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ON THE EXISTENCE AND UNIQUENESS OF SOLUTIONS OF FRACTIONAL ORDER PARTIAL INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract

The main objective of this paper is to investigate and prove the existence and uniqueness theorems of the solution of nonlinear partial integro-differential equations of fractional order. The theorems of two types for the nonlinear fractional order partial integro-differential equations such as one- and two-dimensional are proved by applying the fixed point theorem of Banach space couple with contraction mapping principle, in which the sufficient conditions are presented in order to ensure the existence and uniqueness of a unique fixed point related to the integro-differential equation in operator form.

1. Introduction

In the last three decades, considerable interest has been paid to the so-called fractional calculus, which allows us to consider integration and differentiation of any order, not necessarily integers. For a large extent, this is due to the applications of fractional calculus to problems in different areas of pure and applied sciences, such as physics, chemistry, aerodynamics, electrodynamics of complex medium, viscoelasticity, heat conduction, electricity mechanics, control theory, [1]. The topic fractional calculus can be measured as an old as well as a new subject. Started from some speculations of Leibniz and Euler, followed by other important mathematicians like Laplace, Fourier, Abel, Liouville, Riemann and Holmgren, [12].

Applications of PIDEs can be found in various fields such as, heat conduction of materials with memory, viscoelasticity, nuclear reactor dynamics, jump-diffusion models for pricing of derivatives in finance, financial modelling, electricity swaptions and biofluid flow in fractured biomaterials, [2].

Most of the research papers deal with the existence of unique solutions to equations for different types of differential and/or integral equations of one or multidimension. The Banach fixed point principle which is established by Polish mathematician Stefan Banach in 1922 is an important approach and it is one of the most powerful, fruitful tools of modern mathematics and may be considered as a core subject for nonlinear analysis. It is used by a lot of authors to demonstrate the existence of a unique fixed point of certain self-maps of metric or normed spaces, [3, 6].

In the later years, many theorems concerned with the existence, uniqueness and stability of fractional nonlinear differential and integro-differential equations as a basic theoretical part of some applications are investigated by many authors, for example, Engler [4] studied the existence and regularity results for two classes of semilinear parabolic integro-differential equations on a bounded time-space cylindrical figures, Hadid et al. [5] used the fixed point theorem and the contraction mapping principle to obtain the local existence and uniqueness of solution of differential equations of fractional order differential equations, Yu and Gao [20] obtained a sufficient condition for the existence of the solutions of fractional differential equations by using Schauder fixed point theorem, Momani et al. [10] studied the local and global uniqueness theorems of solutions of the fractional integro-differential equation by using Bihari's and Gronwall's inequalities, Ibrahim and Momani [13] studied the existence and uniqueness of solutions of a class of fractional order differential equations, Su and Liu [16] examined the results of the existence of boundary value problem of a nonlinear fractional differential equation by using Schauder fixed point theorem, Bahuguna and Dabas [2] applied the method of lines to establish the existence and uniqueness of a strong solution for the partial integro-differential equations, Learning et al. [8] establish the existence and uniqueness of mild solutions for a class of semilinear integro-differential equations of fractional order with nonlocal initial conditions by using the fixed point theorem due to Sadovskii, Matar [9] deliberated the existence of solutions for nonlocal fractional semilinear integro-differential equations in Banach spaces via Banach fixed point theorem, Murad et al. [15] studied the existence of a unique solution of boundary value problem for fractional order integro-differential equations in Banach spaces by using Banach and Krasnosel'skii fixed point theorems, Karthikeyan and Trujillo [7] also examined the existence and uniqueness of solutions of fractional order integro-differential equations with boundary conditions via Schaefer's fixed point theorem, Tari [18] proved a theorem and some results related to the existence and uniqueness of solution of two-dimensional Volterra partial integro-differential equation.

