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ON ROBUST NONPARAMETRIC REGRESSION ESTIMATION FOR CENSORED DEPENDENT DATA AND FUNCTIONAL REGRESSOR

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Abstract

In this paper, we study a robust nonparametric estimator of a regression function when the response variable is subject to random right censorship and when the covariate is of functional nature. Under suitable conditions, we establish an almost complete (a.co.) consistency result with rate as well as an asymptotic distribution result of the estimator when observed data exhibit a mixing dependence.

Introduction

The study of the relationship between a variable of interest T and a covariate X is one of the most important problems in statistics. Recent years

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have witnessed a renewal of interest in robust regression estimation. Here, we are interested in a right censorship robust regression model. The estimation of the robust regression is a problem of considerable interest, especially for medical researchers, and reliability engineers. The robust nonparametric estimation of the regression function was developed by Collomb and Härdle [12] in the real-valued data case. They established, under suitable conditions, the almost complete convergence rate of an M-estimator with kernel weights when the observations are independent and identically distributed. Many important results have been developed in this field. For the real-valued data case, there is a huge literature on parametric or nonparametric estimation of the robust regression function, see for instance the key works of Robinson [32], Boente and Fraiman [5, 6], Fan et al. [18] for previous results and Laïb and Ould-Saïd [25], Boente and Rodriguez [7] and Ferraty and Vieu [19] for recent advances and references.

Recently, the statistics of functional data have received a growing attention. The first result on robust estimates of the functional nonparametric function model was given by Azzedine et al. [4]. They established the almost complete convergence of robust estimators of the regression function when the regressors are functional and the observations are i.i.d. These results are extended to the dependent case by Attouch et al. [3] who established the almost complete convergence rate of the robust nonparametric regression estimation when the regressors is functional and the observation are alpha-mixing. Asymptotic normality has been considered in Attouch et al. [1, 2] for i.i.d. and dependent data. Crambes et al. [13] stated the convergence in the L^q norm in both i.i.d. and alpha-mixing cases. Gheriballah et al. [20] established the almost complete convergence of a nonparametric M-regression estimate for functional ergodic data.

To the best of our knowledge, the problem of estimating the robust regression function under dependent right censored data and functional regressors has not been addressed in the literature. Some works were devoted to mean or quantile regression models when data are censored and some of them of functional nature. For instance, El Bahi and Ould Saïd [17] established the strong uniform consistency of the nonparametric estimation of conditional quantile for functional regressors in the i.i.d. case. These results are extended (including an asymptotic normality result) to the dependent case by Horrigue and Ould-Saïd [21, 22]. Chaouch and Khardani [11] investigated asymptotic properties of the conditional quantile function of randomly censored data using functional stationary ergodic property. Under random left truncation (RLT model), Derrar et al. [15] have studied the asymptotic properties of robust nonparametric regression estimation in a context of functional covariate in the i.i.d. case.

The central object of interest of this work is estimating a robust regression function under censored data and functional regressors.

To that aim, let (T, C, X) be a random vector, where T is the variable of interest, called the *lifetime variable* in the literature of censored data, with distribution function F. The variable C is the random right censoring time with distribution function G and X is a covariable that takes values in some semi-metric space $\mathcal{E}$ which is endowed with a semi metric $d(.,.)$. Then in the right censorship model, one only observes $(Y_i, X_i, \delta_i)_{i=1,\dots,n}$, where $Y_i = \min(T_i, C_i)$ and $\delta_i = I\,(T_i \leq C_i)$ is the indicator of censoring status. For any $x \in \mathcal{E}$, we consider a real valued Borel function $\rho(.,.)$ satisfying some regularity conditions to be stated below. The nonparametric robust model studied in this paper, denoted by θ_x, is implicitly defined as a zero with respect to (w.r.t.) the equation

$$\Gamma(x, t) = E[\rho(Y, t) | X = x] = 0. \tag{1}$$

We suppose that, for all $x \in \mathcal{E}$, θ_x exists and is unique (see, for instance, Boente and Fraiman [5]). Our model is a generalization of the classical regression in sense that the latter can be obtained by taking $\rho(Y, t) = (y - t)$.

The aim of this paper is to extend the results of Lemdani and Ould Saïd [26] to the functional case, and those of Azzedine et al. [4] and Attouch et al.

[3] to the incomplete data. More precisely, we propose a smooth estimator of the robust regression (1) for a censoring random model, when the data are both dependent and functional nature. We establish, under suitable conditions, the almost complete convergence and asymptotic normality of the proposed M-estimator with kernel weights. The paper is organized as follows. In Section 2, we define the kernel robust regression estimate in censoring model. Assumptions and main results are given in Section 3. Finally, the proofs of the main results are relegated to Section 4 with some auxiliary results.

The Proposed Estimator

For any distribution function H, let $\tau_H = \sup\{t : H(t) < 1\}$ be the support's right endpoint. Let D be a compact such that $\theta_x \in D \bigcup (\infty, \tau]$, where $\tau < \min(\tau_G, \tau_F)$.

Let $\{(T_i, C_i, X_i), i \geq 1\}$ be a sequence of strictly stationary random vectors where X_i takes the values in some semi-metric space $(\mathcal{E}, d(., .))$ and T_i, C_i are real-valued.

If no censoring is present, then it is well known that a nonparametric estimator of $\Gamma(x, t)$ denoted by $\hat{\Gamma}(x, t)$ (see Attouch et al. [3]) is given by

$$\hat{\Gamma}(x, t) = \frac{\sum_{i=1}^{n} K\left(\frac{d(x, X_i)}{h}\right)\rho(T_i, t)}{\sum_{i=1}^{n} K\left(\frac{d(x, X_i)}{h}\right)},$$

with K is a probability density function and $h = h_n$ is a sequence of positive real numbers which goes to zero as n goes to infinity. In the censorship model, we adapt the idea of Carbonez et al. [10] given in the real-valued context, to our case, we define $\widetilde{\Gamma}(x, t)$ as an estimate of $\Gamma(x, t)$ by

$$\tilde{\Gamma}_n(x, t) = \frac{\sum_{i=1}^{n} \frac{\delta_i}{\overline{G}(Y_i)} K\left(\frac{d(x, X_i)}{h}\right) \rho(Y_i, t)}{\sum_{i=1}^{n} K\left(\frac{d(x, X_i)}{h}\right)}, \tag{2}$$

where $\overline{G}$ is the survival function of the random variable C. In practice, $\overline{G}$ is usually unknown, hence it is impossible to use the estimator (2). Then, we replace $\overline{G}$ by its Kaplan-Meier (KME) [24] estimate $\overline{G}_n$ defined by

$$\overline{G}_n(y) = 1 - G_n(y) = \begin{cases} \prod_{i=1}^{n} \left(1 - \frac{1 - \delta_{(i)}}{n - (i - 1)}\right)^{I_{\{Y_{(i)} \leq y\}}} & \text{if } y < Y_{(n)}, \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

where $Y_{(1)} < Y_{(2)} < \cdots < Y_{(n)}$ are the order statistics of $(Y_i)_{1 \leq i \leq n}$ and $\delta_{(i)}$ is the concomitant of $Y_{(i)}$.

