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A NEW MIXTURE MODEL FROM GENERALIZED POISSON AND GENERALIZED INVERSE GAUSSIAN DISTRIBUTION

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Abstract

In this paper, we propose a new distribution for modeling count datasets with some unique characteristics, obtained by mixing the generalized Poisson distribution (GPD) and the generalized inverse Gaussian distribution (GIGD) and using the framework of the Lagrangian probability distribution. Some structural properties of the proposed new distribution are discussed. Parameter estimates are computed using the method of maximum likelihood. A real-life data set is used to examine the performance of the new distribution.

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1. Introduction

The basic Poisson distribution for analyzing count data can generate biased results and inconsistent parameter estimates when used for modeling observations where over-dispersion and/or under-dispersion may be present. Crucially, the model works based on its restrictive assumption of equality of the mean and variance; which is inconsistent with situations where observations are either over-dispersed or under-dispersed. To overcome the limitation of the model caused by this restrictive assumption, Greenwood and Yule [4] considered the average outcome, say *t*, from the Poisson distribution as a realization of a gamma random variable, and used the convolution to define the negative binomial (NB) distribution $F(\cdot)$ with:

$$F(\cdot) = \int F(\cdot/t) f(t) dt. \tag{1}$$

Although the NB distribution has the major advantage of being able to quantify the amount of variability in the Poisson rate (also called the average outcome), it lacks, in some of the ways, the capability to capture the long tail characteristics of some count datasets. Consul and Jain [1] developed the generalized Poisson distribution from the Lagrangian probability distribution (and discussed their preference for the Lagrangian basis among other possible candidate distributions). But the model still suffers from adequacy problems when used in describing datasets characterized by a large amount of zeros in addition to a very long tail (Geedipally et al. [3] and Li et al. [13]). In this paper, we proposed a new mixture model which addresses these problems.

The paper is organized as follows: Section 2 shows how the framework of Lagrangian probability distribution is used to derive a mixture of the generalized Poisson and the generalized inverse Gaussian (GPGIG) to obtain the new proposed model. In Section 3, we study the statistical properties of the new distribution, and consider maximum likelihood estimates of model parameters in Section 4. A real-life dataset is used to test the adequacy of the new model in Section 5, while Section 6 provides some concluding remarks on the new model.

2. A New Probability Model

In this section, we define a new probability mass function generated from mixing the generalized Poisson distribution with the generalized inverse Gaussian distribution. The work borrows some ideas from the approach of Consul and Jain [1] and several more recent contributions and/or extensions of it. Because the work relies heavily on the Lagrangian formulation, it seems reasonable to situate this aspect of its property in the discussion ahead. Jensen [6] gave the first kind of Lagrange expression using the argument immediately below.

Let $f(z)$ and $g(z)$ be analytic functions of z, which can be successfully differentiated in a closed uniform interval in such a way that $f(1) = g(1) = 1$ and $g(0) \neq 0.$ Then, under the transformation $z = ug(z),$ where $u = 1$ and $z = 1,$ Lagrange inverted the function $f(z)$ in the neighborhood of $z = 0,$ and expressed the inversion as a power series of u as follows:

$$f(u) = \sum_{x=0}^{\infty} \frac{u^y}{y!} D^{y-1}[(g(z))^y f'(z)]_{z=0}, \tag{2}$$

where $D \equiv \dfrac{\partial}{\partial z}$ and $f'(z) = \partial f(z)/\partial z.$

If every term of the series equation (2) is non-negative, then the series becomes a probability generating function in variable u and leads to the probability mass function

$$P(Y = y) = \begin{cases} f(0), & y = 0, \\ \dfrac{1}{y!}\{D^{y-1}[(g(z))^y f'(z)]\}_{z=0}, & y = 1, 2, \ldots. \end{cases} \tag{3}$$

Li et al. [14] relaxed the assumption $g(1) = f(1) = 1$ in the general Lagrangian distribution and defined the class of generalized Lagrangian distributions through the introduction of an extra parameter called - *the Lagrangian expression point* - into the probability mass function using the following argument.

Suppose $f(z)$ and $g(z)$ are analytic functions such that

$$[D^{y-1}[(g(z))^y f'(z)]]_{z=0} \geq 0$$

and $g(0) \neq 0,$ where D is a partial differential operator as defined above, for any point $t > 0,$ such that $f(t) > 0$ and $g(t) > 0,$ $\forall y \in N.$ The generalized Lagrangian probability distribution of the first kind is defined as:

$$P(Y = y|s(t)) = \begin{cases} \dfrac{f(0)}{f(t)}, & y = 0, \\ \dfrac{(t/g(t))^y}{y!\, f(t)} \{D^{y-1}[(g(z))^y f'(z)]\}_{z=0}, & y = 1, 2, \ldots. \end{cases} \tag{4}$$

By letting $f(z) = e^{\theta z}$ and $g(z) = e^{\lambda z},$ we get the probability mass function (pmf) for the generalized Lagrangian distribution to be:

$$P(Y(t) = y) = \frac{\theta t(\theta t + \lambda t y)^{y-1}}{y!} e^{-\theta t - \lambda t y}, \quad y = 0, 1, 2, \ldots \tag{5}$$

when $t = 1,$ the (pmf) formulation in (5) above reduces to the generalized Poisson distribution (GPD). More generally, assuming that the conditions that generated equation (5) hold, and letting the variable t to be a continuous random variable from the generalized inverse Gaussian (GIG) distribution with density

$$s(t) = \frac{\left(\frac{b}{a}\right)^{\frac{r}{2}}}{2K_r\sqrt{(ab)}} t^{r-1} e^{-\frac{1}{2}\left(bt + \frac{a}{t}\right)}, \; t > 0, \; -\infty < r < \infty, \; a \geq 0, \text{ and } b \geq 0, \tag{6}$$

where $K_r(t)$ is the modified Bessel function of the third kind of order r with argument t (Jorgensen [8]). Then the proposed new model obtained from mixing (5) and (6) is:

$$Pr(Y = y)$$

$$= \int_0^\infty \frac{\theta t(\theta t + \lambda t y)^{y-1}}{y!} e^{-\theta t - \lambda t y} \frac{\left(\frac{b}{a}\right)^{\frac{r}{2}}}{2K_r\sqrt{(ab)}} t^{r-1} e^{-\frac{1}{2}\left(bt + \frac{a}{t}\right)} dt, \quad t > 0. \tag{7}$$