Motivated by these works, in this paper, we study and prove the existence and uniqueness of the solution and find the sufficient condition satisfying the Lipschitz condition for the subsequent nonlinear fractional order partial integro-differential equation (NFPIDE):

One-dimensional NFPIDE:

$$ {}_{0}^{C}D_{t}^{\alpha}u(x, t) = g(x, t) + \int_{a}^{x} k(s, t, u(s, t))ds, $$

$$ \alpha \in \mathbb{R}^{+}, \ x \in [a, b], \ t \in [0, T]. \tag{1} $$

Generalized one-dimensional NFPIDE:

$$ {}_{0}^{C}D_{t}^{\alpha}u(x, t) = g(x, t) + {}_{a}I_{x}^{\beta}k(s, t, u(s, t)), $$

$$ \alpha, \beta \in \mathbb{R}^{+}, \ x \in [a, b], \ t \in [0, T]. \tag{2} $$

Two-dimensional NFPIDE:

$$ {}_{0}^{C}D_{t}^{\alpha}(x, t) = g(x, t) + \int_{a}^{x}\int_{0}^{t} k(s, y, u(s, y))dsdy, $$

$$ \alpha \in \mathbb{R}^{+}, \ x \in [a, b], \ t \in [0, T]. \tag{3} $$

Generalized two-dimensional NFPIDE:

$$ {}_{0}^{C}D_{t}^{\alpha}u(x, t) = g(x, t) + {}_{a}I_{x}^{\beta}\,{}_{0}I_{t}^{\gamma}k(s, y, u(s, y)), $$

$$ \alpha, \beta, \gamma \in \mathbb{R}^{+}, \ x \in [a, b], \ t \in [0, T], \tag{4} $$

where k is the kernel function, g is a given function, $u(x, t)$ is an unknown continuous function defined for all x and t, such that $a \le x \le b$, $0 \le t \le T$, $m - 1 < \alpha, \beta, \gamma \le m$, $m \in \mathbb{N}$, ${}_{0}^{C}D_{t}^{\alpha}$ denotes the operator of Caputo fractional order derivative and ${}_aI_x^{\beta}$, ${}_0I_t^{\gamma}$ denote the Riemann-Liouville fractional order integral operators.

In addition, equations (1), (2), (3) and (4) are all considered with the initial condition:

$$u(x, 0) = u_0(x), \quad x \in [a, b]. \tag{5}$$

2. Basic Concepts and Definitions

In order to proceed forward, we give some basic definitions and theorem which are used later on in this paper.

Definition 2.1 [14]**.** Let $T : X \to X$ be a mapping on a normed space $(X, \|\cdot\|)$. A point $x \in X$ for which $Tx = x$ is called a *fixed point* of *T*.

Definition 2.2 [16]**.** The mapping T on a normed space $(X, \|\cdot\|)$ is called *contractive* if there is a non-negative real number $c \in (0, 1)$, such that

$$\| Tx_1 - Tx_2 \| \le c\| x_1 - x_2 \|, \text{ for all } x_1, x_2 \in X.$$

Theorem 2.1 (Banach fixed point theorem) [1]**.** *Let* $(X, \|\cdot\|)$ *be a complete normed space and let the mapping* $T : X \to X$ *be a contraction mapping*. *Then T has exactly one fixed point*.

Definition 2.3 (The Riemann-Liouville fractional integral of two variables) [17, 19]**.** Let $\alpha \ge 0$ be a real number and suppose that for $u \in C[a, b]$, the Riemann-Liouville integral operator ${}_aI_t^{\alpha}$ is defined by:

$${}_aI_t^{\alpha}u(x, t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_a^t (t-s)^{\alpha-1}u(x, s)ds, & \alpha > 0, \\ u(x, t), & \alpha = 0. \end{cases}$$