The properties of the KME for dependent variables can be found in Cai [9]. Then a feasible estimator of $\Gamma(x, t)$ considered in this work is given by

$$\hat{\Gamma}_n(x, t) = \frac{\frac{1}{nE[K_1(x)]} \sum_{i=1}^{n} \frac{\delta_i}{\overline{G}_n(Y_i)} K_i(x) \rho(Y_i, t)}{\frac{1}{nE[K_1(x)]} \sum_{i=1}^{n} K_i(x)} = \frac{\hat{\Gamma}_n^N(x, t)}{\hat{\Gamma}_D(x)}, \tag{4}$$

where $K_i(x) = K\left(\frac{d(x, X_i)}{h}\right)$ for $i = 1, \ldots, n$.

A natural estimator of θ_x denoted by $\hat{\theta}_n(x)$, is a zero w.r.t. t of

$$\hat{\Gamma}_n(x, t) = 0.$$

Assumptions and Main Results

First, let $\{W_i, i \geq 1\}$ be a sequence of random variables and $\mathcal{E}_i^k(W)$

denote the σ-field of events generated by $\{W_j,\ i \leq j \leq k\}$. Given a positive integer n, set

$$\alpha(n) = \sup\{| P(A \cap B) - P(A)P(B) | : A \in \mathcal{E}_1^k(W),\ B \in \mathcal{E}_{k+n}^{\infty}(W),\ k \in \mathbb{N}^*\}.$$

The sequence is said to be α-*mixing* if the mixing coefficient $\alpha(n) \to 0$ as $n \to \infty$. The α-mixing condition is the weakest among mixing conditions known in the literature. Many stochastic processes satisfy the α-mixing condition; the ARMA processes are geometrically strongly mixing, i.e., there exists $\varrho \in (0, 1)$ such that $\alpha(n) = O(\varrho^n)$ (see, e.g., Jones [23]). In what follows, we will use the notation $B(x, r) = \{x' \in \mathcal{E};\ d(x, x') < r\}$ and we will denote by C and C' some strictly positive generic constants; x is a fixed point in $\mathcal{E}$ and N_x denotes a fixed neighborhood of x. For now on, we set

$$\lambda_\gamma(u, t) = E[(\delta_1 \overline{G}^{-1}(Y)\rho(Y, t))^\gamma \,|\, X = u]$$

and

$$\Upsilon_\gamma(u, t) = E[(\delta_1 \overline{G}^{-1}(Y)(\rho'(Y, t)))^\gamma \,|\, X = u], \text{ for } \gamma \in \{1, 2\}.$$

Some assumptions needed to state our results are introduced and gathered below for easy reference:

A1. The process (X_i, Y_i) satisfies:

(i) The function $\phi_x(r) = P[X \in B(x, r)] > 0,\ \forall r > 0.$

(ii) $\forall i \neq j,\ \ E[\rho(Y_i, t)\rho(Y_j, t) | X_i, X_j] \leq C < \infty$ and $P[(X_i, X_j) \in B(x, r) \times B(x, r)] = \zeta_x(r)$, where $\zeta_x(h) \to 0$ as $h \to 0$.

Furthermore, we assume that the ratio $\dfrac{\zeta_x(h)}{\phi_x^2(h)}$ is bounded.

(iii) $(X_i, Y_i)_{i \geq 1}$ is a stationary α-mixing sequence of random variables, with coefficient $\alpha(n) = O(n^{-\nu})$ for some $\nu > 4$.

A2. The function $\Gamma(x, .)$ is of class C^1 and satisfies

$$\forall t \in D, \forall (x_1, x_2) \in N_x \times N_x, \exists b > 0 : |\Gamma(x_1, t) - \Gamma(x_2, t)| \leq C d^b(x_1, x_2).$$

A3. The function ρ is a continuous differentiable function, strictly monotone bounded w.r.t. the second component and its derivative $\frac{\partial \rho(y, t)}{\partial t}$ is bounded and continuous at θ_x uniformly in y, and for each fixed $t \in D$,

$$E[|\rho(Y, t)|^2 | X] < C < \infty.$$

A4. K is a function with support $(0, 1)$ such that $0 < C < K(t) < C' < \infty$.

A5. The bandwidth h satisfies:

(i) $\sqrt{\frac{\log \log n}{n}} = o(\phi_x(h))$;

(ii) $\lim \frac{n(\phi_x(h))^{(\nu+4)/(\nu-4)}}{(\log n)^{\frac{\nu+1}{\nu-4}} (\log \log n)^{\frac{6}{\nu-4}}} \to \infty$ as $n \to \infty$;

(iii) $\lim \phi_x(h) \log(n) \to 0$ as $n \to \infty$;

(iv) $\lim n h^b \phi_x(h) \to 0$ as $n \to \infty$;

(v) $\forall t \in [0, 1]$, $\lim_{h \to 0} \frac{\phi_x(th)}{\phi_x(h)} = \beta_x(t)$.

To establish the asymptotic normality results, the following assumptions are needed:

B1. The function $\lambda_\gamma(., .)$ satisfied the Hölder condition with respect to the first variable, that is: there exists a positive constant b_γ such that

$$\exists C_1 > 0, \forall (u_1, u_2) \in N_x \times N_x, \forall t \in D, |\lambda_\gamma(u_1, t) - \lambda_\gamma(u_2, t)| \leq C_1 d^{b_\gamma}(u_1, u_2).$$

B2. The function $\Upsilon_\gamma(., .)$ satisfied the Hölder condition with respect to the first variable, that is: there exists a positive constant d_γ such that

$$\exists C_2 > 0, \forall (u_1, u_2) \in N_x \times N_x, \forall t \in D,$$

$$| \Upsilon_\gamma(u_1, t) - \Upsilon_\gamma(u_2, t) | \leq C_2 d^{d_\gamma}(u_1, u_2).$$

B3. The derivative of the real function, $\varphi_x(s) = E[\rho(Y, \theta(x)) | d(X, x) = s]$, at 0 exists.

B4. Let (N_n) be a sequence of positive integers tending to infinity such that

$$N_n = o((n\phi_x(h))^{1/2}) \text{ and } (n\phi_x(h))^{1/2}\alpha(N_n) \to 0, \text{ as } n \to \infty.$$

Remark 3.1. Assumption A1 is a standard condition in nonparametric modeling of functional variables under mixing condition, this can be found in Attouch et al. [3]. Assumption A2 is a regularity condition with characterizes the functional nature of the covariate X (see Azzedine et al. [4]). Assumption A3 keeps the same condition on ρ as that given by Collomb and Härdle [12] in the multivariate case. Assumption A4 is common in nonparametric estimation for functional dependent or independent cases, it is stated for simplicity of proofs. Assumptions A5((i), (ii)) gives a condition on the bandwidth which allows to control the covariance term of the estimate. Assumptions A5((iii), (iv), (v)), B1, B2 and B3 are needed in the study of the bias term. Finally, assumption B4 is needed to establish the asymptotic normality (see Masry [29]).