Equation (7) can be transformed using the modified Bessel expansion, denoted by $K_r(\chi, \psi)$, and defined in Paolella [15] as:

$$K_r(\chi, \psi) = \frac{1}{2}\int_0^{\infty} y^{r-1} e^{-\frac{1}{2}(\chi y^{-1} + \psi y)} dy = \eta^r K_r(\zeta), \tag{8}$$

where $\eta = \sqrt{\frac{\chi}{\psi}}$ and $\zeta = \sqrt{\chi\psi}$. In this re-definition, $K_r(\cdot)$ is the modified Bessel function of the third kind with parameters χ and ψ. These parameters regulate both the concentration and the scaling of densities through ζ and η, respectively.

In equation (8), if we define $\chi = a$ and $\psi = (2\theta + 2\lambda y + b)$, then we have

$$\eta = \sqrt{\frac{a}{2\theta + 2\lambda y + b}} \tag{8a}$$

and

$$\zeta = \sqrt{a(2\theta + 2\lambda y + b)}. \tag{8b}$$

Given (8a) and (8b), the distribution of variable Y can be defined as:

$$Pr(Y = y)$$

$$= \frac{\theta(\theta + \lambda y)^{y-1}}{y!} \frac{\left(\frac{b}{a}\right)^{\frac{r}{2}}}{K_r\sqrt{(ab)}} \left(\frac{a}{2\theta + 2\lambda y + b}\right)^{\frac{1}{2}(y+r)} K_{y+r}(\sqrt{a(2\theta + 2\lambda y + b)}). \tag{9}$$

From Jorgensen [8], it is known that if $a \to 0$, then $z \to 0$. When the first part of the modified Bessel arguments is zero, then

$$K_\lambda(z) = 2^{\lambda-1} z^{-\lambda}\Gamma(\lambda),$$

so that we have:

$$K_r\sqrt{(ab)} = 2^{r-1}(\sqrt{ab})^{-r}\Gamma(r)$$

and

$$K_{y+r}\sqrt{a(2\theta + 2\lambda y + b)} = 2^{y+r-1}[a(2\theta + 2\lambda y + b)]^{-\frac{1}{2}(y+r)}\Gamma(y + r).$$

Consequently, we give the new mass function

$$f(y; \theta, \lambda, b, r) = \frac{\Gamma(y + r)}{y!\Gamma(r)}\left(\frac{\theta}{(\theta + \lambda y)}\right)\left(\frac{\frac{b}{2}}{(\theta + \lambda y) + \frac{b}{2}}\right)^{r}\left(\frac{(\theta + \lambda y)}{(\theta + \lambda y) + \frac{b}{2}}\right)^{y} \tag{10}$$

as the limiting function of (9) with parameters bound $(r, \theta, b) > 0$, and $-1 < \lambda < 1$.

3. Statistical Properties

In discussing the properties of the proposed model, we agreed that the two theorems of Li et al. [12] (presented with the proofs) hold. To avoid ambiguity, we restate (without proof) the two theorems here.

3.1. The new model as a probability mass function

A mixture of discrete Lagrangian probability distribution and continuous distribution would form a new probability distribution function if the following (Li et al. [12]) theorems hold:

Theorem 1. *If* $f(z) > 0$ *and* $g(z) > 0$, *where* $z > 0$, *are analytic functions such that* $g(0) \neq 0$, $[D^{y-1}(g(z))^{y} f'(z)]_{z=0} \geq 0$, *and* $f(0) \geq 0$ *and the series in equation* (2) *converges uniformly on any closed and bounded interval*, *then a random variable Y has a uniform mixture of Lagrangian distribution with probability density function*

$$P(Y = y) = \begin{cases} \int_0^1 \left[\frac{f(0)}{f(t)}\right] dt, & y = 0, \\ \int_0^1 \frac{\left(\frac{t}{g(t)}\right)^y}{y!\, f(t)} [D^{y-1}(g(z))^y f'(z)]_{z=0}\, dt, & y > 0. \end{cases} \tag{11}$$

Theorem 2. *Let* $f(t)$ *and* $g(t)$ *satisfy the conditions of Theorem* 1, *and let* $s(t)$ *be a probability density function for some continuous random variable T. Then*:

(i) $$p(y, t) = s(t) \left\{ \frac{\left(\frac{t}{g(t)}\right)^y}{y!\, f(t)} [D^{y-1}(g(z))^y f'(z)]_{z=0} \right\} \tag{12}$$

is the joint probability density function of the random variables (Y, T), *where Y is the discrete and T is the continuous*.

(ii) *The marginal distribution of Y is defined as*:

$$P(Y = y) = \begin{cases} f(0) \int_{-\infty}^{\infty} \left[\frac{s(t)}{f(t)}\right] dt, & y = 0, \\ \int_{-\infty}^{\infty} s(t) \frac{\left(\frac{t}{g(t)}\right)^y}{y!\, f(t)} [D^{y-1}(g(z))^y f'(z)]_{z=0}\, dt, & y \geq 1 \end{cases} \tag{13}$$

(*see Li et al*. [12]) *for the proofs of the theorems*. □

3.2. The non-central moment of the new model

The jth raw (non-central) moment $\mu'_j = E(Y^j)$ of the discrete variable Y from the mass function (12) is:

$$\mu'_j = E(Y^j) = \sum_{y=0}^{\infty} y^j P(Y = y) \tag{14}$$

and

$$E(Y^j) = \sum_{y=0}^{\infty} y^j \int_{-\infty}^{\infty} s(t) \frac{\left(\frac{t}{g(t)}\right)^y}{y!\, f(t)} [D^{y-1}(g(z))^y f'(z)]_{z=0} dt. \tag{15}$$