Definition 2.4 (The Riemann-Liouville fractional differential operator of two variables) [11]**.** Let $\alpha \ge 0$ be any real number and m be the natural number, such that $m - 1 < \alpha \le m$. If $u \in C_t[a, b]$, where $u \in C_t[a, b]$

is the set of all continuous real valued functions on $[a, b]$, which has a continuous nth partial derivative on $[a, b]$ with respect to the variable t, then the Riemann-Liouville differential operator ${}_aD_t^{\alpha}$ is defined by:

$$
{}_aD_t^{\alpha}u(x, t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \dfrac{\partial^n}{\partial t^n}\left(\displaystyle\int_a^t (t-s)^{n-\alpha-1}u(x, s)ds\right), & \alpha > 0, \\ u(x, t), & \alpha = 0. \end{cases}
$$

Definition 2.5 (The Caputo fractional differential operator of two variables) [11]**.** For m to be the smallest integer that exceeds $\alpha \geq 0$, the operator of Caputo fractional order derivative of order $\alpha > 0$ is defined as:

$$
{}_aD_t^{\alpha}u(x, t) = \frac{\partial^{\alpha}u(x, t)}{\partial t^{\alpha}}
$$

$$
= \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_a^t (t-\tau)^{m-\alpha-1} \dfrac{\partial^m u(x, \tau)}{\partial \tau^m} d\tau, & \text{for } m-1 < \alpha < m, \\ \dfrac{\partial^m u(x, t)}{\partial t^m}, & \text{for } \alpha = m \in \mathbb{N} \end{cases}
$$

and the space-fractional derivative operator of order $\beta > 0$ is defined as:

$$
{}_aD_x^{\beta}u(x, t) = \frac{\partial^{\beta}u(x, t)}{\partial x^{\beta}}
$$

$$
= \begin{cases} \dfrac{1}{\Gamma(m-\beta)} \displaystyle\int_a^t (t-\theta)^{m-\beta-1} \dfrac{\partial^m u(x, \theta)}{\partial \theta^m} d\theta, & \text{for } m-1 < \beta < m, \\ \dfrac{\partial^m u(x, t)}{\partial x^m}, & \text{for } \beta = m \in \mathbb{N}. \end{cases}
$$

Definition 2.6 [4]**.** Let $(X, \|\cdot\|)$ be a normed space, a function f defined on the set:

$$
Q = \{(x, t;\ y_1, y_2, \ldots, y_n) : a \leq x, t \leq b, -\infty < y_i < \infty, \text{ for each } i = 1, 2, \ldots, n\}
$$

is said to satisfy *Lipschitz condition* with respect to the variables

$y_1, y_2, ..., y_n$; if a constant $L > 0$ exists with the property that

$$\| f(x, t; y_1, y_2, ..., y_m) - f(x, t; z_1, z_2, ..., z_m) \| \leq L \sum_{j=1}^{n} \| y_i - z_i \|$$

for all $(x, t; y_1, y_2, ..., y_n)$ and $(x, t; z_1, z_2, ..., z_n)$ in Q.

Remark 2.1. The space $C_t([a, b] \times [0, T])$ will be considered in this work as the Banach space of all continuous real valued functions u defined on $[a, b] \times [0, T]$ with continuous nth order partial derivatives with respect to t.

3. Existence and Uniqueness Theorems

In this section, the statement and the proof of the existence and uniqueness of the solution for different types of NFPIDEs are established in which the proof depends on the Banach fixed point theorem in the next theorems.

Now, we consider first the one-dimensional NFPIDE:

$$ {}_0^C D_t^{\alpha} u(x, t) = g(x, t) + \int_a^x k(y, t, u(y, t)) dy,$$

$$0 < \alpha \leq 1, \; x \in [a, b], \; t \in [0, T] \tag{1}$$

with initial condition:

$$u(x, 0) = u_0(x) \tag{5}$$

and recall that for any function $f(x) \in L_1[a, b]$ and $0 < \alpha \leq 1$, then [2]:

$$ {}_a I_x^{\alpha} \, {}_a^C D_x^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{(x-a)^k}{k!}, \quad x > 0, \tag{6}$$

where ${}_a I_x^{\alpha}$ refers to Riemann-Liouville fractional order integral and ${}_a^C D_x^{\alpha}$ refers to the Caputo fractional order derivative.