Our first result, stated in Proposition 3.1, will be needed to prove the main result in Theorem 3.1, it concerns the almost uniform complete convergence of the estimator defined in (4).

Proposition 3.1. *Under assumptions* A1-A4 *and* A5((i),(ii)), *we have*

$$\sup_{t \in D} | \hat{\Gamma}_n(x, t) - \Gamma(x, t) | = O(h^b) + O\left\{\left(\frac{\log n}{n\phi_x(h)}\right)^{1/2}\right\} \text{ a.co. as } n \to \infty.$$

In the following theorem, we prove the consistency of our estimator and give a rate of convergence.

Theorem 3.1. *Assume that* A1-A4 *and* A5((i),(ii)) *are satisfied. Then* $\hat{\theta}_n(x)$ *exists and is unique a.s. for all sufficiently large n. Also*,

$$\hat{\theta}_n(x) - \theta(x) = O(h^b) + O\left\{\left(\frac{\log n}{n\phi_x(h)}\right)^{1/2}\right\} \ a.co. \ as \ n \to \infty.$$

Let us now focus on the asymptotic normality result of our estimate. Notice that a Taylor series expansion of $\hat{\Gamma}_n^N(x, .)$ in the neighborhood of $\theta(x)$, gives

$$\hat{\Gamma}_n^N(x, \hat{\theta}_n(x)) = \hat{\Gamma}_n^N(x, \theta(x)) + (\hat{\theta}_n(x) - \theta(x))(\hat{\Gamma}_n^N)'(x, \xi_n),$$

where ξ_n is between $\hat{\theta}_n(x)$ and $\theta(x)$. We have

$$\hat{\theta}_n(x) - \theta(x) = -\frac{\hat{\Gamma}_n^N(x, \theta(x))}{(\hat{\Gamma}_n^N)'(x, \xi_n)} \tag{5}$$

if the denominator does not vanish. To establish the asymptotic normality, we show that the numerator in (5) suitably normalized, is asymptotically normally distributed and that the denominator converges to $\Upsilon_1(x, \theta(x))$. The result is given in the following theorem.

Theorem 3.2. *Assume that assumptions* A1, A3-A5 *and* B1-B4 *hold. Then we have*

$$\sqrt{n\phi_x(h)}(\hat{\theta}_n(x) - \theta(x)) \overset{D}{\to} N(0, \sigma^2(x, \theta(x))),$$

where $\overset{D}{\to}$ *means the convergence in distribution and*

$$\sigma^2(x, \theta(x)) = \frac{\eta_2\lambda_2(x, \theta(x))}{\eta_1^2(\Upsilon_1(x, \theta(x)))^2} = \frac{\sigma_1^2(x, \theta(x))}{(\Upsilon_1(x, \theta(x)))^2}$$

with

$$\eta_j = -\int_0^1 (K^j)'(s)\beta_x(s)ds, \ for \ j = 1, 2.$$

Conclusion

In this paper, we have proposed a robust nonparametric estimator of a regression function when the response variable is subject to random right censorship, the covariate is of functional nature, and the observations are strictly dependent (strong mixing). Then the almost complete convergence with rates and asymptotic normality results has been proved. The presented methodology can be generalized to other types of processes such as *locally time or spatial dependent or ergodic processes*.

Auxiliary Results and Proofs

The proof of Proposition 3.1 is based on the following decomposition:

$$\begin{aligned}
&\hat{\Gamma}_n(x, t) - \Gamma(x, t) \\
&= \frac{1}{\hat{\Gamma}_D(x)}[(\hat{\Gamma}_n^N(x, t) - \tilde{\Gamma}_n^N(x, t)) + (\tilde{\Gamma}_n^N(x, t) - E[\tilde{\Gamma}_n^N(x, t)])] \\
&\quad + \frac{1}{\hat{\Gamma}_D(x)}(E[\tilde{\Gamma}_n^N(x, t)] - \Gamma(x, t)) \\
&\quad - \frac{\Gamma(x, t)}{\hat{\Gamma}_D(x)}(\hat{\Gamma}_D(x) - E[\hat{\Gamma}_D(x)]),
\end{aligned}$$

where

$$\tilde{\Gamma}_n^N(x, t) = \frac{1}{nE[K_1(x)]}\sum_{i=1}^{n}\frac{\delta_i}{\overline{G}(Y_i)}K_i(x)\rho(Y_i, t).$$

Some auxiliary results and notations will be needed to prove Proposition 3.1. The first lemma deals with the behavior of the difference between $\hat{\Gamma}_D(x)$ and $E[\hat{\Gamma}_D(x)]$.

Lemma 4.1. *Under assumptions* A1 *and* A4-A5((i),(ii)), *we have*

$$\hat{\Gamma}_D(x) - E[\hat{\Gamma}_D(x)] = O\left\{\left(\frac{\log n}{n\phi_x(h)}\right)^{1/2}\right\} \ a.co.$$

Proof. It is similar to that of Lemma 1 of Attouch et al. [3], and then is omitted. □

Lemma 4.2. *Under assumptions* A1, A3-A4 *and* A5((i),(ii)), *we have*

$$\sup_{t\in D}|\hat{\Gamma}_n^N(x, t) - \tilde{\Gamma}_n^N(x, t)| = O\left\{\left(\frac{\log n}{n\phi_x(h)}\right)^{1/2}\right\} \ a.co.$$

Proof. Under A3, we have the following decomposition:

$$|\hat{\Gamma}_n^N(x, t) - \tilde{\Gamma}_n^N(x, t)| \leq \frac{1}{nE[K_1(x)]}\sup_{t\in D}\left|\frac{1}{\overline{G}_n(t)} - \frac{1}{\overline{G}(t)}\right|\sum_{i=1}^n K_i(x)$$

$$\leq \hat{\Gamma}_D(x)\frac{1}{\overline{G}_n(\tau)\overline{G}(\tau)}\sup_{t\in D}|G_n(t) - G(t)|.$$

Since $\overline{G}(\tau) > 0,$ in conjunction with the Law of Iterated Logarithm (LIL) on the censoring law (see formula 4.28 in Deheuvels and Einmahl [14]), we have

$$\sup_{t\in D}|\hat{\Gamma}_n^N(x, t) - \tilde{\Gamma}_n^N(x, t)| \leq \hat{\Gamma}_D(x)\frac{1}{\overline{G}^2(\tau)}\sqrt{\frac{\log\log n}{n}}.$$

Lemma 4.1 and A5(i) conclude the proof. □

Lemma 4.3. *Under assumptions* A1(i), A2 *and* A4, *we have*

$$\sup_{t\in D}|E[\tilde{\Gamma}_n^N(x, t)] - \Gamma(x, t)| = O(h^b) \ a.co.$$

Proof. Using the conditional expectation properties, we get

$$E[\tilde{\Gamma}_n^N(x, t)] = \frac{1}{E[K_1(x)]}E[K_1(x)\rho(T_1, t)].$$

The equiprobability of the couple (X_i, Y_i) and assumption A4 implies

$$E[\tilde{\Gamma}_n^N(x, t)] - \Gamma(x, t)$$

$$= \frac{1}{E[K_1(x)]}E[I_{\{B(x, h)\}}(X_1)(K_1(x))E[\rho(T_1, t)|X = X_1]] - \Gamma(x, t).$$

Under A2, for all $t \in D,$ we get

$$I_{\{B(x,h)\}}(X_1)(K_1(x))|\,\Gamma(X_1, t) - \Gamma(x, t)\,| \leq Ch^b,$$

which completes the proof. □

The following lemma deals with the variance term.