Hence,

$$E(Y^j) = \int_{-\infty}^{\infty} \frac{s(t)}{f(t)} \sum_{y=0}^{\infty} y^j \frac{\left(\frac{t}{g(t)}\right)^y}{y!} [D^{y-1}(g(z))^y f'(z)]_{z=0} dt. \tag{16}$$

Jensen [6] showed that the Lagrange expansion could be written as

$$f(t) = f(0) + \sum_{y=1}^{\infty} \frac{\left(\frac{t}{g(t)}\right)^y}{y!} [D^{y-1}(g(z))^y f'(z)]_{z=0}. \tag{17}$$

By differentiating (17) partially with respect to t, we have

$$Df(t) = \frac{1}{t} g(t) D\left[\frac{t}{g(t)}\right] \sum_{y=1}^{\infty} y \frac{\left(\frac{t}{g(t)}\right)^y}{y!} [D^{y-1}(g(z))^y f'(z)]_{z=0}, \tag{18}$$

so that

$$\frac{tDf(t)}{g(t) D\left[\frac{t}{g(t)}\right]} = \sum_{y=1}^{\infty} y \frac{\left(\frac{t}{g(t)}\right)^y}{y!} [D^{y-1}(g(z))^y f'(z)]_{z=0}. \tag{19}$$

3.3. The first and second moments

Equation (14) gives the general procedure for defining the non-central moments of the new model. We evaluate this for the first two moments.

The first moment is obtained by setting $j = 1$ and using equations (16) and (19), we obtained the first moment of Y as:

$$\mu'_j = E(Y) = \sum_{y=0}^{\infty} yP(Y = y) = \int_{-\infty}^{\infty} \frac{s(t)}{f(t)} \frac{tDf(t)}{g(t)D\left[\frac{t}{g(t)}\right]} dt, \tag{20}$$

$$\mu'_j = E(Y) = \sum_{y=0}^{\infty} yP(Y = y) = \int_{-\infty}^{\infty} \frac{s(t)D\ln f(t)}{D\ln\left[\frac{t}{g(t)}\right]} dt. \tag{21}$$

By differentiating equation (19) partially with respect to t, we obtain:

$$D\left[\frac{tDf(t)}{g(t)D\left[\frac{t}{g(t)}\right]}\right] = \sum_{y=1}^{\infty} y^2 \frac{\left(\frac{t}{g(t)}\right)^{y-1}}{y!} D\left[\frac{t}{g(t)}\right][D^{y-1}(g(z))^y f'(z)]_{z=0}. \tag{22}$$

If both the sides of equation (22) are multiplied by $\dfrac{ts(t)}{\left[f(t)g(t)D\left(\frac{t}{g(t)}\right)\right]}$,

then we have:

$$\frac{ts(t)}{\left[f(t)g(t)D\left(\frac{t}{g(t)}\right)\right]} D\left[\frac{tDf(t)}{g(t)D\left(\frac{t}{g(t)}\right)}\right]$$

$$= \sum_{y=1}^{\infty} y^2 \frac{s(t)\left(\frac{t}{g(t)}\right)^{y}}{y!\,f(t)} D\left[\frac{t}{g(t)}\right][D^{y-1}(g(z))^y f'(z)]_{z=0}. \tag{23}$$

Therefore, the second moment of Y $(\mu'_2 = E(y^2))$ could be written as:

$$\mu'_2 = \sum_{y=0}^{\infty} y^2 P(Y = y) = \sum_{y=0}^{\infty} y^2 \int_{-\infty}^{\infty} s(t) \frac{\left(\frac{t}{g(t)}\right)^{y}}{y!\,f(t)} [D^{y-1}(g(z))^y f'(z)]_{z=0}\, dt$$

$$= \int_{-\infty}^{\infty} \sum_{y=0}^{\infty} y^2 \frac{s(t)\left(\frac{t}{g(t)}\right)^{y}}{y!\,f(t)} [D^{y-1}(g(z))^y f'(z)]_{z=0}\, dt$$

$$= \int_{-\infty}^{\infty} \frac{ts(t)}{\left[f(t)g(t)D\left(\frac{t}{g(t)}\right)\right]} D\left[\frac{tDf(t)}{g(t)D\left(\frac{t}{g(t)}\right)}\right] dt$$

$$= \int_{-\infty}^{\infty} \frac{s(t)}{\left[f(t)D\ln\left(\frac{t}{g(t)}\right)\right]} D\left[\frac{D\ln f(t)}{D\ln\left(\frac{t}{g(t)}\right)}\right] dt. \tag{24}$$

The foregoing can be used to state that the jth non-central moment of the discrete variable Y can be generally defined as:

$$\mu'_j = \int_{-\infty}^{\infty} s(t)K_j(t)dt, \tag{25}$$

where

$$K_1(t) = \frac{D\ln f(t)}{\left[D\ln\left(\frac{t}{g(t)}\right)\right]},\ K_2(t) = \frac{DK_1(t)}{\left[D\ln\left(\frac{t}{g(t)}\right)\right]},$$

$$K_3(t) = \frac{DK_2(t)}{\left[D\ln\left(\frac{t}{g(t)}\right)\right]},\ \ldots,\ K_j(t) = \frac{DK_{j-1}(t)}{\left[D\ln\left(\frac{t}{g(t)}\right)\right]}.$$

3.4. The mean of the new model

Given that $f(t)$ and $g(t)$ are the analytic functions that generate generalized Poisson distribution (GPD) in equation (5) and that equation (6) defines the density function $s(t)$ that leads to the new mass-function equation (10); the mean of the distribution from equation (21) is:

$$\mu = E(Y) = \sum_{y=0}^{\infty} yP(Y = y) = \int_{-\infty}^{\infty} \frac{s(t)D\ln f(t)}{D\ln\left[\frac{t}{g(t)}\right]} dt,$$

where $D \ln f(t) = \frac{\partial}{\partial t} \ln(\exp(\theta t)) = \theta$ and $D \ln\left(\frac{t}{g(t)}\right) = \frac{\partial}{\partial t} \ln\left(\frac{t}{\exp(\lambda t)}\right) = \frac{1}{t} - \lambda.$

