$|N_X(v)| \leq m-1 \text{ for each } v \in V(T)\setminus X. \tag{1}$$

If there exists $z_0 \in V(T_1)\setminus X$, then by (1) and $\delta(T_1) \geq n - \frac{m}{2} - 4$, we have:

$$\begin{aligned} |N_{V(T_1)\setminus X}(z_0)| &= |N_{V(T_1)}(z_0)| - |N_X(z_0)| \\ &\geq \left(n - \frac{m}{2} - 4\right) - (m-1) \\ &= n - \frac{3m}{2} - 3. \end{aligned} \tag{2}$$

Thus, by (2) we have

$$\begin{aligned} |V(T_1)| &\geq |X| + |N_{V(T_1)\setminus X}[z_0]| \\ &\geq (n-1) + \left(\left(n - \frac{3m}{2} - 3\right) + 1\right) = 2n - \frac{3m}{2} - 3 \end{aligned}$$

which implies $|V(T_2)| \leq \frac{3m}{2}$, a contradiction with $|V(T_2)| \geq n - \frac{m}{2} - 3$. We deduce that $|V(T_1)| = n-1$ and $|V(T_2)| = n-2$. Since $\delta(T_2) \geq n - \frac{m}{2}$

$-4 \geq \frac{n-2}{2} = \frac{|V(T_2)|}{2}$, we have that T_2 is pancyclic or $n-2$ is even and $T_2 \cong K_{\frac{n-2}{2}, \frac{n-2}{2}}$. This implies that T_2 also contains C_{n-2}.

Let $D = V(T_2) \bigcup \{v_0\}$ and also T_3 be the subgraph $T[D]$ of T induced by D. Since $\delta(T) \geq n - \frac{m}{2} - 3$ and by (1) we have $|N_{T_1}(v_0)| \leq m-1$, $|N_{T_2}(v_0)| \geq n - \frac{m}{2} - 3 - (m-1) = n - \frac{3m}{2} - 2 > m - 1$. By Lemma 5, we get $T_3 \supseteq C_{n-1}$. The same conclusion holds by observing that $n - \frac{3m}{2} - 2 \geq \frac{n-2}{2}$, hence v_0 is adjacent to two consecutive vertices of C_{n-2} in T_2. Thus, we also have:

$$|N_{T_3}(v)| \leq m - 1 \text{ for each } v \in V(T) \backslash V(T_3). \tag{3}$$

Because $\overline{G}$ contains no $W_{2,m}$, it follows that $\overline{G}$ also contains no K_{m+2}, hence by Lemma 5, we get

$$|N_{T_1}(v)| \leq m \text{ for each } v \in V(G) \backslash V(T_1) \tag{4}$$

and

$$|N_{T_3}(v)| \leq m \text{ for each } v \in V(G) \backslash V(T_3). \tag{5}$$

Claim 1. $H \cong K_{n-1}$.

Suppose H is not complete, so there exist $h_1, h_2 \in V(H)$ such that $h_1 h_2 \notin E(H)$. By (4) and (5), we have $|N_{T_1}(h_1)| + |N_{T_1}(h_2)| \leq 2m$ and $|N_{T_3}(h_1)| + |N_{T_3}(h_2)| \leq 2m$. Let $Y = V(T) \backslash (N_{T_1}(h_1) \cup N_{T_1}(h_2) \cup N_{T_3}(h_1) \cup N_{T_3}(h_2))$, and so $|Y| \geq 2n - 2 - 4m \geq n + \frac{m}{2} - 1$. Let F be the subgraph $T[Y]$ of T induced by Y. By Proposition 1, one deduces that $C_m \subseteq \overline{F}$ which implies $\overline{G} \supseteq W_{2,m}$, a contradiction.

It follows that by Claim 1, we have that $H + \{x\} \cong K_n$, hence $C_n \subseteq G$, a contradiction.

Let $\kappa(T) = 0$. Then T is disconnected and we deduce as above that T has exactly two components, Z_1 and Z_2. Since $\delta(T) \geq n - \frac{m}{2} - 3$, we deduce $\delta(Z_i) \geq n - \frac{m}{2} - 3$ for each $i \in \{1, 2\}$ and so $n - \frac{m}{2} - 2 \leq |V(Z_i)| \leq 2n - 2$. By a similar argument as in the case $\kappa(T) = 1$, we also obtain $Z_1 \supseteq C_{n-1}$, $Z_2 \supseteq C_{n-1}$ and $H + \{x\} \cong K_n$, hence $C_n \subseteq G$, a contradiction.

(b) In the second case, let T be bipartite. Since T is bipartite and $|V(T)| = 2n - 2$, we deduce that T is a spanning subgraph of $K_{j,t}$, where $\max\{j, t\} \geq n - 1$. This implies $\overline{T} \supseteq W_{1,m}$, hence $\overline{G} \supseteq W_{2,m}$ a contradiction, since $E(\overline{T}) \supseteq E(K_j) \cup E(K_t)$ and $n - 1 \geq \frac{9m}{2} > m + 2$.

The proof is complete. □

4. Open Problems

For $t \geq 1$, we define $W_{t,m} = K_t + C_m$. We shall propose some open problems:

(1) Determine the Ramsey numbers $R(C_n, W_{t,m})$ for even $m \geq 4$ and $t \geq 3$.

(2) Determine the Ramsey numbers $R(C_n, W_{t,m})$ for odd $m \geq 5$ and $t \geq 2$.

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