Lemma 4.4. *Under assumptions* A1-A4 *and* A5((i),(ii)), *we have*

$$\sup_{t\in D}|\,\widetilde{\Gamma}_n^N(x, t) - E[\widetilde{\Gamma}_n^N(x, t)]\,| = O\left\{\left(\frac{\log n}{n\phi_x(h)}\right)^{1/2}\right\} \; a.co.$$

Proof. We use covering set techniques. Indeed, since D is a compact subset, so it can be covered by a finite number d_n of intervals centered at t_j of length λ_n such that $\lambda_n = (n\phi_x^{-3}(h))^{-\frac{1}{2}}$. As D is bounded, there exists a constant A such that $d_n\lambda_n \leq A$. Now we put $J(t) = \arg\min_{1\leq j\leq d_n}|\,t - t_j\,|$. We consider the following decomposition:

$$\begin{aligned}
&\sup_{t\in D}|\,\widetilde{\Gamma}_n^N(x, t) - E[\widetilde{\Gamma}_n^N(x, t)]\,| \\
&\leq \max_{j=1,\dots,d_n}\sup_{t\in D}|\,\widetilde{\Gamma}_n^N(x, t) - \widetilde{\Gamma}_n^N(x, t_{J(t)})\,| \\
&\quad + \max_{j=1,\dots,d_n}|\,\widetilde{\Gamma}_n^N(x, t_{J(t)}) - E[\widetilde{\Gamma}_n^N(x, t_{J(t)})]\,| \\
&\quad + \max_{j=1,\dots,d_n}\sup_{t\in D}|\,E[\widetilde{\Gamma}_n^N(x, t_{J(t)})] - E[\widetilde{\Gamma}_n^N(x, t)]\,| \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

Notice that I_1 and I_3 can be treated in the same manner. From Lemma 4.1 and A3, we have

$$\begin{aligned}
I_1 &\leq \frac{1}{\overline{G}(\tau)}\hat{\Gamma}_D(x)\lambda_n \\
&= o(1).
\end{aligned}$$

Now, we deal with I_2, for all $\varepsilon > 0$, we have

$$P(I_2 > \varepsilon) \leq d_n P(|\,\widetilde{\Gamma}_n^N(x, t_{J(t)}) - E[\widetilde{\Gamma}_n^N(x, t_{J(t)})]\,| > \varepsilon).$$

We put

$$\Delta_i(x, t) = \left[\frac{\delta_i}{\overline{G}(Y_i)} K_i(x)\rho(Y_i, t) - E\left[\frac{\delta_i}{\overline{G}(Y_i)} K_i(x)\rho(Y_i, t)\right]\right]. \tag{6}$$

The use of the well-known Fuk-Nagaev's inequality (see Rio [31], p. 87, 6.19b), allows us to get, for all $\varepsilon > 0$ and $r > 1$,

$$\begin{aligned} P(I_2 > \varepsilon) &\leq d_n P\left(\left|\sum_{i=1}^{n} \Delta_i(x, t_{J(t)})\right| > \varepsilon n E[K_1(x)]\right) \\ &\leq A\lambda_n^{-1}\left[\frac{n}{r}\left(\frac{2r}{\varepsilon n E[K_1(x)]}\right)^{\nu+1} + \left(1 + \frac{\varepsilon^2 n^2 E^2[K_1(x)]}{r S_n^2}\right)^{-r/2}\right] \\ &= Q_{1n} + Q_{2n}, \end{aligned}$$

where

$$S_n^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} |\operatorname{cov}(\Delta_i(x, t_{J(t)}), \Delta_j(x, t_{J(t)}))|. \tag{7}$$

We have to calculate the asymptotic behavior of S_n^2. By using A3 and A4, we get

$$\begin{aligned} Var(\Delta_1(x, t_{J(t)})) &\leq E\left[\frac{\delta_1}{\overline{G}^2(Y_1)} K_1^2(x)\rho^2(Y_1, t_{J(t)})\right] \\ &\leq \frac{1}{\overline{G}(\tau)} E[K_1^2(x)E[|\,\rho(T_1, t_{J(t)})\,|^2 \,|\, X_1]] \\ &\leq C[\sup_{s\in[0,1]} K(s)]^2 (E[I_{\{d(X,x)<h\}}]) \\ &= O(\phi_x(h)). \end{aligned} \tag{8}$$

On the other hand, under A1(ii), we have

$$\begin{aligned}
&\operatorname{cov}(\Delta_i(x, t_{J(t)}), \Delta_j(x, t_{J(t)})) \\
&= E[\Delta_i(x, t_{J(t)})\Delta_j(x, t_{J(t)})] \\
&\le C|\, E[K_i(x)K_j(x)]\,| + |\, E[K_i(x)]\,||\, E[K_j(x)]\,| \\
&= O(\phi_x^2(h)). \qquad (9)
\end{aligned}$$

Now, following Masry [28], we define the sets:

$$E_1 = \{(i, j) \text{ such that } 1 \le |\, i - j\,| \le \varphi_n\}$$

and

$$E_2 = \{(i, j) \text{ such that } \varphi_n + 1 \le |\, i - j\,| \le n - 1\},$$

where $\varphi_n \to \infty$ as $n \to \infty$, we can write

$$\sum_{i \ne j} |\operatorname{cov}(\Delta_i(x, t_{J(t)}), \Delta_j(x, t_{J(t)}))| = \mathcal{E}_{1,n} + \mathcal{E}_{2,n},$$

where $\mathcal{E}_{1,n}$ and $\mathcal{E}_{2,n}$ are the sums of covariances over E_1 and E_2, respectively. First, by applying the last upper bound in (9), we get

$$\mathcal{E}_{1,n} = O(n\phi_x^2(h)\varphi_n).$$

For the second term, we use the modified Davydov covariance inequality for mixing processes (see Rio [31], formula 1.12a, p. 10), we have

$$\forall i \ne j, \ |\operatorname{cov}(\Delta_i(x, t_{J(t)}), \Delta_j(x, t_{J(t)}))| \le c\alpha(|\, i - j\,|).$$