Hence, upon applying equation (6) for a function of two variables on equation (1), therefore for $0 < \alpha \leq 1$ and taking ${}_aI_t^{\alpha}$ to both the sides of equation (1) will yield to:

$$u(x, t) - u_0(x) = {}_0I_t^{\alpha} g(x, t) + {}_0I_t^{\alpha} \int_a^x k(y, t, u(y, t)) dy \tag{7}$$

which may be rewritten in operator form as:

$$u(x, t) = u_0(x) + {}_0I_t^{\alpha} g(x, t) + \frac{1}{\Gamma(\alpha)} \int_0^t \int_a^x (t-s)^{\alpha-1} k(y, s, u(y, s)) dyds \tag{8}$$

$$= Tu. \tag{9}$$

Theorem 3.1. *Consider the NFPIDE* (1) *with initial condition* (5) *over the region Q and suppose that k satisfies Lipschitz condition with respect to u with constant L such that* $L < \frac{\Gamma(\alpha+1)}{T^{\alpha}(b-a)}$. *Then equation* (1) (*or equivalently* (8)) *has a unique solution*.

Proof. The set of all continuously differentiable functions defined over the region Q form a complete normed space with supremum norm. Also, as it is seen equation (8) is given in operator form $Tu = u$ and therefore it remains to show that T is a contractive mapping and for this purpose take $u_1, u_2 \in C_t([a, b] \times [0, T])$, and then:

$$\| Tu_1(x, t) - Tu_2(x, t) \|$$

$$= \Bigg\| u_0(x) + {}_0I_t^{\alpha} g(x, t) + \frac{1}{\Gamma(\alpha)} \int_0^t \int_a^x (t-s)^{\alpha-1} k(y, s, u_1(y, s)) dyds$$

$$- u_0(x) - {}_0I_t^{\alpha} g(x, t) - \frac{1}{\Gamma(\alpha)} \int_0^t \int_a^x (t-s)^{\alpha-1} k(y, s, u_2(y, s)) dyds \Bigg\| \tag{10}$$

$$\leq \left\| \frac{1}{\Gamma(\alpha)} \int_0^t \int_a^x (t-s)^{\alpha-1} [k(y, s, u_1(y, s) - k(y, s, u_2(y, s)))] dyds \right\| \tag{11}$$

$$\leq \frac{L}{\Gamma(\alpha)} \int_0^t \int_a^x (t-s)^{\alpha-1} \| u_1(y, s) - u_2(y, s) \| dyds \tag{12}$$

$$\leq \frac{L}{\Gamma(\alpha)} \| u_1 - u_2 \| \int_0^t \int_a^x (t-s)^{\alpha-1} dyds \tag{13}$$

$$= \frac{L}{\Gamma(\alpha)} \| u_1 - u_2 \| \frac{(x-a)t^{\alpha}}{\alpha} \tag{14}$$

$$= \frac{L}{\alpha\Gamma(\alpha)} (x-a)t^{\alpha} \| u_1 - u_2 \| \tag{15}$$

$$= \frac{L}{\Gamma(\alpha+1)} (x-a)t^{\alpha} \| u_1 - u_2 \| \tag{16}$$

$$\leq \frac{L(b-a)T^{\alpha}}{\Gamma(\alpha+1)} \| u_1 - u_2 \| \tag{17}$$

and since $L < \frac{\Gamma(\alpha+1)}{T^{\alpha}(b-a)}$, which implies $\frac{L(b-a)T^{\alpha}}{\Gamma(\alpha+1)} < 1$, T is a contractive mapping and therefore T has a unique fixed point, which means that equation (1) has a unique solution. □

Now, we prove the existence and uniqueness of the second type of one-dimensional NFPIDE that may be considered of the form

$${}_0^C D_t^{\alpha} u(x, t) = g(x, t) + {}_a I_x^{\beta} k(x, t, u(x, t)), \tag{2}$$

where $0 < \alpha \leq 1$, $\beta \in \mathbb{R}^+$, over the region Q, with initial condition $u(x, 0) = u_0(x)$.