Thus, we have

$$\mu = \frac{\theta\left(\frac{b}{2}\right)^r}{\Gamma(r)} \int_0^\infty \frac{t^r}{(1-\lambda t)} e^{-\left(\frac{b}{2}t\right)} dt. \tag{26}$$

The quantity μ diverges when we try to write in compact form; consequently, we write the integral part with parameters restriction as

$$\int_0^\infty \frac{t^r}{(1-\lambda t)} e^{-\left(\frac{b}{2}t\right)} dt = -\frac{r\left(-\frac{1}{\lambda}\right)^r \Gamma(r)\Gamma\left(-r, -\frac{\left(\frac{b}{2}\right)}{\lambda}\right)}{\lambda e^{\left(\frac{\left(\frac{b}{2}\right)}{\lambda}\right)}}, \; r, b > 0, \text{ and } \lambda < 0.$$

After re-parameterization, the μ is defined as:

$$\mu = -\frac{r\left(-\frac{\beta}{2d}\right)^r \Gamma\left(-r, -\left(\frac{\beta}{2d}\right)\right)}{de^{\left(\frac{\beta}{2d}\right)}}. \tag{27}$$

It is important to observe that the second part of the incomplete gamma $\left(\Gamma\left(-r, -\left(\frac{\beta}{2d}\right)\right)\right)$ must not be positive so as to return a real number value. A complex number value will be returned otherwise.

Corollary. *The moments of our distribution do not exist when* $d > 0$.

Remark. These properties in the Corollary are present in Cauchy (a continuous distribution) and quasi-negative binomial distribution (a discrete distribution). Both the distributions are described to be good for data that have very long or heavy tail and highly over-dispersed, respectively.

4. Parameter Estimation

The maximum likelihood method of estimation is used to estimate the parameters of the model. It is well known that the maximum likelihood estimator (MLE) converges in probability to the parameter it is estimating and achieves the Cramer-Rao lower bound on variance (see Cox and Hinkley [2]).

The log-likelihood equation for the independent observations $Y_1, Y_2, Y_3, \ldots, Y_n$ from the new model (equation (10)) with parameter vector $Q = (\theta, \lambda, b, r)^T$ is:

$$\ell(Q) = \ell(\theta, \lambda, b, r \,|\, Y)$$

$$= \sum_{i=1}^{n} \left\{ y_i \log\left(\frac{\theta + \lambda y_i}{\theta + \lambda y_i + \frac{b}{2}} \right) + r \log\left(\frac{\frac{b}{2}}{\theta + d y_i + \frac{b}{2}} \right) \right.$$

$$\left. + \log\left(\frac{\theta}{\theta + \lambda y_i} \right) + \log \Gamma(y_i + r) - \log \Gamma(y_i + 1) - \log \Gamma(r) \right\}. \quad (28)$$

The partial derivatives of the log-likelihood equation with respect to each of the model parameters are:

$$\frac{\partial}{\partial \theta} \ell(Q) = \sum_{i=1}^{n} y_i \left(\frac{4\lambda^2 y_i^2 + 2b\lambda y_i + 8\lambda\theta y_i + b^2 + 2b\theta + 4\theta^2}{b(2\lambda y_i + b + 2\theta)(\lambda y_i + \theta)} \right)$$

$$- \sum_{i=1}^{n} \frac{2r}{2\lambda y_i + b + 2\theta} \overset{set}{\Rightarrow} 0, \quad (29)$$

$$\frac{\partial}{\partial \lambda}\ell(Q) = \sum_{i=1}^{n} y_i\left(\frac{b}{2\lambda y_i(\lambda y_i + \theta)^2 + (2\theta + b)(\lambda y_i + \theta)}\right)^2$$

$$-\sum_{i=1}^{n}\frac{2ry_i}{2\lambda y_i + b + 2\theta} - \sum_{i=1}^{n}\frac{y_i}{\lambda y_i + \theta} \overset{set}{\Rightarrow} 0, \tag{30}$$

$$\frac{\partial}{\partial b}\ell(Q) = \sum_{i=1}^{n} r\left(\frac{2(\lambda y_i + \theta)}{b(2\lambda y_i + b + 2\theta)}\right) - \sum_{i=1}^{n} y_i\left(\frac{1}{2\lambda y_i + b + 2\theta}\right) \overset{set}{\Rightarrow} 0, \tag{31}$$

$$\frac{\partial}{\partial r}\ell(Q) = \sum_{i=1}^{n}\log\left(\frac{b}{2\lambda y_i + b + 2\theta}\right) + \psi(y_i + r) + \psi(r) \overset{set}{\Rightarrow} 0. \tag{32}$$

These equations (29)-(32) cannot be solved analytically, and this means that unique maximum likelihood estimator (UMLE) of the model parameters does not exist. However, by cleverly re-parameterizing the model with $\beta = \frac{b}{\theta}$ and $d = \frac{\lambda}{\theta}$, we reduced the four-parameter model to a three-parameter model

$$f(y; r, \beta, d) = \frac{\Gamma(y + r)}{y!\Gamma(r)}\left(\frac{1}{(1 + dy)}\right)\left(\frac{\frac{\beta}{2}}{(1 + dy) + \frac{\beta}{2}}\right)^r\left(\frac{(1 + dy)}{(1 + dy) + \frac{\beta}{2}}\right)^y. \tag{33}$$

This is the negative-binomial distribution with success probability $\left(\frac{\beta}{2 + \beta}\right)$ when d is forced to be zero.