Then, we get, by A1(iii),

$$\mathcal{E}_{2,n} = O(n\varphi_n^{1-\nu}).$$

Choosing $\varphi_n = (\phi_x(h))^{-2/\nu}$ permits us to get

$$\sum_{i \ne j} |\operatorname{cov}(\Delta_i(x, t_{J(t)}), \Delta_j(x, t_{J(t)}))| = O(n\phi_x(h)). \qquad (10)$$

Finally, combining the previous results yields

$$S_n^2 = O(n\phi_x(h)). \tag{11}$$

Therefore, by putting

$$\varepsilon = \varepsilon_0\sqrt{\frac{\log n}{n\phi_x(h)}}, \quad r = O(\log(n)(\log\log(n))^{\frac{1}{\nu}}), \tag{12}$$

we get

$$Q_{1n} = O(\lambda_n^{-1} n^{(1-\nu)/2} h_n^{-(\nu+1)/2} (\log n)^{(\nu-1)/2} (\log\log n)\varepsilon_0^{-(\nu+1)}).$$

Under A5(ii), $Q_{1n} = O(n^{-1}\log^{-1} n(\log\log^{-2} n))$, this last is the general term of a convergent Bertrand's series. Using a Taylor expansion of $\log(x+1)$ and (12), we have

$$\begin{aligned} Q_{2n} &\le An^{1/2}(\phi_x(h))^{-3/2}\exp\left[-\frac{r}{2}\log\left(1+\frac{\varepsilon_0^2\log n}{r}\right)\right] \\ &\le C(\phi_x(h))^{-3/2} n^{\frac{1-\varepsilon_0^2}{2}}. \end{aligned} \tag{13}$$

In the same way, we can choose ε_0 such that Q_{2n} is the general term of a convergent series. Thus, $\sum_{n\ge 1}(Q_{1n}+Q_{2n}) < \infty$, and the result is then a direct application of Borel-Cantelli's Lemma. □

Proof of Proposition 3.1. It is a consequence of Lemmas 4.1-4.4. □

Proof of Theorem 3.1. We give the proof for the case of an increasing $\rho(Y, .)$, decreasing case being obtained by considering $-\rho(Y, .)$. Therefore, we can write, under this consideration, for all $\varepsilon > 0$,

$$\begin{aligned} &P(|\hat{\theta}_n(x) - \theta(x)| \ge \varepsilon) \\ &\le P(|\hat{\Gamma}_n(x, \theta(x)+\varepsilon) - \Gamma(x, \theta(x)+\varepsilon)| \ge \Gamma(x, \theta(x)+\varepsilon)) \\ &\quad + P(|\hat{\Gamma}_n(x, \theta(x)-\varepsilon) - \Gamma(x, \theta(x)-\varepsilon)| \ge \Gamma(x, \theta(x)-\varepsilon)). \end{aligned}$$

Moreover, under A2, we get that

$$\hat{\theta}_n(x) - \theta(x) = \frac{\Gamma(x, \hat{\theta}_n(x)) - \hat{\Gamma}_n(x, \hat{\theta}_n(x))}{\frac{\partial}{\partial t}\Gamma(x, \varkappa_n)},$$

where $\varkappa_n$ is between $\hat{\theta}_n(x)$ and $\theta(x)$. By the regularity assumption A2 on $\Gamma(x, .)$, we have

$$\exists v > 0, \quad \sum_{n=1}^{\infty} P\left[\frac{\partial}{\partial t}\Gamma(x, \varkappa_n) < v\right] < \infty.$$

The result is then a direct consequence of Proposition 3.1. □

Proof of Theorem 3.2. From (5), we have the following decomposition:

$$\begin{aligned}
\sqrt{n\phi_x(h)}(\hat{\theta}_n(x) - \theta(x)) &= \sqrt{n\phi_x(h)}\,\frac{\hat{\Gamma}_n^N(x, \theta(x)) - \widetilde{\Gamma}_n^N(x, \theta(x))}{(\hat{\Gamma}_n^N)'(x, \xi_n)} \\
&\quad + \sqrt{n\phi_x(h)}\,\frac{\hat{\Gamma}_n^N(x, \theta(x)) - E[\widetilde{\Gamma}_n^N(x, \theta(x))]}{(\hat{\Gamma}_n^N)'(x, \xi_n)} \\
&\quad + \sqrt{n\phi_x(h)}\,\frac{E[\widetilde{\Gamma}_n^N(x, \theta(x))]}{(\hat{\Gamma}_n^N)'(x, \xi_n)} \\
&= \frac{S_1 + S_2 + S_3}{(\hat{\Gamma}_n^N)'(x, \xi_n)}.
\end{aligned}$$

Then, to state the result, we show that S_1 and S_3 are asymptotically negligible, and that S_2 suitably normalized is asymptotically normally distributed and the denominator converges in probability to $\Upsilon_1(x, \theta(x))$.

On one hand, we have

$$S_1 = O\{(\phi_x(h)\log n)^{1/2}\}\ a.co.$$

which goes to zero under A5(iii).

On the other hand, as in Lemma 4.3, we have that

$$S_3 = O(\sqrt{n\phi_x(h)h^b})$$

which goes to zero under A5(iv). □

Lemma 4.5. *Under assumptions* A1, A3-A5 *and* B1-B4, *we have*

$$S_2 \xrightarrow{D} N(0, \sigma_1^2(x, \theta(x))). \tag{14}$$

Proof. In order to establish the asymptotic normality for sums of dependent random variables, we use Doob's small-block and large-block technique (see Doob [16], pp. 228-232). Partition the set $\{1, 2, ..., n\}$ into $2r_n + 1$ subsets with large blocks of size M_n and small blocks of size N_n, with $r_n = \left[\dfrac{n}{M_n + N_n}\right]$. Condition B4 implies that there exists a sequence of positive integers (q_n) tending to infinity such that

$$q_n N_n = o(\sqrt{n\phi_x(h)}),\ q_n\sqrt{n\phi_x(h)}\,\alpha(N_n) \to 0 \text{ as } n \to \infty. \tag{15}$$

Now define the large block size as $M_n = \left[\dfrac{\sqrt{n\phi_x(h)}}{q_n}\right]$. Then using (15) and simple algebra shows that as $n \to \infty$:

$$\frac{N_n}{M_n} \to 0,\ \frac{M_n}{n} \to 0,\ \frac{M_n}{\sqrt{n\phi_x(h)}} \to 0,\ \frac{n}{M_n}\alpha(N_n) \to 0. \tag{16}$$