Therefore, upon taking ${}_0 I_t^{\alpha}$ to both the sides of equation (2), we get:

$$u(x, t) = u_0(x) + {}_0 I_t^{\alpha} g(x, t) + {}_0 I_t^{\alpha} {}_a I_x^{\beta} k(x, t, u(x, t)) \tag{18}$$

$$= u_0(x) + {}_0 I_t^{\alpha} g(x, t)$$

$$+ \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_a^x (t-s)^{\alpha-1} (x-y)^{\beta-1} k(y, s, u(y, s)) dyds \tag{19}$$

$$= Tu. \tag{20}$$

Theorem 3.2. *Consider the equation* (2) *over the region Q and suppose that k satisfies Lipschitz condition with respect to u with constant L such that* $L < \dfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{T^{\alpha}(b-a)^{\beta}}$, *then equation* (2) *has a unique solution.*

Proof. Similarly, as in the proof of Theorem 3.1, let $u_1, u_2 \in C_t([a, b] \times [0, T])$, and then:

$$\| Tu_1(x, t) - Tu_2(x, t) \|$$

$$= \Bigg\| u_0(x) + {}_0I_t^{\alpha} g(x, t) + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_a^x (t-s)^{\alpha-1}(x-y)^{\beta-1} k(y, s, u_1(y, s)) dyds$$

$$- u_0(x) - {}_0I_t^{\alpha} g(x, t)$$

$$- \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_a^x (t-s)^{\alpha-1}(x-y)^{\beta-1} k(y, s, u_2(y, s)) dyds \Bigg\| \tag{21}$$

$$= \left\| \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_a^x (t-s)^{\alpha-1}(x-y)^{\beta-1} [k(y, s, u_1) - k(y, s, u_2)] dyds \right\| \tag{22}$$

$$\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_a^x (t-s)^{\alpha-1}(x-y)^{\beta-1} \| k(y, s, u_1) - k(y, s, u_2) \| dyds \tag{23}$$

$$\leq \frac{L}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_a^x (t-s)^{\alpha-1}(x-y)^{\beta-1} \| u_1 - u_2 \| dyds \tag{24}$$

$$\leq \frac{L}{\Gamma(\alpha)\Gamma(\beta)} \| u_1 - u_2 \| \int_0^t \int_a^x (t-s)^{\alpha-1}(x-y)^{\beta-1} dyds \tag{25}$$

$$= \frac{L}{\Gamma(\alpha)\Gamma(\beta)} \| u_1 - u_2 \| \frac{t^{\alpha}(x-a)^{\beta}}{\alpha\beta} \tag{26}$$

$$= \frac{Lt^{\alpha}(x-a)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)} \| u_1 - u_2 \| \tag{27}$$

$$\leq \frac{LT^{\alpha}(b-a)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)} \| u_1 - u_2 \| \tag{28}$$

and since $L < \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{LT^{\alpha}(b-a)^{\beta}}$, the mapping T is a contractive mapping and therefore it has a unique fixed point, which means that equation (2) has a unique solution. □

Now, we study and prove the existence of a unique solution of the two-dimensional NFPIDE. We start with this type of following two-dimensional NFPIDE:

$$ {}_{0}^{C}D_t^{\alpha}u(x, t) = g(x, t) + \int_a^x \int_0^t k(y, s, u(y, s))ds\,dy, \tag{29} $$

where $0 < \alpha \leq 1$, $(x, t) \in Q$ with the initial condition $u(x, 0) = u_0(x)$.