The log-likelihood for the independent observations $Y_1, Y_2, Y_3, ..., Y_n$ from the new model (equation (32)) with parameter vector $P = (r, \beta, d)^T$ is then:

$$\ell(P) = \ell(r, \beta, d \mid Y)$$

$$= \sum_{i=1}^{n}\left\{ y_i \log\left(\frac{1+dy_i}{1+dy_i+\frac{b}{2}} \right) + r\log\left(\frac{\frac{\beta}{2}}{1+dy_i+\frac{\beta}{2}} \right) \right.$$

$$\left. + \log\Gamma(y_i+r) - \log\Gamma(y_i+1) - \log\Gamma(r) \right\}, \tag{34}$$

we solve for $\hat{r}_{MLE}$, $\hat{\beta}_{MLE}$ and $\hat{d}_{MLE}$ iteratively, using the Newton-Raphson method by estimating $\hat{P} = (\hat{r}, \hat{\beta}, \hat{d})$ using algorithm:

$$\hat{P}_{i+1} = \hat{P}_i - H^{-1}g, \tag{35}$$

where g is the vector of normal equation for which we want the gradient function. Using the idea of minimizing log-likelihood, we have

$$g = \left[\frac{\partial}{\partial r}(-\ell(P)) \;\; \frac{\partial}{\partial \beta}(-\ell(P)) \;\; \frac{\partial}{\partial d}(-\ell(P)) \right]$$

with

$$\frac{\partial}{\partial r}(-\ell(P)) = n\psi(r) - \sum_{i=1}^{n}\psi(y_i+r) - \sum_{i=1}^{n}\log\left(\frac{\beta}{2+\beta+2dy_i}\right), \tag{36}$$

$$\frac{\partial}{\partial \beta}(-\ell(P)) = \sum_{i=1}^{n}\frac{y_i}{(2+\beta+2dy_i)} - 2r\sum_{i=1}^{n}\frac{(1+dy_i)}{\beta(2+\beta+2dy_i)}, \tag{37}$$

$$\frac{\partial}{\partial d}(-\ell(P)) = \sum_{i=1}^{n}\frac{y_i}{(1+dy_i)} + 2r\sum_{i=1}^{n}\frac{y_i}{(2+\beta+2dy_i)}$$

$$- \beta\sum_{i=1}^{n}\frac{y_i^2}{(1+dy_i)(2+\beta+2dy_i)} \tag{38}$$

and H is the Hessian matrix

$$H = \begin{pmatrix} \frac{\partial^2}{\partial r^2}(-\ell(P)) & \frac{\partial^2}{\partial r \partial \beta}(-\ell(P)) & \frac{\partial^2}{\partial r \partial d}(-\ell(P)) \\ \frac{\partial^2}{\partial \beta \partial r}(-\ell(P)) & \frac{\partial^2}{\partial \beta^2}(-\ell(P)) & \frac{\partial^2}{\partial \beta \partial d}(-\ell(P)) \\ \frac{\partial^2}{\partial d \partial r}(-\ell(P)) & \frac{\partial^2}{\partial \beta \partial d}(-\ell(P)) & \frac{\partial^2}{\partial d^2}(-\ell(P)) \end{pmatrix},$$

where

$$\frac{\partial^2}{\partial r^2}(-\ell(P)) = n\psi'(r) - \sum_{i=1}^{n} \psi'(y_i + r), \tag{39}$$

$$\frac{\partial^2}{\partial \beta^2}(-\ell(P)) = \frac{2r}{\beta}\sum_{i=1}^{n} \frac{(1 + dy_i)}{(2 + \beta + 2dy_i)^2} + \frac{2r}{\beta^2}\sum_{i=1}^{n} \frac{(1 + dy_i)}{(2 + \beta + 2dy_i)}$$

$$- \sum_{i=1}^{n} \frac{y_i}{(2 + \beta + 2dy_i)^2}, \tag{40}$$

$$\frac{\partial^2}{\partial d^2}(-\ell(P)) = 2\beta\sum_{i=1}^{n} \frac{y_i^3}{(1 + dy_i)(2 + \beta + 2dy_i)^2} + \beta\sum_{i=1}^{n} \frac{y_i^3}{(1 + dy_i)(2 + \beta + 2dy_i)}$$

$$- 4r\sum_{i=1}^{n} \frac{y_i^2}{(2 + \beta + 2dy_i)^2} - \sum_{i=1}^{n} \frac{y_i^2}{(1 + dy_i)^2}, \tag{41}$$

$$\frac{\partial^2}{\partial \beta \partial r}(-\ell(P)) = \frac{\partial^2}{\partial r \partial \beta}(-\ell(P)) = -2\sum_{i=1}^{n} \frac{(1 + dy_i)}{\beta(2 + \beta + 2dy_i)}, \tag{42}$$

$$\frac{\partial^2}{\partial d \partial r}(-\ell(P)) = \frac{\partial^2}{\partial r \partial d}(-\ell(P)) = 2\sum_{i=1}^{n} \frac{y_i}{(2 + \beta + 2dy_i)}, \tag{43}$$

$$\frac{\partial^2}{\partial \beta \partial d}(-\ell(P)) = \frac{\partial^2}{\partial \beta \partial d}(-\ell(P)) = -2\sum_{i=1}^{n} \frac{y_i(y_i + r)}{(2 + \beta + 2dy_i)} \tag{44}$$

and $\psi(\cdot)$ and $\psi'(\cdot)$ are the di- and tri-gamma functions defined as $\psi(r) = \frac{\Gamma'(r)}{\Gamma(r)}$, $\psi(y_i + r) = \frac{\Gamma'(y_i + r)}{\Gamma(y_i + r)}$, $\psi'(r) = \frac{\Gamma''(r)}{\Gamma(r)} - \frac{\Gamma'(r)^2}{\Gamma(r)^2}$, and $\psi'(y_i + r) = \frac{\Gamma''(y_i + r)}{\Gamma(y_i + r)} - \frac{\Gamma'(y_i + r)^2}{\Gamma(y_i + r)^2}$.

The Newton-Raphson algorithms converge, as our estimates of r, β and d change by less than a tolerated amount with each successive iteration, to $\hat{r}_{MLE}$, $\hat{\beta}_{MLE}$ and $\hat{d}_{MLE}$. The algorithms were implemented in R-program language through subroutine function called non-linear minimization with bound (nlminb).