Put

$$\Psi_i(x, \theta(x))$$

$$= \frac{\sqrt{\phi_x(h)}}{E[K_1(x)]}\left\{\frac{\delta_i}{\overline{G}(Y_i)}K_i(x)\rho(Y_i, \theta(x)) - E\left[\frac{\delta_i}{\overline{G}(Y_i)}K_i(x)\rho(Y_i, \theta(x))\right]\right\},$$

then we have

$$S_2(x,\ \theta(x)) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Psi_i(x,\ \theta(x))$$

$$= \frac{1}{\sqrt{n}}[T_{1,n}(x,\ \theta(x)) + T_{2,n}(x,\ \theta(x)) + T_{3,n}(x,\ \theta(x))],$$

where

$$T_{1,n}(x,\ \theta(x)) = \sum_{m=0}^{r_n-1} L_m(x,\ \theta(x)), \quad T_{2,n}(x,\ \theta(x)) = \sum_{m=0}^{r_n-1} L'_m(x,\ \theta(x))$$

and

$$T_{3,n}(x,\ \theta(x)) = \sum_{i=(M_n+N_n)r_n+1}^{n} \Psi_i(x,\ \theta(x)).$$

Also, let

$$L_m(x,\ \theta(x)) = \sum_{i=m(M_n+N_n)+1}^{m(M_n+N_n)+M_n} \Psi_i(x,\ \theta(x))$$

and

$$L'_m(x,\ \theta(x)) = \sum_{i=m(M_n+N_n)+M_n+1}^{(m+1)(M_n+N_n)} \Psi_i(x,\ \theta(x)).$$

Let us show that as $n \to \infty$:

$$\frac{1}{n}E[T_{2,n}^2(x,\ \theta(x))] \to 0 \text{ and } E[T_{3,n}^2(x,\ \theta(x))] \to 0, \tag{17}$$

$$\left| E\left[\prod_{m=0}^{r_n-1} \exp(itn^{-\frac{1}{2}}L_m)\right] - \prod_{m=0}^{r_n-1} E[\exp(itn^{-\frac{1}{2}}L_m)]\right| \to 0, \tag{18}$$

$$\frac{1}{n}Var(T_{1,n}(x,\ \theta(x))) \to \sigma_1(x,\ \theta(x)), \tag{19}$$

$$\frac{1}{n}\sum_{m=0}^{r_n-1} E[L_m^2 I\{| L_m | > \varepsilon\sigma_1(x, \theta(x))\sqrt{n}\}] \to 0. \tag{20}$$

For every $\varepsilon > 0$, relation (17) implies that $T_{2,n}^2$ and $T_{3,n}^2$ are asymptotically negligible, while (18) shows that the summands L_m in $T_{1,n}$ are asymptotically independent, and (19) and (20) are standard Lindeberg-Feller conditions for asymptotic normality of $T_{1,n}$ under independence. We first establish (17):

$$\frac{1}{n} E[T_{2,n}^2(x, \theta(x))]$$

$$= \frac{1}{n}\sum_{m=0}^{r_n-1} \sum_{i=m(M_n+N_n)+M_n+1}^{(m+1)(M_n+N_n)} E[\Psi_i^2(x, \theta(x))]$$

$$+ \frac{2}{n}\sum_{m=0}^{r_n-1} \sum_{m(M_n+N_n)+M_n+1\le i<j\le(m+1)(M_n+N_n)} \operatorname{cov}(\Psi_i, \Psi_j)$$

$$+ \frac{2}{n}\sum_{0\le i<j\le r_n-1} \operatorname{cov}(L_i', L_j')$$

$$= \Lambda_{1n} + \Lambda_{2n} + \Lambda_{3n}.$$

First, we have

$$E[\Psi_i^2(x, \theta(x))] = \frac{\phi_x(h)E[K_1^2(x)]}{E^2[K_1(x)]} E\left[\frac{\frac{\delta_i}{\overline{G}^2(Y_i)} K_i^2(x)\rho^2(Y_i, \theta(x))}{E[K_1^2(x)]}\right]$$

$$- \phi_x(h)\left\{E\left[\frac{\frac{\delta_i}{\overline{G}(Y_i)} K_i(x)\rho(Y_i, \theta(x))}{E[K_1(x)]}\right]\right\}^2.$$

We show by using analogous arguments as those considered in Lemma 4.3 that

$$E\left[\frac{\frac{\delta_i}{\overline{G}^2(Y_i)}K_i^2(x)\rho^2(Y_i, \theta(x))}{E[K_1^2(x)]}\right] \to \lambda_2(x, \theta(x)),$$

$$E\left[\frac{\frac{\delta_i}{\overline{G}(Y_i)}K_i(x)\rho(Y_i, \theta(x))}{E[K_1(x)]}\right] \to \lambda_1(x, \theta(x)).$$

It is shown in Attouch et al. [1] that

$$\frac{\phi_x(h)E[K_1^2(x)]}{E^2[K_1(x)]} \to \frac{\eta_2}{\eta_1^2}.$$

Finally, we have

$$E[\Psi_i^2(x, \theta(x))] \to \frac{\eta_2}{\eta_1^2}\lambda_2(x, \theta(x)) \tag{21}$$

which yields that

$$\Lambda_{1n}(x) = O\left(\frac{r_n N_n}{n}\right) = o(1) \text{ from (16).}$$

Since, for $k = 2, 3,$ we have

$$|\Lambda_{kn}| \leq \frac{2}{n}\sum_{1 \leq i < j \leq n}|\operatorname{cov}(\Psi_i, \Psi_j)|. \tag{22}$$

Let us prove that the right term in (22) tends to 0 as n tends infinity. In the sequel, we use technique developed by Masry [28]. Let the sets:

$$\Xi_1 = \{(i, j) \text{ such that } 1 \leq |i - j| \leq m_n\}$$

and

$$\Xi_2 = \{(i, j) \text{ such that } m_n + 1 \leq |i - j| \leq n - 1\},$$

where m_n is a sequence of integers such that $m_n = o(n)$ as $n \to \infty$ and for some $\delta \in (0, 1)$. We have

$$(\phi_x(h))^{-\delta} \sum_{l=m_n}^{\infty} (\alpha(l))^{\delta} < \infty. \tag{23}$$

We can write

$$\frac{1}{n} \sum_{1 \leq i < j \leq n} |\operatorname{cov}(\Psi_i, \Psi_j)| = \frac{1}{n}(A_{1,n} + A_{2,n}), \tag{24}$$

where $A_{1,n}$ and $A_{2,n}$ are the sums of covariances over Ξ_1 and Ξ_2, respectively.