We start by taking ${}_0I_t^{\alpha}$ to both the sides of equation (29), will yield to:

$$ \begin{aligned} u(x, t) & \\ &= u_0(x) + {}_0I_t^{\alpha} g(x, t) + {}_0I_t^{\alpha} \int_a^x \int_0^t k(y, s, u(y, s))ds\,dy && (30) \\ &= u_0(x) + {}_0I_t^{\alpha} g(x, t) + \frac{1}{\Gamma(\alpha)} \int_0^t \int_a^x \int_0^{\xi} (t-\xi)^{\alpha-1} k(y, s, u(y, s))dyds d\xi && (31) \\ &= Tu. && (32) \end{aligned} $$

Theorem 3.3. *Consider the two-dimensional nonlinear FPIDE* (3) *and suppose that k satisfies Lipschitz condition with constant L such that* $L < \frac{\Gamma(\alpha+2)}{(b-a)T^{\alpha+1}}$. *Then equation* (3) *has a unique solution.*

Proof. Let $u_1, u_2 \in C_t([a, b] \times [0, T])$. Hence

$$ \begin{aligned} &\| Tu_1(x, t) - Tu_2(x, t) \| \\ &= \Big\| u_0(x) + {}_0I_t^{\alpha} g(x, t) + \frac{1}{\Gamma(\alpha)} \int_0^t \int_a^x \int_0^{\xi} (t-\xi)^{\alpha-1} k(y, s, u_1(y, s))dsdyd\xi \\ &\quad - u_0(x) - {}_0I_t^{\alpha} g(x, t) - \frac{1}{\Gamma(\alpha)} \int_0^t \int_a^x \int_0^{\xi} (t-\xi)^{\alpha-1} k(y, s, u_2(y, s))dsdyd\xi \Big\| \quad (33) \end{aligned} $$

$$= \left\| \frac{1}{\Gamma(\alpha)} \int_0^t \int_a^x \int_0^\xi (t-\xi)^{\alpha-1} [k(y, s, u_1(y, s)) - k(y, s, u_2(y, s))] ds dy d\xi \right\| \quad (34)$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t \int_a^x \int_0^\xi (t-\xi)^{\alpha-1} \| k(y, s, u_1(y, s)) - k(y, s, u_2(y, s)) \| ds dy d\xi \quad (35)$$

$$\leq \frac{L}{\Gamma(\alpha)} \| u_1(y, s) - u_2(y, s) \| \int_0^t \int_a^x \int_0^\xi (t-\xi)^{\alpha-1} ds dy d\xi \quad (36)$$

and upon using the method of integration by parts to evaluate the last integral, it is found that

$$\| Tu_1 - Tu_2 \| \leq \frac{L}{\Gamma(\alpha)} \| u_1(y, s) - u_2(y, s) \| \frac{(x-a)t^{\alpha+1}}{\alpha(\alpha+1)} \quad (37)$$

$$= \frac{L(x-a)t^{\alpha+1}}{\Gamma(\alpha+2)} \| u_1(y, s) - u_2(y, s) \| \quad (38)$$

$$\leq \frac{L(x-a)t^{\alpha+1}}{\Gamma(\alpha+2)} \| u_1 - u_2 \| \quad (39)$$

and since $L < \frac{\Gamma(\alpha+2)}{(b-a)T^{\alpha+1}}$, which implies to $\frac{L(b-a)T^{\alpha+1}}{\Gamma(\alpha+2)} < 1$ and hence T is a contraction mapping, which means that T has a unique fixed point, i.e., equation (3) has a unique solution. □

Now, the second type of two-dimensional NFPIDE which is the most general case has the form:

$${}_0^C D_t^\alpha u(x, t) = g(x, t) + {}_a I_x^\beta {}_0 I_t^\gamma k(s, t, u(x, t)), \quad (4)$$

where $0 < \alpha \leq 1$, $\beta, \gamma \in \mathbb{R}^+$ over the region Q with initial condition $u(x, 0) = u_0(x)$.