5. Zero-inflated of the New Model

Johnson et al. [7] suggested a dual-sate count model which separates the true-zero state process from the parent process in dataset with large number of zeros. The approach is to add extra proportion of zeros, ω, (such that $0 \le \omega < 1$) to the proportion of zeros from the original count distribution. This is done while decreasing the remaining proportions of zeros in an appropriate way. Following that proposition, we defined the zero-inflated version of the new model as:

$$P(Y_i = y_i) = \begin{cases} 1 - p, & \text{if } y_i = 0, \\ p\dfrac{\Gamma(y_i + r)}{y_i!\Gamma(r)}\left(\dfrac{1}{(1 + dy_i)}\right)\left(\dfrac{\frac{\beta}{2}}{(1 + dy_i) + \frac{\beta}{2}}\right)^r \left(\dfrac{(1 + dy_i)}{(1 + dy_i) + \frac{\beta}{2}}\right)^{y_i} \dfrac{1}{\left[1 - \left(\frac{\beta}{2 + \beta}\right)^r\right]} & \text{if } y_i > 0, \end{cases} \tag{45}$$

where $p = (1 - \omega)\left[1 - \left(\frac{\beta}{2 + \beta}\right)^r\right]$ and p is the probability of observing at least one count in a given day. Lambert [9] used the same argument when dealing with datasets in manufacturing equipment. The situation relates the perfect state in which a machine does not produce a defective item and therefore (seen as zero point mass) and an imperfect state in which the machine produces a number of defective items according to a Poisson distribution. The logic generated the zero inflated Poisson (ZIP) model. Other works on the zero inflated models include Heilborn [5] on the negative binomial to get the zero inflated negative binomial (ZINB). Lawal [11] assessed the fit of the generalized Poisson and the zero inflated generalized Poisson (ZIGP) to a dataset on mosquito count in Kenya.

Application of the new count model to "Boko Haram" (BH) attacks data

The "Boko-Haram" insurgency is a well-known security problem in northern-eastern Nigeria. The "Boko-Haram" group comes out with sporadic attacks of various forms against the location population in their domain of operation.

In applying the new model to the "Boko-Haram" data, we used the timeline of "Boko Haram" attacks that was published by "The Nation" (a Nigerian newspaper). The nation's comprehensive data validated the daily newspapers reports already collected by the Department of Mathematics and Statistics at American University of Nigeria, Yola, Adamawa State, Nigeria. The data reported the time and location of every form of attack from 2009 to 2014, as well as the outcome of a particular attack including the number of deaths, injuries, etc. The new model was fitted to data on records of the number of attacks which take place per day, from June 11, 2009 till July 23, 2014. The characteristics of the attacks data include:

(i) The number of daily attacks which leads to deaths.

(ii) The number of daily attacks which results no deaths (which can be described as structural zeros).

(iii) The days when there are no attacks results to sampling zeros.

(iv) Repeated numbers of deaths for some days (same number of deaths are recorded for some days that cause some spikes in the pattern of the distribution).

(v) Since the day when the attacks will end is unknown; the sample size is not fixed.

(vi) A specific attack can result in a very high number of deaths (heavy/long tail in the distributional pattern).

The characteristics of the data listed above clearly influenced a plot of it which was constructed for visualizing many of its spikes, extreme skewness with an unusually long tail. These properties make the usual candidate count data models of Poisson and negative binomial distribution to be unsuitable for analyzing the data. The results of applying all the models to the data are tabulated in Tables 1 and 2.

Table 1. Observed and expected "Boko Haram" death counts with goodness-of-fit tests based on Lawal and Upton [10] rule

Death	Day	P	NB	GP	ZIP	ZINB	ZIGP	QNB	GPGIG	ZIQNB	ZIGPGIG
0	1663	57.3833	1662.1253	1656.8792	1663	1663.0003	1662.9999	1661.743	1661.741	1663.00002	1663.00102
1	27	199.8588	38.4494	75.6638	1.07E-10	30.1981	20.7176	40.03229	40.03241	24.30048	24.30075
2	14	348.0409	19.5417	29.5459	1.70E-09	17.1368	17.1701	20.31627	20.31634	16.24627	16.24628
3	10	404.0603	13.092	16.638	1.80E-08	12.143	14.1472	13.58435	13.5844	12.49793	12.49787
4	9	351.8223	9.8295	10.9907	1.43E-07	9.4612	11.7958	10.17774	10.17777	10.24527	10.24518
5	8	245.0702	7.8568	7.9429	9.06E-07	7.7722	9.9799	8.117571	8.1176	8.70986	8.709764
6	9	142.258	6.5342	6.0816	4.79E-06	6.6041	8.5612	6.736151	6.736176	7.581588	7.581493
7	7	70.7808	5.5851	4.8476	2.17E-05	5.7446	7.435	5.744984	5.745005	6.71002	6.709928
8	9	30.8151	4.8707	3.9804	8.62E-05	5.0839	6.5268	4.998934	4.998953	6.012386	6.012299
9	3	11.925	4.3132	3.3437	3.04E-04	4.5589	5.7834	4.416972	4.416989	5.438955	5.438874
10	8	4.1533	3.8661	2.8599	9.65E-04	4.1311	5.1667	3.950279	3.950295	4.957836	4.95776
11	2	1.315	3.4994	2.4822	2.78E-03	3.7752	4.649	3.56768	3.567695	4.547516	4.547446
12	5	0.3817	3.1933	2.1807	7.36E-03	3.4741	4.2097	3.248325	3.248339	4.192892	4.192827
13	2	0.1023	2.9337	1.9355	1.80E-02	3.416	3.8334	2.97774	2.977753	3.883007	3.882947
14	3	0.0254	2.7109	1.7328	4.08E-02	2.9919	3.5083	2.745561	2.745573	3.609693	3.609637
15	5	0.0059	2.5176	1.5631	8.62E-02	2.7956	3.2252	2.544169	2.54418	3.36672	3.366668
16	3	0.0013	2.3482	1.4193	0.17103	2.622	2.977	2.367839	2.367849	3.14924	3.149193
17	6	0.0003	2.1986	1.2962	0.3192888	2.4673	2.758	2.212184	2.212194	2.95342	2.953376
18	5	0.0001	2.0655	1.1898	0.5629521	2.3286	2.5637	2.073786	2.073795	2.776178	2.776138
19	1	0	1.9463	1.0971	0.9403253	2.2034	2.3904	1.949941	1.94995	2.615009	2.614972
20	4	0	1.8389	1.0158	1.492136	2.0899	2.235	1.838484	1.838493	2.46785	2.467817
21	1	0	1.7412	0.944	2.255015	1.9865	2.0951	1.73766	1.737668	2.332986	2.332955
23	2	0	1.5724	0.8234	4.488687	1.8048	1.8539	1.562399	1.562407	2.094601	2.094575
24	2	0	1.4986	0.7723	5.935651	1.7245	1.7494	1.48578	1.485787	1.988822	1.988798
25	5	0	1.4305	0.7262	7.535093	1.6503	1.654	1.415336	1.415343	1.890745	1.890723
26	4	0	1.3676	0.6845	9.197622	1.5814	1.5665	1.350358	1.350365	1.799598	1.799578
28	1	0	1.2553	0.6121	12.2539	1.4575	1.4121	1.234478	1.234484	1.635497	1.63548