Furthermore, for $i < j$, we observe that

$$\begin{aligned}
&\operatorname{cov}(\Psi_i, \Psi_j) \\
&\leq C \frac{\phi_x(h)}{E^2[K_1(x)]} E\left[\frac{\delta_i \delta_i}{\overline{G}(Y_i)\overline{G}(Y_j)} K_i(x) K_j(x) \rho(Y_i, \theta(x)) \rho(Y_j, \theta(x))\right] \\
&= C \frac{\phi_x(h)}{E^2[K_1(x)]} E\Bigg[K_i(x) K_j(x) \\
&\qquad \cdot E\left[\frac{\delta_i \delta_i}{\overline{G}(Y_i)\overline{G}(Y_j)} \rho(Y_i, \theta(x)) \rho(Y_j, \theta(x)) | X_i, X_j\right]\Bigg].
\end{aligned}$$

By the assumptions made on ρ, it follows that

$$|\operatorname{cov}(\Psi_i, \Psi_j)| \leq C \frac{\phi_x(h)}{E^2[K_1(x)]} P[(X_i, X_j) \in B(x, h) \times B(x, h)].$$

Using assumption A1(ii), it follows that

$$|\operatorname{cov}(\Psi_i, \Psi_j)| = O(\phi_x(h)).$$

Therefore,

$$\frac{1}{n} A_{1,n} = O(\phi_x(h) m_n). \tag{25}$$

Now, choosing m_n such that $m_n \phi_x(h)$ goes to zero as n goes to infinity, we get $\frac{1}{n} A_{1,n} = o(1)$. For the second term in (24), let us use a version of the moment inequality due to Rio [30]. Letting p, q, γ be integer numbers such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{\gamma} = 1$, it follows that

$$\frac{1}{n} A_{2,n} \leq \frac{c}{n} \sum_{(i,j) \in \Xi_2} 2^{1+\frac{1}{\gamma}} (\alpha(|i-j|))^{\frac{1}{\gamma}} (E|\Psi_i^p|)^{\frac{1}{p}} (E|\Psi_i^q|)^{\frac{1}{q}}.$$

Moreover, we get for n large enough,

$$E|\Psi_i^P| \leq \frac{2^{p-1}(\phi_x(h))^{\frac{p}{2}}}{E^p[K_1(x)]} E\left[\left|\frac{\delta_i}{\overline{G}(Y_i)} K_i(x) \rho(Y_i, \theta(x))\right|^P\right]. \tag{26}$$

Conditioning on X_i, we get

$$E\left[\left|\frac{\delta_i}{\overline{G}(Y_i)} K_i(x) \rho(Y_i, \theta(x))\right|\right] \leq C(\phi_x(h)).$$

Therefore,

$$E|\Psi_i^P| = O((\phi_x(h))^{1-\frac{p}{2}}).$$

Hence, using (26), we have

$$\frac{1}{n} A_{2,n} \leq \frac{c}{n} \sum_{(i,j) \in \Xi_2} 2^{1+\frac{1}{\gamma}} (\alpha(|i-j|))^{\frac{1}{\gamma}} ((\phi_x(h))^{1-\frac{p}{2}})^{\frac{1}{p}} ((\phi_x(h))^{1-\frac{q}{2}})^{\frac{1}{q}}$$

$$\leq c(\phi_x(h))^{-\frac{1}{\gamma}} \sum_{l=m_n}^{\infty} (\alpha(l))^{\frac{1}{\gamma}}. \tag{27}$$

Therefore, by (23), we get

$$\frac{1}{n} A_{2,n} \to 0 \text{ as } n \to \infty. \tag{28}$$

Thus, from (25) and (28), we obtain

$$\frac{1}{n} \sum_{1 \le i < j \le n} |\operatorname{cov}(\Psi_i, \Psi_j)| \to 0 \text{ as } n \to \infty. \tag{29}$$

This completes the proof of the first part in (17). For the second part, write

$$\begin{aligned} & n^{-1} E[T_{3,n}^2(x, \theta(x))] \\ &= n^{-1} E\left[\left(\sum_{i=(M_n+N_n)r_n+1}^{n} \Psi_i(x, \theta(x))\right)^2\right] \\ &= n^{-1} E[\Psi_i^2(x, \theta(x))] + 2n^{-1} \sum_{(M_n+N_n)r_n+1 \le i < j \le n} \operatorname{cov}(\Psi_i, \Psi_j) \\ &\le c \frac{n - (M_n + N_n) r_n}{n} + 2n^{-1} \sum_{1 \le i < j \le n} |\operatorname{cov}(\Psi_i, \Psi_j)|. \end{aligned}$$

Therefore, (16) and (29) give the result.

In order to establish (18), we make use of the fact that the process $\{Y_i, X_i\}$ is strongly mixing and Volonskii and Rosanov's Lemma (see Masry [29]). Note that L_m is $F_{i_m}^{j_m}$ measurable with $i_m = m(M_n + N_n) + 1$ and $j_m = m(M_n + N_n) + M_n$. Then we have

$$\begin{aligned} & \left| E\left[\prod_{m=0}^{r_n - 1} \exp(itn^{-\frac{1}{2}} L_m)\right] - \prod_{m=0}^{r_n - 1} E[\exp(itn^{-\frac{1}{2}} L_m)] \right| \\ &\le 16 r_n \alpha(N_n + 1) \approx 16 \frac{n}{M_n} \alpha(N_n + 1) \end{aligned}$$

which tends to zero by (16).

Next, let us establish (19). Since

$$\frac{1}{n}Var(T_{1,n}(x, \theta(x)))$$

$$= \frac{1}{n}\left(\sum_{m=0}^{r_n-1} Var[L_m(x, \theta(x))]\right) + \frac{2}{n}\sum_{0\le i<j\le r_n-1} \operatorname{cov}(L_i, L_j)$$

$$\le \frac{1}{n}\sum_{m=0}^{r_n-1}\sum_{i=m(M_n+N_n)+1}^{m(M_n+N_n)+M_n} E[\Psi_i^2(x, \theta(x))] + \frac{4}{n}\sum_{0\le i<j\le r_n-1} \operatorname{cov}(\Psi_i, \Psi_j).$$

Since $\frac{r_n M_n}{n} \to 1,$ the result follows from (21) and (29).

It remains to establish (20). Now we have to show that the standard Lindeberg-Feller conditions (see Loève [27], p. 280) for asymptotic normality of $\frac{1}{\sqrt{n}}T_{1,n}$ under independence condition are satisfied. So we have to establish

$$\frac{1}{n}\sum_{m=0}^{r_n-1} E[L_m^2 I\{| L_m | > \varepsilon\sigma_1(x, \theta(x))\sqrt{n}\}] \to 0.$$

It suffices to show that, for every $\varepsilon > 0,$ the set $\{| L_m | > \varepsilon\sigma_1(x, \theta(x))\sqrt{n}\}$ is empty for n large enough. Since

$$\frac{1}{\sqrt{n}}| L_m | \le \frac{cM_n}{\sqrt{n\phi_x(h)}},$$

the set $\{| L_m | > \varepsilon\sigma(x)\sqrt{n}\}$ is an empty set by (16). Then this completes the proof of Lemma 4.5. □

Lemma 4.6. *Under assumptions* A1, A2, A5 *and* B1, *we have*

$$(\hat{\Gamma}_n^N)'(x, \xi_n) \xrightarrow{P} \Upsilon_1(x, \theta(x)).$$

Proof. We have

$$\sup_{t\in D}|(\hat{\Gamma}_n^N)'(x, t) - \Upsilon_1(x, t)|$$

$$\leq \sup_{t\in D}|(\hat{\Gamma}_n^N)'(x, t) - (\hat{\Gamma}_n^N)'(x, t)| + \sup_{t\in D}|(\widetilde{\Gamma}_n^N)'(x, t) - \Upsilon_1(x, t)|$$

$$= \gamma_{1,n}(x, t) + \gamma_{2,n}(x, t),$$

where

$$(\hat{\Gamma}_n^N)'(x, t) = \frac{1}{E[K_1(x)]}\sum \frac{\delta_i}{\overline{G}_n(Y_i)} K_i(x)\rho'(Y_i, t).$$