Before we start the statement and the proof of the existence and uniqueness theorem, recall first the following property for Riemann-Liouville fractional order integrals:

$${}_0I_t^{\alpha}\,{}_0I_t^{\gamma} f(t) = {}_0I_t^{\gamma}\,{}_0I_t^{\alpha} f(t) = {}_0I_t^{\alpha+\gamma} f(t) = {}_0I_t^{w} f(t),$$

$$\alpha, \gamma \in \mathbb{R}^+, \alpha + \gamma = w \in \mathbb{R}^+.$$

Therefore, taking ${}_0I_t^{\alpha}$ to both the sides of equation (4), will yield to:

$$u(x, t) = u_0(x) + {}_0I_t^{\alpha} g(x, t) + {}_0I_t^{\alpha}\,{}_aI_t^{\gamma}\,{}_0I_x^{\beta} k(s, t, u(x, t)) \tag{40}$$

$$u(x, t) = u_0(x) + {}_0I_t^{\alpha} g(x, t) + {}_0I_t^{w}\,{}_0I_x^{\beta} k(s, t, u(x, t)) \tag{41}$$

$$= u_0(x) + {}_0I_t^{\alpha} g(x, t)$$

$$+ \frac{1}{\Gamma(w)\Gamma(\beta)} + \int_0^t \int_a^x (t-s)^{w-1}(x-y)^{\beta-1} k(y, s, u(y, s))\,dyds \tag{42}$$

$$= Tu. \tag{43}$$

Theorem 3.4. *Consider the two-dimensional NFPIDE* (4) *and suppose that k satisfies Lipschitz condition with constant L such that* $L < \frac{\Gamma(w+1)\Gamma(\beta+1)}{T(b-a)}$. *Then equation* (4) *has a unique solution.*

Proof. Let $u_1, u_2 \in C_t([a, b] \times [0, T])$, hence:

$$\| Tu_1 - Tu_2 \|$$

$$= \Bigg\| u_0(x) + {}_0I_t^{\alpha} g(x, t)$$

$$+ \frac{1}{\Gamma(w)\Gamma(\beta)} \int_0^t \int_a^x (t-s)^{w-1}(x-y)^{\beta-1} k(y, s, u_1(u, s))\,dsdy$$

$$- u_0(x) - {}_0I_t^{\alpha} g(x, t)$$

$$- \frac{1}{\Gamma(w)\Gamma(\beta)} \int_0^t \int_a^x (t-s)^{w-1}(x-y)^{\beta-1} k(y, s, u_2(u, s))\,dsdy \Bigg\| \tag{44}$$

$$\leq \frac{L}{\Gamma(w)\Gamma(\beta)} \| u_1 - u_2 \| \int_0^t \int_a^x (t-s)^{w-1}(x-y)^{\beta-1}\,dsdy \tag{45}$$

$$= \frac{L}{\Gamma(w)\Gamma(\beta)} \| u_1 - u_2 \| \frac{t^w (x-a)^\beta}{w\beta} \tag{46}$$

$$= \frac{Lt^w (x-a)^\beta}{\Gamma(w+1)\Gamma(\beta+1)} \| u_1 - u_2 \| \tag{47}$$

$$= \frac{LT^w (b-a)^\beta}{\Gamma(w+1)\Gamma(\beta+1)} \| u_1 - u_2 \| \tag{48}$$

and since $L < \frac{\Gamma(w+1)\Gamma(\beta+1)}{T^w (b-a)^\beta}$, then the mapping T is a contractive mapping and therefore it has a unique fixed point. Therefore, equation (4) has a unique solution. □

4. Conclusion

One of the most interesting branches in fractional calculus and fractional order ordinary and/or integral equations is to obtain the solutions of fractional order partial integro-differential equations. Having these things in mind, we study theoretically in this paper the existence and uniqueness of solution for one- and two-dimensional nonlinear fractional order partial integro-differential equations by applying Banach fixed point theorem. In this work we conclude that in equation (4) if $\beta = 1$, $\gamma = 0$, then we get fractional order partial integro-differential equation (1), while if $\gamma = 0$ in equation (4), then we get fractional order partial integro-differential equation (2) and if $\beta = 1$, $\gamma = 1$ in equation (4), then we get fractional order partial integro-differential equation (3). Hence equation (4) and its existence and uniqueness theorem of solutions is considered as a generalization of the existence and uniqueness theorem and the solutions of equations (1), (2) and (3).

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