"Table 1. (Continued)"											
29	1	0	1.205	0.5808	13.41024	1.4015	1.3437	1.182609	1.182616	1.561441	1.561426
30	2	0	1.158	0.5516	14.18651	1.3491	1.2803	1.134253	1.134259	1.492088	1.492075
35	1	0	0.9636	0.4368	11.72477	1.1296	1.0241	0.9346654	0.9346708	1.203229	1.20322
39	3	0	0.8441	0.3707	6.025497	0.9927	0.8722	0.8126988	0.8127038	1.025441	1.025435
40	3	0	0.8181	0.3567	4.780719	0.9626	0.8397	0.7861392	0.786144	0.9867042	0.9866998
42	2	0	0.7697	0.3312	2.796279	0.9065	0.78	0.7369537	0.7369584	0.9150285	0.9150252
43	2	0	0.7472	0.3196	2.063823	0.8803	0.7526	0.7141452	0.7141498	0.881838	0.8818352
44	1	0	0.7258	0.3086	1.488607	0.8553	0.7267	0.6924182	0.6924226	0.8502622	0.8502599
45	1	0	0.7053	0.2982	1.049851	0.8313	0.7021	0.6717005	0.6717049	0.8201998	0.8201979
46	1	0	0.6857	0.2884	0.7243194	0.8084	0.6787	0.6519264	0.6519307	0.7915572	0.7915558
48	1	0	0.6491	0.2703	0.3233788	0.7653	0.6355	0.6149726	0.6149767	0.7381939	0.7381932
50	2	0	0.6155	0.254	0.1329432	0.7256	0.5963	0.5811317	0.5811357	0.6895579	0.6895578
51	2	0	0.5997	0.2464	8.27E-02	0.7069	0.5781	0.5652638	0.5652678	0.6668448	0.666845
52	1	0	0.5845	0.2392	5.05E-02	0.6889	0.5607	0.5500435	0.5500474	0.6451216	0.645122
54	1	0	0.5559	0.2258	1.78E-02	0.6549	0.5282	0.5214011	0.5214048	0.6044294	0.6044303
55	1	0	0.5424	0.2196	1.03E-02	0.6388	0.5131	0.5079132	0.5079169	0.5853618	0.5853629
56	1	0	0.5294	0.2136	5.81E-03	0.6232	0.4985	0.494941	0.4949446	0.5670863	0.5670875
58	1	0	0.5048	0.2024	1.77E-03	0.5938	0.4713	0.4704361	0.4704396	0.5327478	0.5327494
59	2	0	0.4931	0.1972	9.52E-04	0.5798	0.4585	0.4588542	0.4588577	0.5166097	0.5166114
60	3	0	0.4819	0.1922	5.04E-04	0.5663	0.4462	0.4476893	0.4476927	0.5011127	0.5011146
62	1	0	0.4605	0.1828	1.34E-04	0.5406	0.4231	0.426529	0.4265323	0.4719163	0.4719184
65	1	0	0.4312	0.17	1.64E-05	0.5051	0.3917	0.3974393	0.3974425	0.4321915	0.4321939
86	1	0	0.2851	0.1104	1.90E-13	0.3277	0.2444	0.2556044	0.2556068	0.2479121	0.2479149
88	1	0	0.2751	0.1065	2.50E-14	0.3154	0.2348	0.2460381	0.2460405	0.2362264	0.2362291
90	1	0	0.2656	0.1028	3.14E-15	0.3038	0.2258	0.2369619	0.2369642	0.2252494	0.2252521
101	1	0	0.2205	0.086	1.63E-20	0.2487	0.184	0.1944531	0.1944552	0.1754152	0.1754177
108	1	0	0.1971	0.0774	3.76E-24	0.2202	0.163	0.172673	0.172675	0.151016	0.1510183
125	1	0	0.1526	0.0616	8.89E-34	0.1661	0.1243	0.1318421	0.1318437	0.1077426	0.1077445
126	1	0	0.1504	0.0608	2.24E-34	0.1635	0.1225	0.1298604	0.129862	0.1057339	0.1057357
151	1	0	0.1067	0.0456	2.13E-50	0.1112	0.0862	0.09086944	0.09087074	6.82E-02	6.82E-02
166	1	0	0.088	0.0392	6.81E-61	0.0893	0.0713	0.07460202	0.07460317	5.38E-02	5.38E-02