Using the same steps as in the proof of Lemma 4.2, we get

$$\gamma_{1,n}(x, t) = O\left(\left(\frac{\log n}{n\phi_x(h)}\right)^{1/2}\right).$$

On the other hand, we consider the following decomposition:

$$\gamma_{2,n}(x, t) \leq \sup_{t\in D}|(\widetilde{\Gamma}_n^N)'(x, t) - E[(\widetilde{\Gamma}_n^N)'(x, t)]|$$

$$+ \sup_{t\in D}|E[(\widetilde{\Gamma}_n^N)'(x, t)] - \Upsilon_1(x, t)|. \tag{30}$$

By the continuity of $\frac{\partial\rho(y, t)}{\partial t}$ at $\theta(x)$ and the convergence results given in Lemma 4.4, we have that the first term of (30) converges in probability to 0. On the other hand, similar arguments to those used in the proof of Lemma 4.3, can be used to obtain

$$E[(\widetilde{\Gamma}_n^N)'(x, t)] = \Upsilon_1(x, t) + O(h_n^{b_2}).$$

The proof of the lemma is then complete. □

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References

[1] M. Attouch, A. Laksaci and E. Ould Saïd, Asymptotic distribution of robust estimator for functional nonparametric models, Comm. Statist. Theory Methods 38 (2009), 1317-1335.

[2] M. Attouch, A. Laksaci and E. Ould Saïd, Asymptotic normality of a robust estimator of the regression function for functional time series data, J. Korean Statist. Soc. 39 (2010), 489-500.

[3] M. Attouch, A. Laksaci and E. Ould Saïd, Robust regression for functional time series data, J. Japan Statist. Soc. 42 (2012), 125-143.

[4] N. Azzedine, A. Laksaci and E. Ould Saïd, On robust nonparametric regression estimation for a functional regressor, Statist. Probab. Lett. 78 (2008), 3216-3221.

[5] G. Boente and R. Fraiman, Robust nonparametric regression estimation, J. Multivariate Anal. 29 (1989), 180-198.

[6] G. Boente and R. Fraiman, Asymptotic distribution of robust estimators for nonparametric models from mixing processes, Ann. Statist. 18 (1990), 891-906.

[7] G. Boente and D. Rodriguez, Robust estimators of high order derivatives of regression function, Statist. Probab. Lett. 76 (2006), 1335-1344.

[8] D. Bosq, Linear processes in function spaces. Theory and application, Lectures Notes in Statistics, 149, Springer, New York, 2000.

[9] Z. Cai, Asymptotic properties of Kaplan-Meier estimator for censored dependent data, Statist. Probab. Lett. 37 (1998), 381-389.

[10] A. Carbonez, L. Györfi and E. C. Van der Meulen, Partition-estimates of a regression function under random censoring, Statist. Decisions 13 (1995), 21-37.

[11] M. Chaouch and S. Khardani, Randomly censored quantile regression estimation using functional stationary ergodic data, J. Nonparametr. Stat. 27 (2015), 65-87.

[12] G. Collomb and W. Härdle, Strong uniform convergence rates in robust nonparametric time series analysis and prediction: kernel regression estimation from dependent observations, Stochastic Process. Appl. 23 (1986), 77-89.

[13] C. Crambes, L. Delsol and A. Laksaci, Robust nonparametric estimation for functional data, J. Nonparametr. Stat. 20 (2008), 573-598.

[14] P. Deheuvels and J. H. J. Einmahl, Functional limit laws for the increments of Kaplan-Meier product-limit processes and applications, Ann. Probab. 28 (2000), 1301-1335.

[15] S. Derrar, A. Laksaci and E. Ould Saïd, On the nonparametric estimation of the functional ψ-regression for a random left-truncation model, J. Stat. Theory Pract. 9 (2015), 823-849.

[16] J. L. Doob, Stochastic Processes, Wiley, New York, 1953.

[17] M. El Bahi and E. Ould Saïd, Strong uniform consistency of nonparametric estimation of the censored conditional quantile for functional regressors, Technical Report, No. 382, L.M.P.A., ULCO, 2000.

[18] J. Fan, T. C. Hu and Y. K. Truong, Robust non-parametric function estimation, Scand. J. Statist. 21 (1994), 433-446.

[19] F. Ferraty and P. Vieu, Nonparametric Functional Data Analysis: Theory and Practice, Springer, New York, 2006.

[20] A. Gheriballah, A. Laksaci and S. Sekkal, Nonparametric *M*-regression for functional ergodic data, Statist. Probab. Lett. 83 (2013), 902-908.

[21] W. Horrigue and E. Ould-Saïd, Strong uniform consistency of a nonparametric estimator of a conditional quantile for censored dependent data and functional regressors, Random Oper. Stoch. Equ. 19 (2011), 131-156.

[22] W. Horrigue and E. Ould-Saïd, Nonparametric regression quantile estimation for dependant functional data under random censorship: asymptotic normality, Comm. Statist. Theory Methods 44 (2015), 4307-4332.

[23] D. A. Jones, Nonlinear autoregressive processes, Proc. Roy. Soc. London 360 (1978), 71-95.

[24] E. L. Kaplan and P. Meier, Nonparametric estimation from incomplete observations, J. Amer. Statist. Assoc. 53 (1958), 457-481.

[25] N. Laïb and E. Ould Saïd, A robust nonparametric estimation of the autoregression function under an ergodic hypothesis, Canad. J. Statist. 28 (2000), 817-828.

[26] M. Lemdani and E. Ould Saïd, Nonparametric robust regression estimation for censored data, Statistical Papers 58(2) (2017), 505-525. DOI: 10.1007/s00362-015-0709-8.

[27] M. Loève, Probability Theory, Springer-Verlag, New York, 1963.

[28] E. Masry, Recursive probability density estimation for weakly dependent stationary processes, IEEE Trans. Inform. Theory 32 (1986), 254-267.

[29] E. Masry, Nonparametric regression estimation for dependent functional data: asymptotic normality, Stochastic Process. Appl. 115 (2005), 155-177.

[30] E. Rio, Covariance inequalities for strongly mixing processes, Ann. Inst. Henri Poincaré Probab. Statist. 29 (1993), 587-597.

[31] E. Rio, Théorie asymptotique des processus aléatoires faiblement dépendants, Mathématiques et Applications, Springer, New York, 2000.

[32] P. M. Robinson, Robust nonparametric autoregression, Lecture Notes in Statistics, 26, Springer, New York, 1984, pp. 247-255.