"Table 1. (Continued)"											
188	1	0	0.0673	0.0321	2.46E-77	0.0656	0.055	0.05690352	0.05690449	3.91E-02	3.91E-02
189	1	0	0.0665	0.0319	4.14E-78	0.0647	0.0544	0.05623344	0.0562344	3.86E-02	3.86E-02
193	1	0	0.0635	0.0308	3.12E-81	0.0612	0.052	0.05365109	0.05365203	3.66E-02	3.66E-02
200	1	0	0.0585	0.0291	8.79E-87	0.0557	0.0482	0.04948275	0.04948364	3.33E-02	3.33E-02
300	1	0	0.0202	0.0149	3.24E-176	0.0157	0.0187	0.01812126	0.01812173	1.12E-02	1.12E-02
600	1	0	0.0012	0.0044	1.00E-30	0.0005	0.0024	0.002261967	0.002262072	1.70E-03	0.0017021
800	1	0	0.0003	0.0025	1.00E-48	0.0001	0.0008	0.000850388	0.000850438	7.95E-04	7.95E-04
Poisson											
13+	95	0.1353									
χ^2		111726.9									
Se		0									
Df		12									
ZIP											
NB											
126+	10		41.3514								
χ^2			132.6886								
Se			5.5E-10								
Df			53								
GP											
65+	17			22.465							
χ^2				325.6848							
Se				3.20E-44							
Df				46							
ZINB											
126+	10					44.5085					
χ^2						107.861					
Se						3.94E-06					
Df						51					
ZIGP											
126+	10						38.7585				
χ^2							121.217				

"Table 1. (Continued)"

Se							7.55E-06				
Df							51				
QNB											
126+	10							38.27786			
χ^2								138.9827			
Se								4.39E-10			
Df								52			
GPGIG											
126+	10								38.27986		
χ^2									138.9833		
Se									4.38E-10		
Df									52		
ZIQNB											
108+	12									38.631845	
χ^2										90.71846	
Se										0.000134314	
Df										48	
ZIGPGIG											
108+	12										38.631922
χ^2											90.71879
Se											0.000134302
Df											48

Table 2. Parameters estimate

Parameters	P	NB	GP	ZIP	ZINB	ZIGP	QNB	GPGIG	ZIQNB	ZIGPGIG
b0	1.2479 (0.01239775)*	1.2479 (0.1521243)*	1.2479 (0.3600339)*	3.4575 (0.393462347)*	2.7534 (4.70663024)	3.3537 (4.478877)	NA	NA	NA	NA
a0	NA	NA	NA	2.0934	1.2546	1.976	NA	NA	1.772733	1.772723
ω	NA	NA	NA	0.8903 (0.007231989)*	0.7781 (0.06079543)*	0.8783 (0.0404466)*	NA	NA	0.8547973 (0.01461306)*	0.8547959 (0.018235)*
r	NA	NA	NA	NA	NA	NA	0.024312111 (0.0020412)*	0.0243122 (0.0020413)*	0.37823923 (0.05623418)*	0.3782221 (0.116138)*
β	NA	NA	NA	NA	NA	NA	0.008194277 (0.0017523)*	0.01638839 (0.0035045)*	0.02465585 (0.0038739)*	0.0493092 (0.016008)*
d	NA	NA	NA	NA	NA	NA	0.000979062 (0.000825)	0.00097908 (0.000826)	0.00460702 (0.00177941)*	0.0046067 (0.002264)*
Dispersion parameter(k)	NA	42.9418 (3.3745795)*	8.0509 (0.5728409)*	NA	6.874 (2.82541550)	0.3966 (0.0085575)*	NA	NA	NA	NA
MLL	-22015.82	-1498.084	-1547.42	-8286.358	-1495.792	-1491.7	-1496.938	-1496.938	-1489.37	-1489.37
Proportion of zero	0.0307	0.8898	0.887	0.8903	0.8903	0.8903	0.8895838	0.8895832	0.890257	0.8902575
Deviance(D)	43167.68	455.1858	405.7667	14802.82	471.4131	519.995	NA	NA	NA	NA
Chi-Sq	414541.7	4395.836	1253.807	29361.04	4083.511	4241.389	1878.15	1878.057	2151.835	2151.109
P-value	0	1.98E-37	0	0	0	0	0.4107075	0.4113037	3.63E-06	3.82E-06
AIC	44033.64	3000.167	3098.84	16576.72	2997.584	2989.396	2999.877	2999.877	2984.741	2984.741
Df	1867	1866	1866	1866	1865	1865	1865	1865	1865	1865
Measure of Dispersion	23.1214	0.2439	0.2174	7.9329	0.2528	0.2788	NA	NA	NA	NA

The true proportion of is 0.89026
*** Indicates significant at 5% point**

6. Conclusion

In this article, we have proposed a new more flexible three-parameter model for analyzing the types of count data which may be characterized by attributes that were observed in the "Boko-Haram" insurgency data.

The model allows the probability of "success" to vary depending not only on the parameters of the distribution, but also on the value of the random variable. We have evaluated the performance of the model and its zero-inflated extension in comparison with several established distributions and their zero-inflated using the "Boko Haram" dataset as inputs. The performance of the model fittings are compared by using maximized log-likelihood statistic (MLL), the Chi-squared goodness-of-fit statistic, the Akaike information criteria (AIC), and the deviance statistic (to an extent). Table 1 shows the grouped expected count under the different models. It is important to also note that to generate the expected probabilities for each of the models, we note that $\sum_{i=0}^{800} \hat{\pi}_{\iota} = 1,$ but there were no observed values at a number of design points like $Y = 22, 27, 31, 32, 33, 34, 36, 37, 38, 41,$ etc. In particular, for example, in the case of the Poisson model, all the probabilities are accounted for, at $Y \leq 18\left(\text{that is, } \sum_{i=0}^{18} \hat{\pi}_{\iota} = 1\right),$ whereas the remaining probabilities become zeros. However, none of the other model's probability also sum to 1 and by extension the sum of fitted values did not sum to $n = 1868.$ Furthermore, the expected values under the models are relatively small. Consequently, we decided for other models (except the ZIP that could not follow the procedure) to truncate the values of Y satisfying Lawal and Upton [10] rule which states the condition under which the Chi-square approximation will be appropriately used in grouped data which have a number of small expectations. Our results show that the data is actually zero-inflated since the extra parameter for all zero-inflated models (ω) are significant. The new model performed like the QNBD but better than all the other established models.

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